Quasigroups and Related Systems **34** (2026), 355 – 363

https://doi.org/10.56415/qrs.v34.24

Ideal-primeness in the category of acts

Mohammad Roueentan and Roghaieh Khosravir

Abstract. The main purpose of this article is to introduce and study the concept of ideal-prime subacts. For fully invariant subacts of an S-act, some equivalent conditions are given to being ideal-prime, which gives us conditions for the primeness of an ideal on a commutative monoid S.

1. Introduction

In the category of rings the study of the properties of prime ideals is an important classical research subject. In fact some of the important properties of a ring R are known by studying its ideals which justifies the study of prime ideals. Also several attempts have been made to extend the concept of prime ideals to the concept of prime submodules (see for example [3], [8], [9]). Prime modules and submodules notions in module theory, originated in the work of Dauns, ([5]). We recall that an ideal P of a ring R is said to be *prime* if for any $a, b \in R$, whenever $aRb \subseteq P$, either $a \in P$ or $b \in P$. By a similar way the concept of a prime ideal is provided over a semigroup S, see for example [4]. The first investigation into prime acts and prime subacts goes back to Ahsan and Zhongkui ([2]) and was followed by some other authors, for instance [6, 10, 13]. From [10] recall that over a semigroup S, a proper subact B of an S-act A is said to be *prime* if for any $a \in A$ and $t \in S, aSt \subseteq B$ implies that $a \in B$ or $At \subseteq B$. Recall that from [12], a subact B of an S-act A is called *fully invariant* if for any endomorphism f of $A, f(B) \subseteq B$ and also A is called *duo* if any subact of A is fully invariant. Let S be a monoid and A be an S-act. If $T = \mathrm{End}(A)$ and B is a fully invariant subact of A, then we say that B is an *ideal-prime* subact of A if for any ideal I of T and any fully invariant subact C of A, whenever $I(C) \subseteq B$, then either $I(A) \subseteq B$ or $C \subseteq B$ where for any subact D of $A, I(D) = \bigcup_{f \in I} f(D)$. In this paper, over a monoid S, we study the concept of ideal-primeness for fully invariant subacts of an S-act A. Several

2010 Mathematics Subject Classification: 20M30
Keywords: S-act, monoid, ideal-prime

equivalent cnditions to being ideal prime are given and in particular as a result, we provide conditions under which an ideal is prime. In this paper we assume that S denotes a monoid and an S-act A_S (or in the short form A) is a unitary right S-act. An equivalence relation ρ on an S-act A is called a *congruence* on A if $a\ \rho\ b$ implies $(as)\ \rho\ (bs)$ for every $a, b \in A, s \in S$ and also the equivalence class of $a \in A$ is denoted by $[a]_\rho$. For an S-act A the diagonal relation $\{(a, a) \mid a \in A\}$ on A is a congruence on A which is denoted by Δ_A. Also if B is a subact of A, then the congruence $(B \times B) \cup \Delta_A$ on A is denoted by ρ_B and is called the *Rees congruence* by the subact B. The resulting factor act is denoted byA/B and is called the *Rees factor act* of A by the subact B. From [1] an S-act A is called *quasi-projective* if for every S-act C, every S-homomorphism $f : A \longrightarrow C$ can be lifted with respect to every epimorphism $g : A \longrightarrow C$, i.e., there exists a homomorphism $h : A \longrightarrow A$ such that $f = gh$. Necessary backgrounds about acts can be found in [7].

2. Ideal-prime subacts

In this section we investigate preliminary and basic properties of ideal-prime subacts over a monoid S. From [7] for S-acts A and B, the trace of B in A is defined as $\mathrm{Tr}(B,A) := \bigcup_{f\in \mathrm{Hom}(B,A)} f(B)$ where $\mathrm{Hom}(B,A)$ denotes the set of homomorphisms from B to A. If $T = \mathrm{End}(A)$ is the set of endomorphisms of an S-act A and I is a subset of T, then for any non-empty subset C of A, we denote $I(C) := \bigcup_{f\in I} f(C)$ which is a subact of $\mathrm{Tr}(C,A)$ whenever C is a subact of A.

Definition 2.1. Suppose S is a monoid and A is an S-act and also $T = End(A)$. A proper fully invariant subact B of A is called *ideal-prime* if for any fully invariant subact C of A and any ideal I of T, if $I(C) \subset B$, then either $I(A) \subset B$ or $C \subset B$.

Theorem 2.2. *Suppose S is a monoid and A is an S-act and also $T = End(A)$. If B is a proper fully invariant subact of A, then the following conditions are equivalent:*

(i) *B is an ideal-prime subact of A.*

(ii) *For any subact C of A and any right ideal I of T, if $I(C) \subset B$, then either $I(A) \subset B$ or $C \subset B$.*

(iii) *For any $f \in T$ and any fully invariant subact C of A if $f(C) \subset B$, then either $f(A) \subset B$ or $C \subset B$.*

(iv) *For any subset D of A and any left ideal I of T, if $IT(D) \subset B$, then either $I(A) \subset B$ or $D \subset B$.*

(v) *For any $f \in T$ and any $a \in A$ if $f(T(a)) \subset B$, then either $f(A) \subset B$ or $a \in B$.*

Proof. (i) $\Rightarrow$ (ii): Suppose B is an ideal-prime subact of A and I is a right ideal of T. Thus $I = IT$ and TI is an ideal of T. Since C is a subact of A, we can see that $T(C)$ is a fully invariant subact of A. If $I(C) \subset B$, then $(TI)(T(C)) = TIT(C) = T(I(C)) \subset B$. Thus $TI(A) \subset B$ or $T(C) \subset B$ and so $I(A) \subset B$ or $C \subset B$ as desired.

(ii) $\Rightarrow$ (iii): Suppose $f \in T$ and C is a fully invariant subact of A such that $f(C) \subset B$. Since $f(C) = f(T(C)) = fT(C)$, by assumption $fT(A) \subset B$ or $C \subset B$ and consequently $f(A) \subset B$ or $C \subset B$.

(iii) $\Rightarrow$ (v): Let $f \in T$ and $a \in A$ such that $f(T(a)) \subset B$. Then $f(T(a))S \subset B$ and hence $f(T(aS)) \subset B$. Thus by assumption, we have $f(A) \subset B$ or $T(aS) \subset B$ which impies that $f(A) \subset B$ or $a \in B$.

(ii) $\Rightarrow$ (iv): If I is a left ideal of T and $IT(D) \subset B$, then $IT(D)S \subset B$. It follows that $IT(DS) \subset B$. Thus by assumption $IT(A) \subset B$ or $DS \subset B$ proving that $I(A) \subset B$ or $D \subset B$.

(iv) $\Rightarrow$ (v): Since B is a fully invariant subact of A, the condition $f(T(a)) \subset B$ implies that $T(fT(a)) \subset B$ and hence $(Tf)(T(a)) \subset B$. Thus by (iv), $Tf(A)) \subset B$ or $\{a\} \subset B$ and codequently $f(A) \subset B$ or $a \in B$.

(v) $\Rightarrow$ (i): Suppose I is an ideal of T and C is a fully invariant subact of A such that $I(C) \subset B$ such that $I(A) \nsubseteq B$ and $C \nsubseteq B$. Then there exists $f \in I$ with $f(A) \nsubseteq B$ and $a \in C \backslash B$. Thus by (v), $fT(a) \nsubseteq B$, a contradiction. □

From [12] an S-act A is called *duo* if any subact of A is fully invariant. It is easy to see that over a commutative monoid S any cyclic S-act is duo.

Example 2.3. Suppose $S = \{1, e, f, 0\}$ is a monoid with zero such that $e^2 = e, f^2 = f$ and $ef = fe = 0$. If $A = (e, f)S = \{(e, f)s | s \in S\}$, then the proper subacts of A are $\{(0, 0)\}$, $B = (e, 0)S = \{(0, 0), (e, 0)\}$ and $C = (0, f)S = \{(0, 0), (0, f)\}$. We show that B is an ideal-prime subact of A. First note that S is commutative and so A is a duo S-act. Suppose $T = End(A)$ and for a subact D of $A, I(D) \subset B$, Where I is an ideal of T. According to the subacts of A, it is sufficient to prove the result for the case $D = C$. Since $C \nsubseteq B$, we should prove that $I(A) \subset B$. Suppose $g \in I$ is a homomorphism of A and $g((e, f)) = (e, f)s$ for some $s \in S$. If $s = 0$

or $s = e$, then clearly the result follows. If $s = 1$, then $g((e, f)f) = (e, f)f$ and so $g((0, f)) = (0, f)$ which contradicts $I(C) \subset B$. Also if $s = f$, then we have the same contradiction. Thus $I(A) \subset B$ and we are done.

Recall that for an element a of an S-act A, the homomomorphism $\lambda_a : S_S \longrightarrow A$ is defined by $\lambda_a(s) = as$ for every $s \in S$. From [10] a subact B of an S-act A is called *prime* if for any $a \in A$ and $t \in S, aSt \subseteq B$ implies that $a \in B$ or $At \subseteq B$.

Remark 2.4. If S is a commutative monoid and A is an S-act, then for any $s \in S$ we can define the homomorphism $f : A \longrightarrow A$ by $f(a) = as$ for any $a \in A$. By a routin argument and due to the previous theorem, we can see that, if B is an ideal-prime subact of A, then B is a prime subact of A.

From [11], an S-act A is called *fully idempotentet* if for any subact B of $A, B = \bigcup_{f \in Hom(A,B)} f(B)$.

Corollary 2.5. *Suppose S is a monoid and A is a quasi-projective S-act and also $T = End(A)$. If B is a proper fully invariant subact of A, then the following conditions are equivalent:*

(i) *B is an ideal-prime subact of A.*

(ii) *For the factor act A/B, the subact $\bar{0}$ is ideal-prime.*

Moreover if A is a fully idempotent S-act, then the above conditions are equivalent to:

(iii) *If I is an ideal of T and C is a fully invariant subact of A such that $I(A)$ and C properly contain B, then $I(C) \nsubseteq B$.*

Proof. To prove (i) $\Leftrightarrow$ (ii) we use part (iii) of the previous theorem.

(i) $\Rightarrow$ (ii): Let $\bar{C}$ be a fully invariant subact of A/B and $g \in \mathrm{End}(A/B)$ such that $g(\bar{C}) = \bar{0}$ and $g \neq 0$. Since A is quasi-projective, there exists $f \in T$ such that $\pi f = g\pi$ where $\pi : A \longrightarrow A/B$ is the natural epimorphism. Since $g \neq 0, f(A) \nsubseteq B$. Let $K = \pi^{-1}(\bar{C})$. Then $\pi f(K) = \bar{0}$ and hence $f(K) \subset B$. Clearly K is a fully invariant subact of A and by the properties of B, we must have $K \subset B$ proving that $\bar{C} = \bar{0}$ and the result follows.

(ii) $\Rightarrow$ (i): Let $f \in T$ and C be a fully invariant subact of A such that $f(C) \subset B$ and $f(A) \nsubseteq B$. Define $g : A/B \longrightarrow A/B$ by $g(\bar{a}) = \overline{f(a)}$ for any $a \in A$. Since B is a fully invariant subact of A, g is a well-defined homomorphism. Also since $\pi f = g\pi$, $g\pi(C) = \pi f(C) = \bar{0}$ and consequently

by part (iii) of Theorem 2.2 and assumption, we have $\pi(C) = \bar{0}$ proving that $C \subset B$ as desired.

(i) $\Rightarrow$ (iii): Holds regarding the concept of ideal-primeness subacts.

(iii) $\Rightarrow$ (ii): Since B is a fully invariant subact of A, for any $f \in T$ we can find $\bar{f} \in \mathrm{End}(A/B)$ such that $\bar{f}\pi = \pi f$. Let $\bar{I}$ be an ideal of $\mathrm{End}(A/B)$ and $\bar{C}$ a fully invariant subact of A/B such that $\bar{I}(\bar{C}) = \bar{0}$. If $I = \{f \in T |\ \exists \bar{f} \in \bar{I} : \pi f = \bar{f}\pi\}$, then I is an ideal of T. Moreover there exists a fully invariant subact C of A such that $\bar{C} = C/B$. Suppose on the contrary that $\bar{I}(A/B) \neq \bar{0}$ and $\bar{C} \neq \bar{0}$. Then there exists $\bar{f} \in \bar{I}$ such that $\bar{f}(A/B) \neq \bar{0}$. Since A is quasi-projective, we can find $f \in T$ such that $\bar{f}\pi = \pi f$. Hence $f \in I$ and $\pi f(A) = \bar{f}\pi(A) = \bar{f}\bar{A} \neq \bar{0}$. It follows that $f(A) \nsubseteq B$. Thus $I(A) \cup B \supsetneq B$. Since $\bar{C} \neq \bar{0}$, $B \subsetneq C$. Put $J = I \cup I_B$ where $I_B = \{f \in T | f(A) \subset B\}$. Note that since A is fully idempotent, $B = \bigcup_{f \in Hom(A,B)} f(B)$ and so $I_B \neq \emptyset$. It is easy to check that$J$ is an ideal of T and $J(A) = I(A) \cup I_B(A) = I(A) \cup B \supsetneq B$. If $g \in I$, then there exists $\bar{g} \in \bar{I}$ such that $\bar{g}\pi = \pi g$ and hence $\pi g(C) = \bar{g}\pi(C) = \bar{g}(\bar{C}) \subset \bar{I}(\bar{C}) = \bar{0}$. Thus $g(C) \subset B$ showing that $I(C) \subset B$. Thus if $f \in J$, then we can easily see that $f(C) \subset B$ and so $J(C) \subset B$ which is a contradiction. □

From [4], recall that an ideal I of a semigroup S is called prime if whenever $ab \in I$, either $a \in I$ or $b \in I$.

Corollary 2.6. *Suppose S is a monoid and P is a proper ideal of S, then the following conditions are equivalent:*

(i) *P is an ideal-prime subact of S_S.*

(ii) *For the monoid S/P, the ideal $\bar{0}$ is ideal-prime.*

(iii) *If I and J are ideals of S such that $IJ \subset P$, then either $I \subset P$ or $J \subset P$.*

(iv) *If $a, b \in S$ and $aSb \subset P$, then either $a \in P$ or $b \in P$.*

(v) *If $a \in S$ and I is an ideal of S such that $aI \subset P$, then either $I \subset P$ or $aS \subset P$.*

Moreover, if S is commutative, then the above conditions are equivalent to:

(vi) *P is a prime ideal of S.*

Proof. (i) $\Leftrightarrow$ (ii) hold by Corollary 2.5.
(i) $\Rightarrow$ (iii): First note that since S is a monoid, $\mathrm{End}(S_S) = \{\lambda_x | x \in S\}$.

Suppose for ideals I and J of S, $IJ \subset P$. Let $I' = \{\lambda_s | s \in I\}$. Since I is an ideal of S, it is easy to see that I' is an ideal of $\mathrm{End}(S_S)$. Also since J is an ideal of S, J is a fully invariant subact of S_S and we can check that $I'(J) = \bigcup_{\lambda_s \in I'} \lambda_s(J) = \{xy | x \in I, y \in J\} = IJ \subset P$. Thus by assumption $J \subset P$ or $I'(S_S) \subset P$. Since $I'(S_S) = I$, thus $J \subset P$ or $I \subset P$ and the result follows.
(iii) $\Rightarrow$ (i): Let I be an ideal of $\mathrm{End}(S_S)$ and J is a fully invariant subact of S_S such that $I(J) \subset P$. If $I' = \{s | \lambda_s \in I\}$, then $I(J) = I'J$ and so $I(J) \subset P$ implies that $I'J \subset P$ and consequently by assumption, $I' \subset P$ or $J \subset P$. Since $I' = I'S = I(S_S)$, we have $I(S_S) \subset P$ or $J \subset P$ as desired.

The proofs of the other implications are straightforward and will be omitted. □

Note that in part (iii) of the above corollary, we can replace ''ideals" by "left (right) ideals".

Corollary 2.7. *Suppose S is a monoid and B is an ideal-prime subact of an S-act A such that $P = \{s | As \subset B\} \neq \emptyset$. If A is a fully idempotent S-act, then P is a prime ideal of S.*

Proof. First note that P is an ideal of S. Let $T = \mathrm{End}(A)$ and I, J be two ideals of S. If $IJ \subset P$ and $J \nsubseteq P$, then $AIJ \subset B$ and $AJ \nsubseteq B$. Note that AI and AJ are fully invariant subacts of A. Since A is fully idempotent, $AJ = \bigcup_{f \in Hom(A,AJ)} f(AJ)$. Since $AJ \nsubseteq B$, there exists $f : A \longrightarrow AJ$ such that $f(A) \nsubseteq B$ and $f(A) \subset AJ$. Thus $f(AI) \subset AJI \subset B$. Hence by Theorem 2.2 (iii), we imply that $AI \subset B$ and consequently $I \subset P$ as desired. □

Proposition 2.8. *Suppose A is an S-act over a monoid S. Then the following hold:*

(i) *If B is a subact of A which is maximal among fully invariant subact of A, then B is an ideal-prime subact of A. In particular if I a maximal ideal of S, then I is ideal-prime.*

(ii) *If A is a duo S-act, then any maximal subact of A is an ideal-prime subact.*

Proof. (i): Let C be a fully invariant subact of A and $f \in \mathrm{End}(A)$ such that $C \nsubseteq B$ and $f(C) \subset B$. Thus clearly $A = C \cup B$ and so $f(A) \subset B$. Thus by Theorem 2.2, B is ideal-prime.

(ii): Any subact of A is fully invariant and so by part (i) we have the result. □

Proposition 2.9. *Suppose A is an S-act such that the intersection of any family of ideal-prime subacts of A is non-empty. Then any ideal-prime subact B of A contains a subact B^* which is minimal in the family of ideal-prime subacts of A.*

Proof. Let $\mathcal{H}$ be the set of all ideal-prime subacts of A which are contained in B. Thus $B \in \mathcal{H}$ and so $\mathcal{H} \neq \emptyset$. Let $\mathcal{X} \subset \mathcal{H}$ be a non-empty chain and $Q = \bigcap_{D \in \mathcal{X}} D$. Then Q is a fully invariant subact of A and $Q \subset B$. Suppose $f \in \mathrm{End}(A) = T$ and $a \in A \backslash Q$ such that $f(T(a)) \subset Q$. Since $a \notin Q$, there exists $D \in \mathcal{X}$ with $a \notin D$. By the ideal-primness of D, we have $f(A) \subset D$. If $C \in \mathcal{X}$, then either $C \subset D$ or $D \subset C$. If $C \subset D$, then $a \notin C$ and so $f(A) \subset C$ by ideal-primness of C. Also if $D \subset C$, then $f(A) \subset D \subset C$. Thus for any $C \in \mathcal{X}$ we have $f(A) \subset C$. Hence $f(A) \subset Q$ and by Theorem 2.2, we imply that Q is an ideal-prime subact of A. Since Q is a lower bound for $\mathcal{X}$, by Zorn's lemma, there exists an ideal-prime subact B^* which is minimal in $\mathcal{H}$. Also since any ideal-prime subact of B^* is an element of $\mathcal{H}$, we conclude that B^* is a minimal ideal-prime subact of A. □

As an example, if A is the S-act which is mentioned in Example 2.3, then we can see that A satisfies the conditions of Proposition 2.9.

Proposition 2.10. *Suppose A is an S-act over a monoid S and $T = End(A)$. If B is a fully invariant subact of A and $I_B = \{f \in T | f(A) \subset B\}$, then if B is an ideal-prime subact of A, then I_B is a prime ideal of T. Converesly if A is fully idempotent and I_B is a prime ideal of T, then B is an ideal-prime subact of A.*

Proof. First note that I_B is an ideal of T. Let J_1 and J_2 be ideals of S such that $J_1 J_2 \subset I_B$. Then $J_1 J_2(A) \subset I_B(A) \subset B$. If $J_1 \nsubseteq I_B$, then $J_1(A) \nsubseteq B$ and by ideal-primeness of B, we have $J_2(A) \subset B$ which implies that $J_2 \subset I_B$ and hence I_B is a prime ideal of T. Conversely suppose I_B is a prime ideal of S and $f \in T$ and also C is a fully invariant subact of A with $f(C) \subset B$. If $f(A) \nsubseteq B$, then $f \notin I_B$. Since A is fully idempotent, $C = \bigcup_{h \in Hom(A,C)} h(C)$. Suppose $g \in T$ and $h \in \mathrm{Hom}(A, C)$. Then $gh(A) \subset g(C) \subset C$ and so $Th(A) \subset C$. Thus $f(Th(A)) \subset B$ and consequently $fTh \subset I_B$. Since $f \notin I_B$, by Corollary 2.6, $h \in I_B$ and hence for any $h \in \mathrm{Hom}(A, C), h(A) \subset B$. Thus $C = \bigcup_{h \in Hom(A,C)} h(C) \subset B$ and so by Theorem 2.2, B is an ideal-prime subact of A. □

Lemma 2.11. *Suppose A is a quasi-projective S-act and for a subact C of $A, \pi : A \longrightarrow A/C = \bar{A}$ is the nutural epimorphism. Then the following hold:*

(i) *If B is an ideal-prime subact of A and $C \subset B$ is a fully invariant subact of A, then $\bar{B} = B/C$ is an ideal-prime subact of $\bar{A}$.*

(ii) *If C is a fully invariant subact of A and $\bar{B}$ is an ideal-prime subact of $\bar{A}$, then $\pi^{-1}(\bar{B})$ is an ideal-prime subact of A.*

Proof. (i): Suppose $f \in T = \mathrm{End}(\bar{A})$ and $\bar{D} = D/C$ is a fully invariant subact of $\bar{A}$ such that $f(\bar{D}) \subset \bar{B} = B/C$. Since A is quasi-projective we can find an element $g \in \mathrm{End}(A)$ such that $f\pi = \pi g$. Then $f(\bar{D}) = f\pi(D) = \pi g(D) = (g(D) \cup C)/C \subset \bar{B}$ and so $g(D) \subset B$. Since C is a fully invariant subact of A, and $\bar{D}$ is a fully invariant subact of $\bar{A}$, we can easily see that D is a fully invariant subact of A. Thus by ideal-primeness of B, we have $g(A) \subset B$ or $D \subset B$. Thus $(g(D) \cup C)/C \subset \bar{B}$ or $\bar{D} \subset \bar{B}$ and it follows that $f(\bar{A}) = (g(D) \cup C)/C \subset \bar{B}$ or $\bar{D} \subset \bar{B}$ which means that $\bar{B}$ is an ideal-prime subact of $\bar{A}$.
(ii): Let $B = \pi^{-1}(\bar{B})$. Suppose $f \in \mathrm{End}(A)$ and D is a fully invariant subact of A such that $f(D) \subset B$. Since C is a fully invariant subact of A, we can define an endomorphism $\bar{f}$ of $\bar{A}$ such that $\bar{f}\pi = \pi f$. From $f(D) \subset B$, we have $\pi f(D) \subset \pi(B) = \bar{B}$ or $\bar{f}\pi(D) \subset \bar{B}$. Since A is quasi-projective, there exists $g \in \mathrm{End}(A)$ such that $\pi g = f\pi$ and hence we can see that $\pi(D)$ is a fully invariant subact of $\bar{A}$. Thus by assumption, $\bar{f}\pi(A) \subset \bar{B}$ or $\pi(D) \subset \bar{B}$. If $\bar{f}\pi(A) \subset \bar{B}$, then $\bar{f}(\pi(A)) \subset \bar{B}$ and so $\pi f(A) \subset \bar{B}$ that is $f(A) \subset B$. Also in the case that $\pi(D) \subset \bar{B}$ we can see that $D \subset B$ and hence by Theorem 2.2, the result follows. □

For an S-act A, by $In_P(A)$ we denote the intersection of all ideal-prime subacts of A.

Theorem 2.12. *Let A be a quasi-projective S-act and $In_P(A) \neq \emptyset$. Then $In_P(A/In_P(A)) = \bar{0}$.*

Proof. Let $\bar{A} = A/In_P(A)$. By the above lemma, we have
$$In_P(\bar{A}) = \bigcap_{\substack{\bar{B} \subset \bar{A} \\ \bar{B} \text{ is ideal prime}}} \bar{B} = \bigcap_{\substack{B \subset A \\ B \text{ is ideal prime}}} (B/In_P(A)) = \Big(\bigcap_{\substack{B \subset A \\ B \text{ is ideal prime}}} B\Big)/In_P(A)$$
$$= In_P(A)/In_P(A) = \bar{0}.$$ □

Example 2.13. Suppose $S = \mathbb{Z}$ is the set of integers with the multiplication. If $I = 2\mathbb{Z}$, then by Proposition 1.12 of [12], the S-act $A = S/I$ is

a duo quasi-projective S-act. Since A contains no non-zero subacts other than itself, $In_P(A) = 0$ and so $In_P(A/In_P(A)) = \bar{0}$.

References

[1] **J. Ahsan and K. Saifullah**, *Completely quasi-projective monoids*, Semigroup Forum **38** (1989), 123–126.

[2] **J. Ahsan, L. Zhongkui**, *Prime and semiprime acts over monoids with zero*,. Math. J. Ibaraki Univ. **33** (2001), 9–15.

[3] **R.L. Casland, P.F. Smith**, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math. **23** (1993), 1041–1062.

[4] **A.H. Clifford, G.B. Preston**, *The algebraic theory of semigroups*, Math. Surveys, **7**, Providence, AMS 91961).

[5] **J. Dauns**, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156–181.

[6] **A.A. Estaji, S. Tajnia**, *Prime subacts over commutative monoids with zero*. Lobachevski J. Math. **32** (2011), 358–365.

[7] **M. Kilp, U. Knauer, A.V. Mikhalev**, *Monoids, Acts and Categories, With Application to Wreath Product*, Walter de Gruyter, Berlin. (2000).

[8] **C.P. Lu**, *Prime submodules*. Comm. Math. Univ. St. Paul. **33** (1984), 61–69.

[9] **R.L. McCasland, M.E. Moore**, *Prime submodules*. Comm. Algebra. **20** (1992), 1803–11817.

[10] **M.A. Naghipoor**, *On prime and radical subacts of acts over semigroups*, Comm. Algebra **51** (2023), 1779–1799.

[11] **M. Roueentan, M. Ershad**, *A note on fully idempotent acts*, Honam Math. J. **35** (2013), 381–388.

[12] **M. Roueentan, M. Ershad**, *Strongly duo and duo right S-acts*. Ital. J. pure Appl. Math. **32** (2014), 143–154.

[13] **H. Wu, Q. Li**, *On the spectra of commutative semigroups*. Semigroup Forum **101** (2020), 465–485.

Received November 06, 2025

M. Roueentan
Lamerd Higher Education Center, Shiraz University of Technology, Lamerd, Iran.
E-mail: rooeintan@sutech.ac.ir

R. Khosravi
Department of Mathematics, Faculty of Sciences, Fasa University, Fasa, Iran.
E-mail: Khosravi@fasau.ac.ir