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The algebra and combinatorics of travel groupoids on finite trees

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Abstract. We study travel groupoids on finite trees. We describe the structure of left and right cosets with respect to a subgroupoid and characterize all commuting pairs, flexible pairs, associative triples, as well as left and right self-distributive triples in such groupoids. We also count the number of each of these configurations. Furthermore, we characterize homomorphisms and several other types of maps between travel groupoids on trees. Finally, we describe left and right congruences on such groupoids and introduce the generalized notions of weak left and weak right congruences, providing their complete characterization.

1. Introduction

The notion of a travel groupoid, introduced by Nebeský in [14], offers an elegant algebraic framework for studying paths in geodetic graphs through binary operations. Having a geodetic graph G and its two distinct vertices $u, v \in V(G)$, we define $u + v$ to be the first vertex other than u lying on the unique shortest path from u to v. Also, setting $u + u := u$ for all $u \in V(G)$, we obtain a correctly defined binary operation $+$ on the vertex set $V(G)$. The pair $(V(G), +)$ is then called a travel groupoid. As can be seen from their definition, travel groupoids capture directional movement along shortest paths which give rise to rich algebraic structures on graphs.

Using this approach, it was shown that travel groupoids on geodetic graphs can be characterized as abstract groupoids satisfying several axioms [14]. This result established a bridge between combinatorial and algebraic structures on graphs and laid the groundwork for later developments

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in the theory of travel groupoids. In the follow-up paper [15], using three natural axioms, Nebeský characterized travel groupoids on finite trees.

In [16] Nebeský defined travel groupoids abstractly as groupoids $(X, *)$ which satisfy the next two conditions for all $u, v \in X$: $(u * v) * u = u$; if $(u * v) * v = u$, then $u = v$. In this way, every travel groupoid naturally induces a graph G with the vertex set $V(G) = X$ and the edge set $E(G) = \{\{u, v\} : u \neq v, u * v = v\}$. Several extensions have been proposed in [16], including notions such as *non-confusing*, *smooth*, and *semi-smooth* travel groupoids, each imposing additional structural properties.

Since the formal introduction of travel groupoids, a growing body of research has focused on their structure and existence on various classes of graphs. The authors of [12] constructed smooth travel groupoids on finite graphs, answering one of the open problems from [16]. Other results include the classification of non-confusing travel groupoids on finite connected graphs [4], and the characterization and enumeration of travel groupoids on complete multipartite graphs [13].

In this paper, we focus on travel groupoids on finite trees, a class of graphs where the interplay between algebraic operations and graph-theoretic properties is particularly transparent. We analyze several structural and combinatorial properties of travel groupoids on trees. Our main contributions include:

1. a description of left and right cosets for subgroupoids (Proposition 2.6);
2. a characterization and enumeration of commuting pairs, flexible pairs, associative triples, left and right self-distributive triples (Section 3.1);
3. a characterization of homomorphisms and several other types of maps between travel groupoids on trees (Section 3.2);
4. a classification of left and right congruences, along with a new notion of weak congruences, which we also fully characterize (Section 3.3).

We note that some of the results from this paper were announced in [1].

2. Definitions and preliminary results

2.1. Graphs

In this paper, all graphs are assumed to be finite, simple and undirected. Thus, a *graph* G is a pair (V, E), where $V = V(G)$ is the set of its *vertices*,

and $E = E(G) \subset \binom{V(G)}{2}$ is the set of its *edges*. Two vertices $u, v \in V(G)$ are *adjacent* provided there is an edge $\{u, v\} \in E(G)$. By $N_G(u) = \{v \in V(G) : \{u, v\} \in E(G)\}$ and $N_G[u] = N_G(u) \cup \{u\}$ we denote the *open neighborhood* and the *closed neighborhood* of a vertex $u \in V(G)$, respectively. The degree of u is defined as $\deg_G(u) = |N_G(u)|$. A vertex u is called a *leaf* provided $\deg_G(u) = 1$. By $\mathrm{Leaf}(G)$ we denote the set of all leaves in G.

For a graph G, its *first Zagreb index* [8] is the number

$$M_1(G) = \sum_{u \in V(G)} \deg_G^2(u).$$

Let G be a graph and $A \subset V(G)$. By $G[A]$ we denote the corresponding induced subgraph.

For a graph G and an equivalence relation $\sim$ on $V(G)$, define the corresponding *factor-graph* G' as the graph having equivalence classes of $\sim$ as its vertices, with two such classes A and B being adjacent provided $\{a, b\} \in E(G)$ for some $a \in A$, $b \in B$. The corresponding map $p : V(G) \to V(G')$, where $p(a) = [a]_\sim$ for all $a \in V(G)$, is called the *projection* onto a factor-graph.

A graph is *connected* provided each pair of its vertices can be joined by a path. A *connected component* in a graph is its maximal connected subgraph. A set of vertices $A \subset V(G)$ is called *connected* if $G[A]$ is connected. A *forest* is a graph without cycles. A *tree* is a connected forest.

The vertex set $V(G)$ of any connected graph G is naturally endowed with the "shortest path" metric d_G, where $d_G(u, v)$ equals the number of edges on a shortest path between u and v in G. The famous *Wiener index* [18] of a connected graph G is the number

$$W(G) = \sum_{\{u,v\} \subset V(G)} d_G(u, v).$$

For a pair of vertices $u, v \in V(G)$ in a connected graph G, the *metric interval* between u, v is the set

$$[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}.$$

A set of vertices $A \subset V(G)$ in a connected graph G is said to be in *general position* if for all $a, b \in A$, we have $[a, b]_G \cap A = \{a, b\}$. A set A is *convex* provided $[a, b]_G \subset A$ for all $a, b \in A$. The *convex hull* $\mathrm{Conv}_G(A)$ of A is the smallest convex set containing A.

For a triple of vertices $a, b, c \in V(G)$ in a connected graph, their *median* is any vertex from the intersection of three metric intervals $[a, b]_G \cap [a, c]_G \cap [b, c]_G$. Hence, a triple of vertices can have several medians or no medians at all. A connected graph G is called *median* provided every triple of its vertices has a unique median. In this case, the unique vertex from $[a, b]_G \cap [a, c]_G \cap [b, c]_G$ is called the *median* of the triple a, b, c, and it is denoted by $m_G(a, b, c)$. For example, trees and hypercubes are median graphs.

A set of vertices $A \subset V(G)$ in a connected graph G is called *Chebyshev* provided for any vertex $x \in V(G)$ there exists a unique vertex $a \in A$ with $d_G(x, a) = \min\{d_G(x, u) : u \in A\}$. The corresponding vertex a is called the *projection* of x onto A, and it is denoted by $\mathrm{pr}_A(x)$. A set A is *gated* provided for any $x \in V(G)$ there is $a \in A$ such that for all $u \in A$, it holds $a \in [x, u]_G$. The corresponding vertex a is necessarily unique and is called the *x-gate* in A. One can observe that every gated set is both convex and Chebyshev. In contrast, Chebyshev sets may be disconnected.

Proposition 2.1. [9, Proposition 3.7] *A connected graph is a tree if and only if every connected set of its vertices is Chebyshev.*

A connected graph is called *geodetic* provided there is a unique shortest path between each pair of vertices. It is clear that trees are geodetic.

2.2. Maps between graphs

Let G and H be two graphs. A map $f : V(G) \to V(H)$ is called a *homomorphism* if $\{f(u), f(v)\} \in E(H)$ for all edges $\{u, v\} \in E(G)$. An *isomorphism* between two graphs is a bijective homomorphism f such that f^{-1} is also a homomorphism.

A map $f : V(G) \to X$ is called a *proper coloring* provided $f(u) \neq f(v)$ for all edges $\{u, v\} \in E(G)$. In other words, proper colorings of graphs can be thought as homomorphisms into complete graphs on X.

A map $f : V(G) \to V(H)$ between two connected graphs G and H is called *metric* if $d_H(f(u), f(v)) \leqslant d_G(u, v)$ for all pairs $u, v \in V(G)$. For example, the projection on any factor-graph is metric by construction. Also, in [7, Lemma 1] it was proved that the projection on any gated set is metric as well.

The next simple result shows that metric maps are generalization of homomorphisms.

Proposition 2.2. [10, Proposition 2.2] *A map $f : V(G) \to V(H)$ is metric if and only if for every edge $\{u, v\} \in E(G)$, we have $d_H(f(u), f(v)) \leqslant 1$.*

In the proof of Proposition 3.14 below we will need another characterization of metric maps.

Proposition 2.3. [6] *A map $f : V(G) \to V(H)$ is metric if and only if every connected set $A \subset V(G)$ has a connected image $f(A)$ in H.*

A map $f : V(G) \to V(H)$ between two graphs G and H is called *monotone* provided every vertex $y \in V(H)$ has a connected pre-image $f^{-1}(y)$ in G. Further, a map $f : V(G) \to V(H)$ between two connected graphs G and H is *linear* if $f([u, v]_G) \subset [f(u), f(v)]_H$ for all pairs $u, v \in V(G)$. It can be observed that any linear map is monotone. We also note that linear maps between median graphs (in particular, between trees) can be characterized as those maps which backward preserve convex sets (see [17, Proposition 3.7]).

2.3. Travel groupoids

In this paper, by a *groupoid* we will mean a pair $(X, \circ)$, where $\circ$ is a binary operation on a set X.

A subset $A \subset X$ is *closed* provided $a \circ b \in A$ for all $a, b \in A$. In this case, the pair $(A, \circ)$ is called a *subgroupoid*. A set $A \subset X$ is *generating* if X is the smallest closed set containing A.

Let $(X, \circ)$ and $(Y, *)$ be two groupoids. A map $f : X \to Y$ is called a *homomorphism* provided $f(a \circ b) = f(a) * f(b)$ for all $a, b \in X$. A homomorphism from a groupoid to itself is called its *endomorphism*.

Let G be a geodetic graph and $u, v \in V(G)$ be two distinct vertices. Let $P = \{u - x - \cdots - v\}$ be the unique shortest path between u and v. Consider the binary operation $+$ on $V(G)$ defined as $u + v := x$, and $u + u := u$ for all $u \in V(G)$. The corresponding groupoid $(V(G), +)$ is called the *travel groupoid* [14] on G. Note that geodetic graphs can be characterized as connected graphs having Chebyshev closed neighborhoods $N_G[u]$ for all $u \in V(G)$ (see [9, Theorem 3.9]). Hence, one can define the travel operation on geodetic graphs by putting $u + v = \mathrm{pr}_{N_G[u]}(v)$.

As an example, consider the tree T with $V(T) = \{a, b, c, x, y\}$ and $E(T) = \{\{a, b\}, \{b, c\}, \{c, x\}, \{b, y\}\}$. Then $(x + (a + y)) + y = (x + b) + y = c + y = b$.

The abstract characterization of travel groupoids on geodetic graphs was obtained in [14] (see [14, Lemma 1] for the four axioms that completely describe these groupoids). Similarly, travel groupoids on trees were characterized in [15] via three simple axioms. The characterization of travel

groupoids by the so–called partition systems was presented in [5]. For the study of travel groupoids on infinite graphs we refer to the paper [3].

The next simple technical result will be used in the proofs of our results below.

Lemma 2.4. *Let T be a tree, and $x, y, z \in V(T)$ with $y \in [x, z]_T \setminus \{x\}$. Then $x + y = x + z$.*

Proof. Follows from the fact that $[x, y]_T \subset [x, z]_T$ for all $y \in [x, z]_T$. □

The next result describes subgroupoids of travel groupoids, which unsurprisingly, are precisely the subtrees of a tree.

Proposition 2.5. *Let T be a tree. Then a set $A \subset V(T)$ is closed under $+$ if and only if A is connected.*

Proof. Necessity. If A is disconnected, then there is a pair of vertices $a, b \in A$ having $[a, b]_T \setminus A \neq \emptyset$. Fix a vertex $u \in [a, b]_T \setminus A$ with the shortest distance $d_T(a, u)$ among all such vertices. It is clear that $u \neq a$. Consider the vertex $v := u + a$ (see Figure 1). Clearly, $v \in A$. But then $u = v + b \notin A$, which is a contradiction.

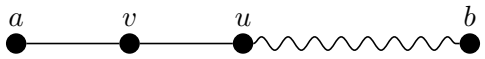


Figure 1: Illustration from the proof of the necessity in Proposition 2.5.

Sufficiency. If A is a connected set, then for all $a, b \in A$, we have $a + b \in [a, b]_T \subset A$. Thus, A is closed under $+$. □

It is clear that the collection of all closed subsets in a given groupoid forms a lattice. For travel groupoids this lattice was studied in [19]. In particular, it was proved that this lattice determines the tree up to isomorphism [19, Theorem 11].

For a subset $A \subset X$ and an element $x \in X$, the left coset is the set $x \circ A = \{x \circ a : a \in A\}$, and the right coset is the set $A \circ x = \{a \circ x : a \in A\}$.

The left and right cosets of subgroupoids in travel groupoids on trees can be described explicitly as follows.

Proposition 2.6. *Let $A \subset V(T)$ be a connected subset with $|A| \geqslant 2$ in a tree T, and $x \in V(T)$ be some vertex. Then the next statements hold:*

1. *if* $x \in A$, *then* $x + A = N_T[x] \cap A$ *and* $A + x = \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{x\}$;
2. *if* $x \notin A$, *then* $x + A = \{y\}$, *where* $y = x + a$ *for any* $a \in A$, *and*

$$A + x = \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{\mathrm{pr}_A(x), \mathrm{pr}_A(x) + x\}.$$

Proof. 1. Assume that $x \in A$. It is clear that any vertex of the form $u = x + a$, $a \in A$, must lie in $N_T[x]$. Also, as A is connected, any such u belongs to A as well. Hence, $x + A \subset N_T[x] \cap A$. Similarly, if $u \in N_T[x] \cap A$, then $u = x + u$. Thus, $x + A = N_T[x] \cap A$ in this case.

Now, let $a \in A$. Since, also $x \in A$, we have $a + x \in [a, x]_T \subset A$. If $a + x \neq x$, then $a + x$ cannot be a leaf in the induced subtree $T[A]$. Therefore, $A + x \subset \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{x\}$. Conversely, it is evident that $x \in A + x$ since $x \in A$ and $x + x = x$. Thus, fix a vertex $u \in A \setminus \mathrm{Leaf}(T[A])$. Since $\deg_{T[A]}(u) \geqslant 2$, we can choose a vertex $v \in N_{T[A]}(u)$ with $v \neq u + x$ (see Figure 2). Then $u = v + x \in A + x$. Thus, $A + x = \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{x\}$ in this case.

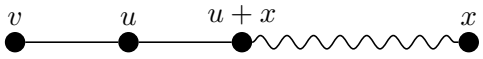


Figure 2: Illustration from the proof of the first statement of Proposition 2.6.

2. Assume that $x \notin A$. Since A is connected and T is a tree, A is gated. Denote by $g := \mathrm{pr}_A(x) \in A$ the x-gate in A. Use Lemma 2.4 for x, $y = g$ and any $a \in A$, to obtain the equality $x + g = x + a$. This means that $x + A = \{x + g\}$.

In order to prove the desired equality for the right coset $A + x$, let us fix a vertex $a \in A \setminus \{g\}$. Use Lemma 2.4 for the triple (a, g, x), to deduce that $a + x = a + g$. It is clear that the vertex $a + x$ cannot be a leaf in $T[A]$ unless it equals g, and g is a leaf in $T[A]$. Therefore, $A + x \subset \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{g, g + x\}$.

Conversely, it is clear that $g + x \in A + x$, and $g \in A + x$ (as for any vertex v from $N_{T[A]}(g) \neq \emptyset$, it holds $v + x = g$). Similarly, for every $u \in A \setminus \mathrm{Leaf}(T[A])$, we can choose a neighbor $v \in N_{T[A]}(u)$ with $v \neq u + x$. Then $v + x = u \in A + x$. Thus, in this case, $A + x = \big(A \setminus \mathrm{Leaf}(T[A])\big) \cup \{g, g + x\}$. □

Note that Propositions 2.5 and 2.6 imply that in travel groupoids on trees, every coset of a subgroupoid is a subgroupoid itself.

The next corollary states that travel groupoids on trees T always have a unique minimal generating set, being the set of leaves $\mathrm{Leaf}(T)$.

Corollary 2.7. *A set $A \subset V(T)$ is generating for $+$ on a tree T if and only if $\mathrm{Leaf}(T) \subset A$.*

Proof. It is clear that for any leaf vertex $x \in \mathrm{Leaf}(T)$, the equality $a+b=x$ necessarily implies that $b=x$. Hence, each generating set $A \subset V(T)$ must include all of the leaves in T. Similarly, if $\mathrm{Leaf}(T) \subset A$, then Proposition 2.5 immediately implies that A is a generating set. □

3. Main results

3.1. Combinatorics of pairs and triples of elements in travel groupoids

A pair of elements $a, b \in X$ in a groupoid $(X, \circ)$ is *commuting* provided $a \circ b = b \circ a$.

Let T be a tree. It is clear from the definition of a travel groupoid that the pair of vertices $a, b \in V(T)$ is commuting for the operation $+$ if and only if $a = b$ or $d_T(a,b) = 2$. This observation leads to the following result.

Proposition 3.1. *The number of (unordered) commuting pairs of vertices for the operation $+$ on a tree T is equal to $1 + \frac{1}{2}M_1(T)$.*

Proof. Assume that T has n vertices. It is clear that there are n pairs of vertices a, b with $a = b$. The number of pairs a, b having $d_T(a,b) = 2$ can be computed as follows. Each such a pair corresponds to the unique vertex u from $N_T(a) \cap N_T(b)$. Hence, the number of such pairs a, b equals $\sum_{u \in V(T)} \binom{\deg_T(u)}{2}$.

Therefore, the number of commuting pairs for $+$ on T is equal to

$$\begin{aligned} n + \sum_{u \in V(T)} \binom{\deg_T(u)}{2} &= n + \sum_{u \in V(T)} \frac{\deg_T(u)(\deg_T(u)-1)}{2} \\ &= n + \frac{1}{2}\sum_{u \in V(T)} \deg_T^2(u) - \frac{1}{2}\sum_{u \in V(T)} \deg_T(u). \\ &= n + \frac{1}{2}M_1(T) - |E(T)| = n + \frac{1}{2}M_1(T) - (n-1) \\ &= 1 + \frac{1}{2}M_1(T). \end{aligned}$$

□

An ordered pair $(x, y) \in X^2$ in a groupoid $(X, \circ)$ is called *flexible* provided $(x \circ y) \circ x = x \circ (y \circ x)$. In case of travel groupoids, we always have $(x + y) + x = x$. Hence, a pair $(x, y) \in V(T)^2$ is flexible for $+$ if and only if $x = x + (y + x)$, which is equivalent to $x = y + x$. The latter equality holds if and only if $x = y$ or $\{x, y\} \in E(T)$. In particular, (x, y) is flexible for a travel groupoid if and only if (y, x) is flexible. And the number of (ordered) flexible pairs for the operation $+$ on an n-vertex tree is equal to $n + 2(n - 1) = 3n - 2$.

An ordered triple of elements $(a, b, c) \in X^3$ in a groupoid $(X, \circ)$ is *associative* if $a \circ (b \circ c) = (a \circ b) \circ c$.

The question of characterization of associative triples for travel groupoids on trees is answered in the following result.

Proposition 3.2. *Let T be a tree. The triple $(a, b, c) \in V(T)^3$ is associative for $+$ if and only if $a = b$, or $a + b = c$, or $b + c = a$.*

Proof. Necessity. Set $u := (a+b)+c = a+(b+c)$. If $a+b \neq c$ and $b+c \neq a$, then $\{u, a+b\}, \{u, a\} \in E(T)$. If $a+b = a$, then $a = b$. Otherwise, $a+b \neq a$ would imply the existence of and edge $\{a, a + b\} \in E(T)$. This, in turn, means that T contains a triangle induced by the vertices $a, a + b, u$. A contradiction.

Sufficiency. We consider three cases separately. If $a = b$, then $(a + b) + c = a + c$ and $a + (b + c) = a + (a + c) = a + c$. Further, if $a + b = c$, then $(a + b) + c = c + c = c$ and $a + (b + c) = c$ since $b + c \in [a, b]_T$. Finally, assume that $b+c = a$. Then $\{a, b\} \in E(T)$, implying $(a+b)+c = b+c = a$. On the other hand, $a + (b + c) = a + a = a$. □

Corollary 3.3. *The number of (ordered) associative triples for $+$ on an n-vertex tree T is equal to $3n^2 - 2n$.*

Proof. We divide associative triples (a, b, c) into two types: $a = b$ and $a \neq b$. Clearly, there are n^2 triples of the first type. If $a \neq b$, then Proposition 3.2 asserts that $a + b = c$ or $b + c = a$. Note that these two possibilities cannot hold at the same time (as $a \neq b$). Thus, there are exactly $n(n - 1)$ triples (a, b, c) with $a \neq b$ and $a+b = c$. Similarly, there are $n(n-1)$ triples (a, b, c) with $a \neq b$ and $b + c = a$. Therefore, the number of associative triples for $+$ on T equals $n^2 + 2n(n - 1) = 3n^2 - 2n$. □

An ordered triple $(a, b, c) \in X^3$ in a groupoid $(X, \circ)$ is called *left self-distributive* provided $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$.

Proposition 3.4. *Let T be a tree. The triple $(a,b,c) \in V(T)^3$ is left self-distributive for $+$ if and only if $a = b$, or $b + c = a$, or $a \notin [b,c]_T$.*

Proof. Necessity. Assume that $a \neq b$ and $a \in [b,c]_T$. Then $a+b \neq a$ and $a+b \neq a+c$. Further, since a is adjacent to both $a+b$ and $a+c$, we obtain $(a+b)+(a+c) = a$. But since the triple (a,b,c) is left self-distributive, this implies $a+(b+c) = a$, which is equivalent to $a = b+c$.

Sufficiency. We consider three cases separately. If $a = b$, then $a+(b+c) = a+(a+c) = a+c$ and $(a+b)+(a+c) = a+(a+c) = a+c$. Further, if $a = b+c$, then $a+(b+c) = a+a = a$. Also, $(a+b)+(a+c) = b+(a+c) = b+c = a$. Finally, assume $a \notin [b,c]_T$. Then $a \neq m := m_T(a,b,c)$. Thus, $(a+b)+(a+c) = (a+m)+(a+m) = a+m$ and $a+(b+c) = a+m$ (here we combined the fact that $m \in [a, b+c]_T$ together with Lemma 2.4). □

Corollary 3.5. *The number of (ordered) left self-distributive triples for $+$ on an n-vertex tree T is equal to $n^3 + n^2 - n - 2W(T)$.*

Proof. Again, let us divide left self-distributive triples (a,b,c) into two types: $a = b$ and $a \neq b$. There are n^2 triples with $a = b$. If $a \neq b$, then Proposition 3.4 asserts that $b + c = a$, or $a \notin [b,c]_T$. Observe that these two possibilities cannot hold simultaneously.

As before, there are $n(n-1)$ triples having $a \neq b$ and $b+c = a$. Hence, let us calculate the number of (ordered) triples (a,b,c) with $a \notin [b,c]_T$ (the condition $a \neq b$ is redundant here). For any fixed ordered pair of vertices $(b,c) \in V(T)^2$, we have exactly $d_T(b,c)+1$ vertices from the metric interval $[b,c]_T$. Thus, we have $n - (d_T(b,c)+1)$ possibilities to choose the vertex a to form an admissible triple. This means that the number of such triples equals

$$\sum_{(b,c)\in V(T)^2} (n - d_T(b,c) - 1) = n^2(n-1) - \sum_{(b,c)\in V(T)^2} d_T(b,c) = n^2(n-1) - 2W(T).$$

Therefore, the number of left self-distributive triples for $+$ on T is equal to $n^2 + n(n-1) + n^2(n-1) - 2W(T) = n^3 + n^2 - n - 2W(T)$. □

An ordered triple $(a,b,c) \in X^3$ in a groupoid $(X, \circ)$ is called *right self-distributive* provided $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$. It turns out, characterizing right self-distributive triples for travel groupoids on trees requires more work.

Theorem 3.6. *Let T be a tree. The triple $(a, b, c) \in V(T)^3$ is right self-distributive for $+$ if and only if none of the following conditions holds:*

1. *$a = c$ and $d_T(b, c) \geqslant 2$;*
2. *$a + b = c$ and $d_T(b, c) \geqslant 2$;*
3. *a, b, c lie in general position and $\{a, m\} \in E(T)$, where $m = m_T(a, b, c)$.*

Proof. Necessity. We show that a triple satisfying one of the three conditions is not right self-distributive for $+$.

At first, assume $a = c$ and $d_T(b, c) \geqslant 2$. Then $(a+b)+c = (a+b)+a = a$, but $(a + c) + (b + c) = (a + a) + (b + a) = a + (b + a)$. Equating these parts, we obtain $a = a + (b + a)$, implying $a = b + a$. This would mean $d_T(b, c) = d_T(b, a) \leqslant 1$, a contradiction.

Further, let $a + b = c$ and $d_T(b, c) \geqslant 2$. Then $(a + b) + c = c + c = c$, but $(a + c) + (b + c) = c + (b + c) \neq c$ (as otherwise, $c + (b + c) = c$ would imply $c = b + c$ which would mean that $d_T(b, c) \leqslant 1$).

Now let a, b, c be in general position and $d_T(a, m) = 1$. Then $(a+b)+c = m + c \in [m, c]_T \setminus \{m\}$ and $(a + c) + (b + c) = m + (b + c) \in [m, b]_T \setminus \{b\}$ (see Figure 3). And since $[m, c]_T \cap [m, b]_T = \{m\}$, we can conclude that $m + c \neq m + (b + c)$.

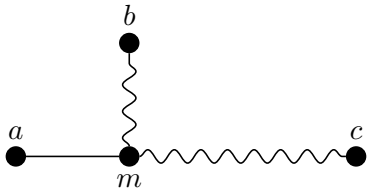


Figure 3: Illustration of the third condition from the proof of necessity in Theorem 3.6.

Sufficiency. Let us assume that the triple (a, b, c) satisfies none of the three conditions. We consider several cases below.

Case 1: the vertices a, b, c are not pairwise distinct.

If $a = b$, then $(a + b) + c = (a + a) + c = a + c$ and $(a + c) + (b + c) = (a + c) + (a + c) = a + c$. Now let $a = c$ and $d_T(b, c) = d_T(b, a) \leqslant 1$. We can assume that $\{a, b\} \in E(T)$. Here $(a + b) + c = (a + b) + a = a$ and $(a+c)+(b+c) = (a+a)+(b+a) = a+(b+a) = a+a = a$. Finally, suppose $b = c$. Then $(a + b) + c = (a + b) + b$ and $(a + c) + (b + c) = (a + b) + b$.

Case 2: the vertices a, b, c are pairwise distinct, but are not in general position.

Subcase 2.1: $a \in [b, c]_T$.

We use Lemma 2.4 for $x = a + b$, $y = a$, $z = c$ in order to deduce $(a + b) + c = (a + b) + a = a$. Similarly, we use Lemma 2.4 for $x = a + c$, $y = a$, $z = b + c$ to obtain $(a + c) + (b + c) = (a + c) + a = a$. See Figure 4 for the illustration.

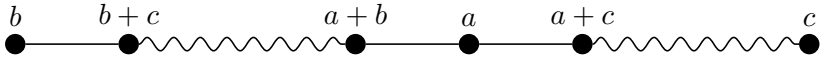


Figure 4: Illustration of Subcase 2.1 from the proof of Theorem 3.6.

Subcase 2.2: $b \in [a, c]_T$.

We use Lemma 2.4 for $x = a$, $y = b$, $z = c$ to obtain the equality $a+b = a+c$. Hence, $(a+c)+(b+c) = (a+b)+(b+c)$. Again, using Lemma 2.4 for $x = a + b$, $y = b + c$, $z = c$, we obtain $(a + b) + (b + c) = (a + b) + c$. See Figure 5 for the illustration.

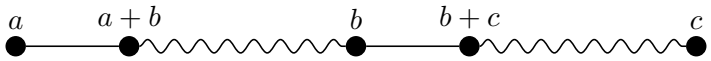


Figure 5: Illustration of Subcase 2.2 from the proof of Theorem 3.6.

Subcase 2.3: $c \in [a, b]_T$ and $a + b \neq c$.

In this subcase, $d_T(a, c) \geqslant 2$. Similarly to the previous case, use Lemma 2.4 for $x = a$, $y = c$, $z = b$ to obtain the equality $a + c = a + b$. Hence, $(a + c) + (b + c) = (a + b) + (b + c)$. Now, using Lemma 2.4 for $x = a + b$, $y = c$, $z = b + c$, we obtain $(a + b) + (b + c) = (a + b) + c$. See Figure 6 for the illustration.



Figure 6: Illustration of Subcase 2.3 from the proof of Theorem 3.6.

Subcase 2.4: $a + b = c$ and $d_T(b, c) \leqslant 1$.

In this subcase, $b + a = c$ (as a, b, c are pairwise different). Thus, we have $(a + b) + c = c + c = c$ and $(a + c) + (b + c) = c + c = c$.

Case 3: the vertices a, b, c are in general position and $d_T(a, m) \geqslant 2$ (recall that $m = m_T(a, b, c)$ is the median of this triple).

In this case, we have $m \notin \{a, b, c\}$. Also, $a + m \neq m$. Use Lemma 2.4 for $x = a$, $y = m$, $z = b$ to obtain the equality $a + b = a + m$. Thus, $(a + b) + c = (a + m) + c$. Again, use Lemma 2.4 for $x = a + m$, $y = m$, $z = c$ to obtain $(a + m) + c = (a + m) + m$.

Further, using Lemma 2.4 for $x = a$, $y = m$, $z = c$, we obtain $a + c = a + m$. Hence, $(a + c) + (b + c) = (a + m) + (b + c)$. Finally, invoke Lemma 2.4 once again for $x = a + m$, $y = m$, $z = b + c$. We obtain $(a + m) + (b + c) = (a + m) + m$. See Figure 7 for the illustration.

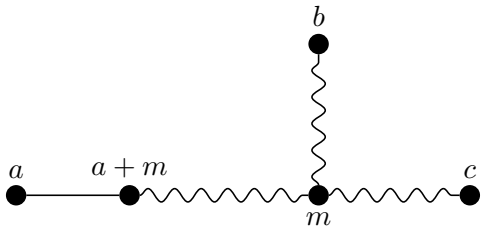


Figure 7: Illustration of Case 3 from the proof of Theorem 3.6. □

To count the number of right self-distributive triples for travel groupoids on a tree, we need one more notation. For a vertex $x \in V(T)$, by $M(x)$ denote the multiset of numbers of vertices in connected components of the forest $T - x$.

Proposition 3.7. *Let T be a tree with $n \geqslant 2$ vertices and $l = |\operatorname{Leaf}(T)|$ leaves. Then the number of ordered triples which are not right self-distributive for $+$ on T is equal to*

$$(n-1)^2 l - M_1(T) - \sum_{x \notin \operatorname{Leaf}(T)} \big((\deg_T(x) - 2) \cdot \sum_{k \in M(x)} k^2 \big). \tag{1}$$

Proof. Of course, we use Theorem 3.6. To start, we note that the three conditions from Theorem 3.6 are pairwise incomparable. Hence, we must calculate the numbers of triples satisfying each of these conditions separately.

It is clear that the number of ordered triples $(a,b,c) \in V(T)^3$ with $a = c$ and $d_T(b,c) \geqslant 2$ equals the number of ordered pairs $(b,c) \in V(T)^2$ having $d_T(b,c) \geqslant 2$. The latter equals $n^2 - n - 2(n-1) = n^2 - 3n + 2$.

Further, let us calculate the number of ordered triples $(a,b,c) \in V(T)^3$ with $a + b = c$ and $d_T(b,c) \geqslant 2$. Observe that this number equals the number of ordered pairs $(a,b) \in V(T)^2$ having $d_T(a,b) \geqslant 3$. Indeed, if the condition is satisfied, then $d_T(a,b) = d_T(a,c) + d_T(b,c) = 1 + d_T(b,c) \geqslant 3$. Similarly, given a pair (a,b) with $d_T(a,b) \geqslant 3$, the vertex c is determined uniquely.

Overall we have n^2 ordered pairs of vertices $(a,b) \in V(T)^2$. Among them, n pairs having $d_T(a,b) = 0$, also $2(n-1)$ pairs having $d_T(a,b) = 1$, and $\sum_{u \in V(T)} \deg(u)(\deg(u) - 1)$ pairs are with $d_T(a,b) = 2$. Hence, the number of ordered pairs $(a,b) \in V(T)^2$ having $d_T(a,b) \geqslant 3$ equals

$$\begin{aligned} & n^2 - \big(n + 2(n-1) + \sum_{u \in V(T)} \deg_T(u)(\deg_T(u) - 1)\big) \\ &= n^2 - (n + 2(n-1) + M_1(T) - 2(n-1)) \\ &= n^2 - n - M_1(T). \end{aligned}$$

Finally, we must calculate the number of triples $(a,b,c) \in V(T)^3$ such that a, b, c lie in general position and $\{a, m\} \in E(T)$ for $m = m_T(a,b,c)$. Such triples can be encoded by the choice of the pair $(b,c) \in V(T)^2$ along with the choice of an element $x \in [b,c]_T \setminus \{b,c\}$ and the choice of its neighbor y from $N_T(x) \setminus [b,c]_T$ (here x corresponds to m, and y corresponds to a). Therefore, the number of these triples is equal to

$$\sum_{(b,c) \in V(T)^2} \left(\sum_{x \in [b,c]_T \setminus \{b,c\}} (\deg_T(x) - 2) \right). \tag{2}$$

In what follows, we will rearrange this double sum in terms of the multiset $M(x)$ for $x \notin \mathrm{Leaf}(T)$. Now, for a fixed $x \notin \mathrm{Leaf}(T)$, let us calculate the number of ordered pairs $(b,c) \in V(T)^2$ having $x \in [b,c]_T \setminus \{b,c\}$. For each such a pair (b,c), the vertices b, c must lie in different connected components in a forest $T - x$. And vice versa, each pair of vertices from different components in $T - x$ serves for our purpose. Hence, the number of such pairs equals the number of all ordered pairs of different vertices (being $(n-1)(n-2)$) minus the number of pairs from the same component in $T - x$:

$$(n-1)(n-2) - \sum_{k \in M(x)} k(k-1).$$

This leads to the following rearrangement of the formula (2):

$$\sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2)\cdot \left((n-1)(n-2) - \sum_{k\in M(x)} k(k-1) \right) \right). \qquad (3)$$

The formula (3) can be further simplified as follows:

$$\begin{aligned}
&\sum_{x\notin \mathrm{Leaf}(T)} (\deg_T(x)-2)(n-1)(n-2) \\
&\quad - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \sum_{k\in M(x)} k(k-1) \right) \\
&= (n-1)(n-2) \left(\sum_{x\notin \mathrm{Leaf}(T)} \deg_T(x) - 2(n-l) \right) \\
&\quad - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \left(\sum_{k\in M(x)} k^2 - \sum_{k\in M(x)} k \right) \right) \\
&= (n-1)(n-2)(2(n-1)-l-2(n-l)) \\
&\quad - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \left(\sum_{k\in M(x)} k^2 - (n-1) \right) \right) \\
&= (n-1)(n-2)(l-2) - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \sum_{k\in M(x)} k^2 \right) \\
&\quad + \sum_{x\notin \mathrm{Leaf}(T)} (\deg_T(x)-2)(n-1) \\
&= (n-1)(n-2)(l-2) - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \sum_{k\in M(x)} k^2 \right) \\
&\quad + (n-1)(2(n-1)-l-2(n-l)) \\
&= (n-1)^2(l-2) - \sum_{x\notin \mathrm{Leaf}(T)} \left((\deg_T(x)-2) \sum_{k\in M(x)} k^2 \right).
\end{aligned}$$

Taking the sum of the three computed numbers above, we obtain

$$
\begin{aligned}
&n^2 - 3n + 2 + n^2 - n - M_1(T) + (n-1)^2(l-2) \\
&- \sum_{x \notin \mathrm{Leaf}(T)} \left((\deg_T(x) - 2) \sum_{k \in M(x)} k^2 \right) \\
&= (n-1)^2 l - M_1(T) - \sum_{x \notin \mathrm{Leaf}(T)} \left((\deg_T(x) - 2) \sum_{k \in M(x)} k^2 \right). \qquad \square
\end{aligned}
$$

For example, if $T = P_n$ is a path with $n \geqslant 2$ vertices, then $l = 2$, $M_1(T) = 1 + 1 + 2^2(n-2) = 4n - 6$, and $\deg_T(x) = 2$ for all $x \notin \mathrm{Leaf}(T)$. Using formula (1), we obtain that the number of triples which are not right self-distributive for $+$ on P_n equals

$$(n-1)^2 \cdot 2 - (4n-6) = 2n^2 - 4n + 2 - 4n + 6 = 2n^2 - 8n + 8 = 2(n-2)^2.$$

Further, if $T = K_{1,n-1}$ is a star with $n \geqslant 3$ vertices, then T has a unique non-leaf vertex x_0 (known as its center). This means that $l = n-1$, $M_1(T) = n-1+(n-1)^2$, and $\sum_{k \in M(x_0)} k^2 = n-1$. Plugging these numbers into the formula (1), we obtain that the number of triples which are not right self-distributive for $+$ on $K_{1,n-1}$ equals

$$
\begin{aligned}
&(n-1)^2(n-1) - (n-1+(n-1)^2) - (n-1-2)(n-1) \\
&= (n-1)^3 - (n-1) - (n-1)^2 - (n-3)(n-1) \\
&= (n-1)((n-1)^2 - 1 - (n-1) - (n-3)) \\
&= (n-1)(n^2 - 2n + 1 - 1 - n + 1 - n + 3) \\
&= (n-1)(n^2 - 4n + 4) = (n-1)(n-2)^2.
\end{aligned}
$$

3.2. Maps between travel groupoids

The following result shows that the condition of being a homomorphism between travel groupoids on trees imposes a strong restriction on the underlying map.

Theorem 3.8. *Let T_1 and T_2 be two trees. A map $f : V(T_1) \to V(T_2)$ is a homomorphism between the corresponding travel groupoids if and only if f is constant or an injective graph homomorphism.*

Proof. Necessity. Let f be a homomorphism between two travel groupoids. At first, we prove that f is monotone. Indeed, assume that $u, v \in f^{-1}(y)$

for some $y \in V(T_2)$. Fix a vertex $x \in [u,v]_{T_1} \setminus \{u,v\}$. We use induction on $d_{T_1}(u,x)$ in order to prove that $f(x) = y$. If x is adjacent to u, then $f(x) = f(u+v) = f(u)+f(v) = y+y = y$. If $d_{T_1}(u,x) \geqslant 2$, then fix a vertex $x' \in [u,x]_{T_1} \cap N_{T_1}(x)$. By induction assumption, $f(x') = y$. Since $x'+v = x$, the same argument yields $f(x) = f(x'+v) = f(x') + f(v) = y + y = y$.

Further, we consider two cases.

Case 1: the map f is not a proper coloring of T_1.

In this case, there is an edge $\{u_0, v_0\} \in E(T_1)$ with $f(u_0) = f(v_0) =: y$. Let us prove that in this case, f is constant. Indeed, since $|f^{-1}(y)| \geqslant 2$ and f is monotone, the pre-image $f^{-1}(y)$ is connected. If $f^{-1}(y) = V(T_1)$, then f is constant. Otherwise, there exists $x \in V(T_1) \setminus f^{-1}(y)$ and $a, b \in f^{-1}(y)$ with $\{a,b\}, \{b,x\} \in E(T_1)$. However, then we have $y = f(b) = f(a+x) = f(a) + f(x) = y + f(x)$, implying $f(x) = y$, which is a contradiction.

Case 2: the map f is a proper coloring of T_1.

Since f is monotone, in this case, f must be injective. Let us prove that f is a graph homomorphism. Indeed, for any edge $\{u,v\} \in E(T_1)$, it holds $f(u) \neq f(v)$ and $f(v) = f(u+v) = f(u) + f(v)$. This implies that $\{f(u), f(v)\} \in E(T_2)$.

Sufficiency. It is clear that any constant map f is a homomorphism between travel groupoids. Similarly, suppose that f is an injective graph homomorphism. Then f induces an isomorphism between T_1 and the subtree $T_2[\operatorname{Im} f]$. Hence, f is a homomorphism between travel groupoids of T_1 and $T_2[\operatorname{Im} f]$. However, the travel groupoid of $T_2[\operatorname{Im} f]$ is a subgroupoid of the travel groupoid of T_2 (see Proposition 2.5). Thus, f is a homomorphism between the travel groupoids of T_1 and T_2. □

Corollary 3.9. *A non-constant map $f : V(T) \to V(T)$ is an endomorphism of the travel groupoid on a tree T if and only if f is a graph automorphism.*

Note that the dynamical structure of tree automorphisms was described in [11, Corollary 3.1] (also, see [11] for the description of the dynamical structure of tree endomorphisms, metric, and linear maps on trees), where it was shown that for a permutation σ of a set V, there exists a tree T on V such that σ is an automorphism of T if and only if σ has a fixed point, or there is a σ-periodic point with period two, in which case all σ-periodic points have even periods.

One way to relax the standard notion of a homomorphism between groupoids is to specify only a subset of pairs of elements for which the

homomorphism condition must hold. More precisely, let $(X, \circ)$ and $(Y, *)$ be two groupoids, and R be a binary relation on X. A map $f : X \to Y$ is said to be an R-homomorphism provided $f(a \circ b) = f(a) * f(b)$ for all $(a, b) \in R$. Clearly, an ordinary homomorphism corresponds to the special case where $R = X^2$. In a similar way, we can define R-metric and R-linear maps between connected graphs by requiring the respective structural condition to hold only on the relation R.

For a map $f : X \to Y$, by $R_f = \{(a, b) \in X^2 : f(a) = f(b)\}$ we denote the corresponding equivalence relation on X known as the *kernel* of f.

Proposition 3.10. *Let T_1 and T_2 be a pair of trees. For a map $f : V(T_1) \to V(T_2)$, the next conditions are equivalent:*

1. *f is an R_f-homomorphism between travel groupoids;*
2. *f is monotone;*
3. *f is R_f-linear.*

Proof. 1 $\Rightarrow$ 2: If f is an R_f-homomorphism, then for all $y \in V(T_2)$ and $u, v \in f^{-1}(y)$, we have $f(u+v) = f(u)+f(v) = y+y = y$. This means that $f^{-1}(y)$ is closed under $+$. Proposition 2.5 implies that $f^{-1}(y)$ is connected.

2 $\Rightarrow$ 3: Similarly, if f is monotone, then for all $(u, v) \in R_f$, we have $[u, v]_{T_1} \subset f^{-1}(y)$ for $y := f(u) = f(v)$. This means that $f([u, v]_{T_1}) = \{y\} = [f(u), f(v)]_{T_2}$. Thus, f is R_f-linear.

3 $\Rightarrow$ 1: Let $(u, v) \in R_f$. Then $f(u) = f(v) =: y$. Since f is R_f-linear, we have $f([u, v]_{T_1}) \subset \{y\}$. Thus, $f(u + v) = y = y + y = f(u) + f(v)$. Hence, f is an R_f-homomorphism. □

For an undirected graph G, an *orientation* of G is any irreflexive and antisymmetric binary relation R on $V(G)$ such that for every edge $\{u, v\} \in E(G)$, exactly one of $(u, v) \in R$ or $(v, u) \in R$ holds.

Proposition 3.11. *Let T_1 and T_2 be a pair of trees, and R be an orientation of T_1. Then $f : V(T_1) \to V(T_2)$ is an R-homomorphism between travel groupoids if and only if f is a metric map.*

Proof. Necessity. Let $\{u, v\} \in E(T_1)$ be an edge in T_1. Then $(u, v) \in R$ or $(v, u) \in R$. Without loss of generality, suppose $(u, v) \in R$. In this case, $f(v) = f(u + v) = f(u) + f(v)$, implying that $d_{T_2}(f(u), f(v)) \leqslant 1$. By Proposition 2.2, f is metric.

Sufficiency. Assume f is a metric map. Then for all $(u, v) \in R$, we have $\{u, v\} \in E(T_1)$. By Proposition 2.2, this implies that $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(T_2)$. Therefore, $f(u) + f(v) = f(v) = f(u + v)$. □

3.3. Congruences on travel groupoids

For a groupoid $(X, \circ)$, an equivalence relation $\sim$ on X is called a *left congruence* provided for all $a, b, x \in X$ with $a \sim b$, it holds $x \circ a \sim x \circ b$. Similarly, one can define *right congruences* on groupoids.

Each set X is naturally equipped with two *trivial* equivalence relations: the *diagonal* $\Delta(X) = \{(a, a) : a \in X\}$, and the relation with a unique equivalence class (being the whole set X). It is clear that trivial equivalence relations are both left and right congruences for any groupoid.

Proposition 3.12. *Let T be a tree, and $\sim$ be an equivalence relation on $V(T)$. Then the following statements hold:*

1. *$\sim$ is a left congruence for $+$ on T if and only if each equivalence class of $\sim$ is a connected subset;*
2. *$\sim$ is a right congruence for $+$ on T if and only if $\sim$ is trivial.*

Proof. 1. *Necessity.* Let $x, y \in [a]_\sim$ be two vertices from the equivalence class of a vertex $a \in V(T)$. Then $x + x \sim x + y$, meaning that $x \sim x + y$. Therefore, $x + y \in [a]_\sim$. By Proposition 2.5, $[a]_\sim$ is connected.

Sufficiency. Let $a, b, x \in V(T)$ with $a \sim b$. If $x \in [a, b]_T$, then $x + a \in [x, a]_T \subset [a, b]_T \subset [a]_\sim$ as $[a]_\sim$ is connected. Similarly, $x + b \in [b]_\sim$. Hence, $x + a \sim x + b$.

Now assume that $x \notin [a, b]_T$. Then the median $m := m_T(a, b, x)$ is different from x. Applying Lemma 2.4 twice for x, $y = m$, $z \in \{a, b\}$, we deduce that $x + a = x + m = x + b$.

2. The sufficiency of this condition is clear. Thus, let us prove its necessity. At first, note that similarly to left congruences, it can be proved that a right congruence for $+$ on T has connected equivalence classes.

Further, assume that $\sim$ is not the diagonal $\Delta(V(T))$. Then there exist two distinct vertices $a, b \in V(T)$ with $a \sim b$. If $[a]_\sim \neq V(T)$, then there is a vertex $x \in V(T) \setminus [a]_\sim$ adjacent to some vertex $y \in [a]_\sim = [b]_\sim$. Since $a \neq b$, we have $a \neq y$ or $b \neq y$. Without loss of generality, assume that $a \neq y$. Then $a \sim y$, implying $a + x \sim y + x$. However, $a + x \in [a]_\sim$, but $y + x = x$. Therefore, $a + x \sim x$, implying that $x \in [a]_\sim$ as well. The obtained contradiction proves that $\sim$ is trivial. □

Corollary 3.13. *Let T_1 and T_2 be a pair of trees. A map $f : V(T_1) \to V(T_2)$ is*

1. *monotone if and only if R_f is a left congruence for $+$ on T_1;*

2. *injective or constant if and only if R_f is a right congruence for $+$ on T_1.*

Proof. 1. Follows from the first part of Proposition 3.12, and the fact that $[a]_{R_f} = f^{-1}(f(a))$ for all $a \in V(T_1)$.

2. Immediately follows from the second part of Proposition 3.12. □

Hence, left congruences on travel groupoids on trees T correspond to the partitions of the vertex set $V(T)$ into non-empty connected subsets. This means that if we take a factor-graph of T by a left congruence $\sim$, then we obtain another tree T'. The corresponding projection $p : V(T) \to V(T')$ (recall that $p(a) = [a]_\sim$ for all $a \in V(T)$) will be simultaneously a linear and a metric map by the construction of T'.

The next result can be seen as an analogue of the first isomorphism theorem for travel groupoids on trees.

Proposition 3.14. *Let $f : V(T_1) \to V(T_2)$ be a linear metric map between two trees T_1 and T_2. Then the factor-graph of T_1 by the kernel R_f is isomorphic to the induced subgraph $T_2[\operatorname{Im} f]$.*

Proof. Denote by T' the factor-graph of T_1 by R_f. First, note that since f is linear, it is monotone. This means that each equivalence class of R_f is connected. Thus, T' is a tree. Similarly, as f is metric, $\operatorname{Im} f$ is connected (see Proposition 2.3). Thus, $T_2[\operatorname{Im} f]$ is also a tree.

We construct the desired isomorphism between T' and $T_2[\operatorname{Im} f]$ in the most natural way. Fix an equivalence class $A \in V(T')$ and any its vertex $a \in A$. By $\varphi(A)$ denote the image $f(a)$. It is clear that $\varphi(A)$ is independent from the choice of a (for any $b \in A$, we have $(a, b) \in R_f$, hence $f(a) = f(b)$). Thus, $\varphi : V(T') \to \operatorname{Im} f$ is a correctly defined map.

It is evident that φ is a bijection. Let us show that φ is an isomorphism of trees T' and $T_2[\operatorname{Im} f]$. Indeed, if $\{A, B\} \in E(T')$ is an edge, then $\{a, b\} \in E(T)$ for some $a \in A$ and $b \in B$. Since $(a, b) \notin R_f$, we have $f(a) \neq f(b)$. And since f is metric, we conclude that $\{f(a), f(b)\} \in E(T_2)$. Thus, φ is a bijective homomorphism between trees. Therefore, φ is an isomorphism. □

Remark 3.15. The proof of Proposition 3.14 in fact can give us a characterization of linear metric maps between trees. Namely, a map $f : V(T_1) \to V(T_2)$ is linear and metric if and only if there exists a left congruence $\sim$ on $V(T_1)$ such that the corresponding map $\varphi : V(T') \to \operatorname{Im} f$ is an isomorphism between the factor-tree T' and the induced subtree $T_2[\operatorname{Im} f]$.

Further, examining closely the proof of Proposition 3.12, we can see that it mainly exploited the fact that a travel groupoid is idempotent. Hence, it would be natural to consider the following relaxation for the definition of a congruence on a groupoid. For a groupoid $(X, \circ)$, an equivalence relation $\sim$ on X will be called *weak left congruence* provided for all $a, b, x \in X$ with $a \sim b$ and $x \neq a, b$, it holds $x \circ a \sim x \circ b$. Similarly, we define *weak right congruences* to be equivalence relations $\sim$ on X satisfying $a \circ x \sim b \circ x$ for all $a, b, x \in X$ with $a \sim b$ and $x \neq a, b$.

At first, we present a characterization of weak left congruences on travel groupoids on trees. To do so, we will need one more definition. Let G be a connected graph. The corresponding 2-*distance graph* is the graph $D_2(G)$ with the same vertex set $V(D_2(G)) = V(G)$, and the edge set $E(D_2(G)) = \{\{u, v\} : d_G(u, v) = 2\}$ (see the paper [2] for characterization of 2-distance graphs). For example, the 2-distance graph of any connected bipartite graph has two connected components.

Theorem 3.16. *Let T be a tree, and $\sim$ be an equivalence relation on $V(T)$. Then $\sim$ is a weak left congruence for $+$ on $V(T)$ if and only if the following two conditions hold:*

1. *each equivalence class of $\sim$ is either a connected set in T or in $D_2(T)$;*
2. *for each equivalence class A of $\sim$ which is a connected set in $D_2(T)$, all the vertices from $\mathrm{Conv}_T(A) \setminus A$ are equivalent under $\sim$.*

Proof. Necessity. Let $A \subset V(T)$ be an equivalence class of $\sim$. We consider two cases depending on parities of the distances between vertices from A.

Case 1. there exists a pair $a, b \in A$ with odd distance $d_T(a, b)$.

Let us prove that in this case, A is a connected set in T. To do so, fix a pair $a_0, b_0 \in A$ with the smallest odd distance among such pairs. We aim to show that $d_T(a_0, b_0) = 1$. To the contrary, assume $d_T(a_0, b_0) \geqslant 3$. Denote $x = a_0 + b_0$ and $y = x + b_0$. Then a_0, b_0, x, y are pairwise distinct vertices, and $a_0 = x + a_0 \sim x + b_0 = y$. Hence, $y \in A$. But $d_T(y, b_0) = d_T(a_0, b_0) - 2$ is a smaller odd distance, which is a contradiction.

Thus, a_0 and b_0 are adjacent. Fix a vertex $x \in A \setminus \{a_0, b_0\}$ (if $A = \{a_0, b_0\}$, then A is trivially connected). Without loss of generality, assume that $a_0 \in [x, b_0]_T$ (similarly, we can consider the case $b_0 \in [x, a_0]_T$). Consider with the first vertex $y := a_0 + x$ on the path from a_0 to x (see Figure 8). Then $x \sim b_0$ implies $y = a_0 + x \sim a_0 + b_0 = b_0$. Hence, $y \in A$. Using a simple inductive argument, we can conclude that $[x, a_0]_T \subset A$. This proves that A is a connected set in T.

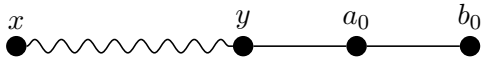


Figure 8: Illustration for the Case 1 from the proof of Theorem 3.16.

Case 2: for all $a, b \in A$, the distance $d_T(a, b)$ is even.

Fix $a, b \in A$. Assume $d_T(a, b) \geqslant 4$. Again, put $x = a + b$ and $y = x + b$. Then $a = x + a \sim x + b = y$. Hence, $y \in A$. Using an inductive argument, we can conclude that for every vertex $z \in [a, b]_T$ having the even distance $d_T(a, z)$, it holds $z \in A$. This asserts that A is a connected set in $D_2(T)$.

Finally, we need to prove that for such an equivalence class A of $\sim$, the set $B := \mathrm{Conv}_T(A) \setminus A$ is a subset of some other equivalence class of $\sim$. First, observe that B is also a connected set in $D_2(T)$. Hence, it is sufficient to prove that any pair of vertices from B lying at the distance 2 in T are equivalent under $\sim$. Fix such a pair $u, v \in B$, and the unique vertex $x \in N_T(u) \cap N_T(v)$. It is clear that $x \in A$. Moreover, by the definition of the convex hull, there exist $a \in A \cap (N_T(u) \setminus \{x\})$ and $b \in A \cap (N_T(v) \setminus \{x\})$. Now, $a \sim b$ immediately implies $u = x + a \sim x + b = v$. Therefore, B indeed lies in a separate equivalence class of $\sim$.

Sufficiency. Let $a, b \in V(T)$ with $a \sim b$, and $x \neq a, b$. If $x \notin [a, b]_T$, then Lemma 2.4 asserts that $x + a = x + m = x + b$ (where $m = m_T(a, b, x)$). Hence, trivially $x + a \sim x + b$.

Thus, assume that $x \in [a, b]_T$. We consider two cases depending on the connectivity of $A = [a]_\sim$. If A is connected in T, then $[a, b]_T \subset A$. Hence, $x + a \in [a, x]_T \subset [a, b]_T$ and $x + b \in [x, b]_T \subset [a, b]_T$ also implies $x + a \sim x + b$.

Now, suppose that A is connected in $D_2(T)$. Denote $B = \mathrm{Conv}_T(A) \setminus A$. It is clear that $[a, b]_T \subset A \cup B$.

If $x \in A$, then the distances $d_T(a, x)$ and $d_T(x, b)$ are both even. Therefore, the distances $d_T(a, x + a)$ and $d_T(x + b, b)$ are both odd. This means that the vertices $x + a$ and $x + b$ belong to B. However, B must lie in some equivalence class of $\sim$. Hence, $x + a \sim x + b$.

Similarly, if $x \in B$, then $d_T(a, x)$ and $d_T(x, b)$ are odd. Thus, $d_T(a, x+a)$ and $d_T(x + b, b)$ are even. This implies that $x + a, x + b \in A$, hence, $x + a \sim x + b$ as well. □

Example 3.17. Consider a tree T with $V(T) = \{1, \ldots, 9\}$ and $E(T) = \{\{1, 2\}, \{2, 3\}, \{2, 6\}, \{3, 4\}, \{3, 7\}, \{4, 5\}, \{7, 8\}, \{7, 9\}\}$. Let $\sim$ be the equivalence relation on $V(T)$ induced by the partition $V(T) = \{1, 3, 8, 9\} \sqcup \{4, 5\} \sqcup$

$\{2,7\} \sqcup \{6\}$. Then the classes $\{4,5\}$, $\{6\}$ are connected in T, and the classes $\{1,3,8,9\}$, $\{2,7\}$ are connected in $D_2(T)$. Theorem 3.16 ensures that $\sim$ is a weak left congruence for $+$ on T.

To characterize weak right congruences on travel groupoids on trees, we will also need several new definitions.

A pair of distinct vertices $u, v \in V(G)$ in a graph G are called *false twins* provided $\{u,v\} \notin E(G)$ and $N_G(u) = N_G(v)$. Denote by $\sim_{ftw}$ the equivalence relation "being equal or being false twins". It is clear that for two vertices $u, v \in V(T)$ in a tree T, we have $u \sim_{ftw} v$ if and only if $u = v$ or u and v are leaves lying at distance 2.

Having two equivalence relations R, Q on a set X, the relation R is called a *refinement of* Q provided $R \subset Q$.

Theorem 3.18. *Let T be a tree, and $\sim$ be an equivalence relation on $V(T)$. Then $\sim$ is a weak right congruence for $+$ on $V(T)$ if and only if $\sim$ is trivial, or $\sim$ correspond to the (unique) bipartition of T, or $\sim$ is a refinement of $\sim_{ftw}$ on $V(T)$.*

Proof. Sufficiency. It is clear that a trivial $\sim$ is a weak right congruence. Similarly, assume $\sim$ is given by the bipartition of T. Then for any pair $a, b \in V(T)$ with $a \sim b$ and any $x \in V(T) \setminus \{a,b\}$, we have $a + x, b + x$ lie in the opposite part than a, b. Thus, $a + x \sim b + x$.

Finally, suppose that $\sim$ is a refinement of $\sim_{ftw}$. Then $a \sim b$ with $a \neq b$ implies that $a, b \in \mathrm{Leaf}(T)$ with $d_T(a,b) = 2$. Let $v \in V(T)$ be the unique vertex from $N_T(a) \cap N_T(b)$. Then for all $x \in V(T) \setminus \{a,b\}$, we have $a + x = v = b + x$.

Necessity. Assume that $\sim$ is not the diagonal on $V(T)$. Then there exist two distinct vertices $a, b \in V(T)$ with $a \sim b$. We consider several cases.

Case 1: there exist distinct vertices $a_0, b_0 \in V(T)$ with $a_0 \sim b_0$ having the odd distance $d_T(a_0, b_0)$ between them.

Let us show that in this case, every two vertices are equivalent under $\sim$. Without loss of generality, we can assume that a_0, b_0 has the smallest odd distance among such pairs of vertices. If $d_T(a_0, b_0) \geqslant 3$, then consider the vertex $x = a_0 + b_0$. It is clear that $x \neq a_0, b_0$. Hence, $x = a_0 + x \sim b_0 + x$ and $x \neq b_0 + x$. But then $d_T(x, b_0 + x) = d_T(a_0, b_0) - 2$ is an odd smaller distance, which is a contradiction.

Therefore, assume a_0 and b_0 are adjacent in T. Fix a vertex $x \in N_T(a_0) \setminus \{b_0\}$. Then $x = a_0 + x \sim b_0 + x = a_0$, implying $x \sim a_0$. Similarly, one can

show that any vertex y from $N_T(b_0) \setminus \{a_0\}$ is equivalent to b_0 (and hence, to a_0). Using an inductive argument, we can conclude proving that any vertex in T is equivalent to a_0 under $\sim$.

Case 2: for all pairs of distinct vertices $a, b \in V(T)$ with $a \sim b$, the distance $d_T(a, b)$ is even.

Subcase 2.1: there exist two distinct vertices $a_0, b_0 \in V(T)$ with $a_0 \sim b_0$, one of which is non-leaf.

We show that in this case, $\sim$ must be given by the bipartition of T. For definiteness, suppose that a_0 is a non-leaf vertex in T. Also, without loss of generality, we can assume that a_0, b_0 has the smallest (even) distance among such pairs of vertices. Similarly to the Case 1, one can show that $d_T(a_0, b_0) = 2$. Indeed, assume $d_T(a_0, b_0) \geqslant 4$. Consider the vertex $x = a_0 + b_0$. Now, we use the fact $x = a_0 + x \sim b_0 + x$ together with the observation that neither x nor $b_0 + x$ cannot be leaves in T, in order to obtain a contradiction with the minimality of $d_T(a_0, b_0)$.

Thus, $d_T(a_0, b_0) = 2$. As before, set $x = a_0 + b_0$. Fix a vertex $v \in V(T) \setminus \{a_0, x\}$. Using induction on $d_T(a_0, v)$, we aim to prove the following: for even distances $d_T(a_0, v)$, it holds $a_0 \sim v$; for odd distances $d_T(a_0, v)$, it holds $x \sim v$.

For the induction basis, we assume $d_T(a_0, v) \leqslant 2$. At first, let $d_T(a_0, v) = 1$. Then $v = a_0 + v \sim b_0 + v = x$, implying that $v \sim x$.

Now let $d_T(a_0, v) = 2$. If $a_0 + v = x$, then fix a vertex $u \in N_T(a_0) \setminus \{x\}$ (we can do this since a_0 is non-leaf). We have, $u = a_0 + u \sim b_0 + u = x$ (see the left part of Figure 9). Thus, $a_0 = u + v \sim x + v = v$, implying $a_0 \sim v$. If $a_0 + v \neq x$, then denote $u = a_0 + v$ (see the right part of Figure 9). Again, use the fact that $u \sim x$, to deduce $v = u + v \sim x + v = a_0$. Thus, in this case, we have $v \sim a_0$ as well.

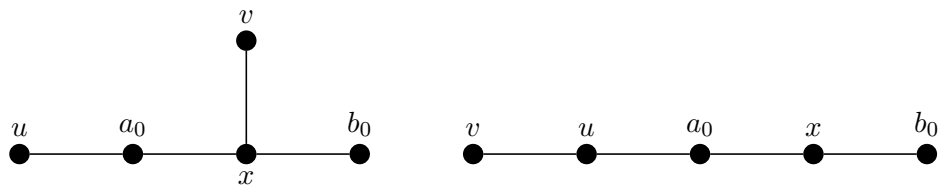


Figure 9: Illustration for the induction basis in the Subcase 2.1 from the proof of Theorem 3.18.

To prove the induction step, assume $d_T(a_0, v) \geqslant 3$. Define $t = v + a_0$. Clearly, $d_T(a_0, t) = d_T(a_0, v) - 1$.

At first, suppose that $a_0 = x + v$ (see Figure 10). If $d_T(a_0, v)$ is even, then $d_T(a_0, t)$ is odd. Thus, by induction assumption, $x \sim t$. We have, $a_0 = x + v \sim t + v = v$, implying $a_0 \sim v$. Similarly, if $d_T(a_0, v)$ is odd, we conclude that $d_T(a_0, t)$ is even. By induction assumption, this means that $t \sim a_0 \sim b_0$. Then $v = t + v \sim b_0 + v = x$. Hence, $v \sim x$ in this case.



Figure 10: Illustration for the induction step in the Subcase 2.1 from the proof of Theorem 3.18, the case $a_0 = x + v$.

Now, assume that $a_0 \neq x + v$. Then $a_0 + v = x$. Fix a vertex $u \in N_T(a_0) \setminus \{x\}$ (again, we can do this since a_0 is non-leaf). Then $u = a_0 + u \sim b_0 + u = x$ (see Figure 11). If $d_T(a_0, v)$ is even, then $d_T(a_0, t)$ is odd. Thus, by induction assumption, $x \sim t$. In this case, we have $t \sim u$. Thus, $v = t + v \sim u + v = a_0$. Similarly, if $d_T(a_0, v)$ is odd, we have that $d_T(a_0, t)$ is even. By induction assumption, this means that $t \sim a_0$. We have $v = t + v \sim a_0 + v = x$. Therefore, $v \sim x$ in this case.

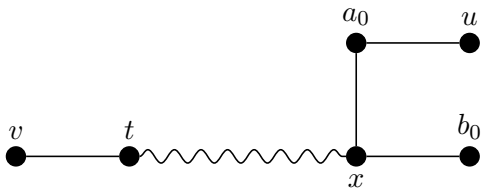


Figure 11: Illustration for the induction step in the Subcase 2.1 from the proof of Theorem 3.18, the case $a_0 \neq x + v$.

Further, since a_0 and x are adjacent, they are not equivalent under $\sim$. Thus, we can conclude that $\sim$ has only two distinct equivalence classes, being

$$[a_0]_\sim = \{v \in V(T) : d_T(a_0, v) \text{ is even}\} \text{ and}$$
$$[x]_\sim = \{v \in V(T) : d_T(a_0, v) \text{ is odd}\}.$$

And therefore, in this case, $\sim$ is indeed given by the bipartition of T.

Subcase 2.2: for all distinct vertices $a, b \in V(T)$ with $a \sim b$, both a and b are leaves.

Let us prove that any such a pair of vertices a, b has the distance $d_T(a, b) = 2$. Indeed, assuming there is such a pair $a_0, b_0 \in \mathrm{Leaf}(T)$ having $d_T(a_0, b_0) \geqslant 4$, we conclude that $x = a_0 + b_0$ and $b_0 + x$ is also a pair of distinct vertices with $x = a_0 + x \sim b_0 + x$. However, neither of x, $b_0 + x$ are leaves in T. This contradiction shows that $\sim$ is indeed a refinement of $\sim_{ftw}$ on $V(T)$. □

Remark 3.19. Theorems 3.16 and 3.18 suggest that each weak right congruence for $+$ on a tree is also its weak left congruence (the same trivially holds for congruences as can be seen from Proposition 3.12).

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