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On the number of normal subgroup chains in some classes of finite groups

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Abstract. Computing numerical invariants of finite groups plays a fundamental role in understanding their structure. In this paper, we study one such invariant: the number of chains of normal subgroups of a finite group G that terminate in G. Our investigation focuses on four distinct families of non-abelian groups: the 2-group $H(n)$, the semi-dihedral group SD_{8n}, and the groups U_{6n} and V_{8n}. For each family, we derive explicit formulas, providing a detailed description of their normal subgroup lattices and offering deeper insight into their structural properties.

1. Introduction

Group theory typically involves the computation of numerical invariants associated with a given group, whereas asymptotic group theory is concerned with analyzing the asymptotic behavior of these invariants, as detailed in [10]. This topic has attracted considerable attention from researchers; see, for example, [2–4, 6, 8, 11, 12, 14, 16]. One of the numerical invariants is the number of subgroups of a finite group G, which corresponds to the order of the subgroup lattice of G. In [4], Calhoun generalized the formula for the number of subgroups in a dihedral group to the class of groups that can be formed as cyclic extensions of cyclic groups such as the abelian groups of the form $\mathbb{Z}_m \times \mathbb{Z}_n$. In [14], Tărnăuceanu presented an arithmetic method for determining the number of certain types of subgroups of finite abelian groups. In [16], he and Tóth studied the number of subgroups of a given exponent in a finite abelian group and obtained formulas for the number of subgroups of rank two and three. In [6], Elaydi studied one of the other numerical invariants, the number of normal subgroups of a finite group G and proved that this number is the same as the number of all topologies on G which make G a topological group.

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Building upon these studies, we introduce and investigate a new numerical invariant, namely the number of chains of normal subgroups of a group G terminating in G. Throughout the paper, all groups are assumed to be finite, and G always denotes a finite group with identity element e. The notation $\mathcal{L}(G)$ ($\mathcal{LN}(G)$, respectively) represents the lattice of all subgroups (normal subgroups, respectively) of G. Moreover, $\mathcal{N}_{cn}(G)$ denotes the number of chains of normal subgroups of G terminating in G. In [8], the value of $\mathcal{N}_{cn}$ has been computed for dihedral and dicyclic groups with the following presentations:

$$D_{2n} = \langle x, y \mid x^n = y^2 = e,\ y^{-1}xy = x^{-1} \rangle,$$
$$T_{4n} = \langle x, y \mid x^{2n} = y^4 = e,\ y^2 = x^n,\ y^{-1}xy = x^{-1} \rangle.$$

The structure of the normal subgroup lattices of D_{2n} and T_{4n} is described in [8, 11]. For the dihedral group D_{2n} we have

$$\mathcal{LN}(D_{2n}) = \begin{cases} \mathcal{L}(M) \cup \{D_{2n}\}, & n \equiv 1 \pmod 2, \\ \mathcal{L}(M) \cup \{M_1, M_2, D_{2n}\}, & n \equiv 0 \pmod 2, \end{cases}$$

where $M = \langle x \rangle \cong \mathbb{Z}_n$, $M_1 = \langle x^2, y \rangle \cong D_{2n/2}$, and $M_2 = \langle x^2, xy \rangle \cong D_{2n/2}$ are maximal normal subgroups of D_{2n}.

For the dicyclic group T_{4n} we have

$$\mathcal{LN}(T_{4n}) = \begin{cases} \mathcal{L}(M) \cup \{T_{4n}\}, & n \equiv 1 \pmod 2, \\ \mathcal{L}(M) \cup \{M_1, M_2, T_{4n}\}, & n \equiv 0 \pmod 2, \end{cases}$$

where $M = \langle x \rangle \cong \mathbb{Z}_{2n}$, $M_1 = \langle x^2, y \rangle \cong T_{4n/2}$, and $M_2 = \langle x^2, xy \rangle \cong T_{4n/2}$ are maximal normal subgroups of T_{4n}.

This paper focuses on determining $\mathcal{N}_{cn}$ for the 2-group $H(n)$, the semi-dihedral group SD_{8n}, together with the groups U_{6n} and V_{8n}, each of which possesses the following representations:

$$H(n) = \langle x, y, z \mid x^{2^{n-2}} = y^2 = z^2 = e,\ [x, y] = [y, z] = e,\ x^z = xy \rangle,\ n \geqslant 4;$$
$$SD_{8n} = \langle x, y \mid x^{4n} = y^2 = e,\ yxy = x^{2n-1} \rangle,\ n \geqslant 2;$$
$$U_{6n} = \langle x, y \mid x^{2n} = y^3 = e,\ yxy = x \rangle,\ n \geqslant 2;$$
$$V_{8n} = \langle x, y \mid x^{2n} = y^4 = e,\ xyx = y^{-1},\ xy^{-1}x = y \rangle,\ n \geqslant 2.$$

Some numerical invariants related to the aforementioned groups are computed in [11], including maximal, normal maximal, self-normalizing

maximal, maximal normal, minimal normal, normal minimal and Sylow subgroups. Furthermore, a matrix model for determining the number of subgroups, normal subgroups, and characteristic subgroups of these groups is presented in [12].

The number of chains of normal subgroups of a group G terminating in G is equal to the number of distinct fuzzy subgroups of G with respect to the natural equivalence relation $\sim$ defined in [15]. Furthermore, the number of non-isomorphic chains of normal subgroups of G terminating in G which are invariant under automorphisms coincides with the number of distinct fuzzy subgroups of G with respect to the equivalence relation $\approx$ introduced in [8]. Since the equivalence relations $\sim$ and $\approx$ coincide, and every subgroup of a cyclic group is normal, we have the following theorem from [13].

Theorem 1.1. *Let G be a finite cyclic group of order n and $p_1^{m_1}p_2^{m_2}\dots p_s^{m_s}$ be the decomposition of n as a product of prime factors. Then, the number of chains of (normal) subgroups of G terminating in G is given by the equality:*

$$\mathcal{N}_{cn}(G)=2^{\sum\limits_{\alpha=1}^{s} m_\alpha}\sum_{i_2=0}^{m_2}\sum_{i_3=0}^{m_3}\dots\sum_{i_s=0}^{m_s}(-1/2)^{\sum\limits_{\alpha=2}^{s} i_\alpha}\prod_{\alpha=2}^{s}\binom{m_\alpha}{i_\alpha}\binom{m_1+\sum\limits_{\beta=2}^{\alpha}(m_\beta-i_\beta)}{m_\alpha},$$

and the above iterated sums are equal to 1 for $s=1$.

Specifically, when G is a cyclic group of order p^m, pq^m, p^2q^m, or pqr (where p,q,r are prime numbers), the corresponding values of $\mathcal{N}_{cn}$ are 2^m, $2^m(m+2)$, $2^{m-1}(m^2+7m+8)$, and 26, respectively.

2. Main Results

In this section, we extend the investigations presented in [11, 12] by computing the numerical invariant $\mathcal{N}_{cn}$ for the groups $H(n)$, SD_{8n}, U_{6n}, and V_{8n}. The computational details for each group are provided in individual subsections.

2.1. Group $H(n)$

The 2-group $H(n)$ was introduced by Abbaspour and Behravesh in [1]. In that work, the authors computed its character table and investigated some of its fundamental properties. Later, in [12], it was established that for $n\geqslant 5$, the group $H(n)$ possesses $11+5(n-4)$ normal subgroups of order 2^k,

where $0 \leqslant k \leqslant n-1$. The structural organization of the elements of $\mathcal{LN}(G)$ is summarized in Table 1. From this table, we deduce that the maximal

Table 1: Normal subgroups of a given order in the group $H(n)$, $n \geqslant 5$

Order	1	2	2^i, $(2 \leqslant i \leqslant n-3)$	2^{n-2}	2^{n-1}
	$\{e\}$	$\langle x^{2^{n-3}} \rangle$	$\langle x^{2^{n-i-2}} \rangle$	$\langle x^2, y \rangle$	$\langle x, y \rangle$
		$\langle x^{2^{n-3}} y \rangle$	$\langle x^{2^{n-i-2}} y \rangle$	$\langle x^2 z, y \rangle$	$\langle xz, y \rangle$
		$\langle y \rangle$	$\langle x^{2^{n-i-1}}, y \rangle$	$\langle x^4, y, z \rangle$	$\langle x^2, y, z \rangle$
			$\langle x^{2^{n-i-1}} z, y \rangle$		
			$\langle x^{2^{n-i}}, y, z \rangle$		

normal subgroups of $H(n)$ are $M_1 = \langle x, y \rangle$, $M_2 = \langle xz, y \rangle$, $M_3 = \langle x^2, y, z \rangle$, and for every $2 \leqslant i \leqslant n-4$, the following relations hold:

$$\langle x^{2^{n-i-2}} \rangle \leqslant \langle x^{2^{n-i-3}} \rangle, \langle x^{2^{n-i-3}} y \rangle, \langle x^{2^{n-i-2}}, y \rangle$$
$$\langle x^{2^{n-i-2}} y \rangle \leqslant \langle x^{2^{n-i-2}}, y \rangle$$
$$\langle x^{2^{n-i-1}}, y \rangle \leqslant \langle x^{2^{n-i-2}}, y \rangle, \langle x^{2^{n-i-2}} z, y \rangle, \langle x^{2^{n-i-1}}, y, z \rangle$$
$$\langle x^{2^{n-i-1}} z, y \rangle \leqslant \langle x^{2^{n-i-1}}, y, z \rangle$$
$$\langle x^{2^{n-i}}, y, z \rangle \leqslant \langle x^{2^{n-i-1}}, y, z \rangle.$$

For a clearer understanding of the structure of the normal subgroup lattice of $H(n)$, the case $H(7)$ is illustrated in Figure 1.

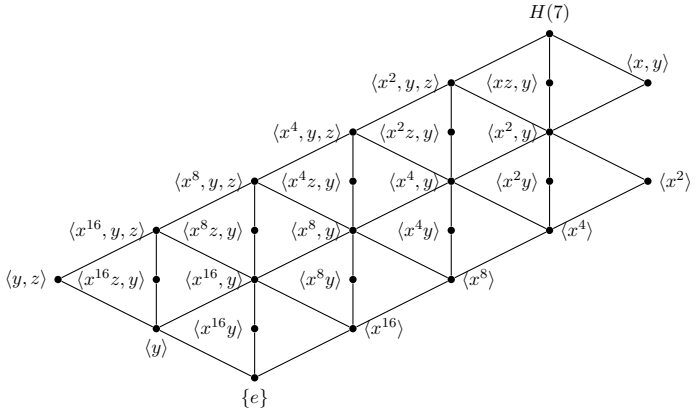


Figure 1: The normal subgroup lattice of $H(7)$.

Theorem 2.1. *The value of $\mathcal{N}_{cn}(H(n))$, which is the number of chains of normal subgroups of $H(n)$ terminating in $H(n)$, where $n \geqslant 5$, is $2^{n-1}(n^2 - n - 2)$.*

Proof. Assume that $\mathcal{C}_i$, where $2 \leqslant i \leqslant n-3$, is the set of chains of subgroups $P_{i_j} \in \mathcal{LN}(H(n))$ of $\langle x^{2^i}, y\rangle \cong \mathbb{Z}_{2^{n-i-2}} \times \mathbb{Z}_2$ terminating in $\langle x^{2^i}, y\rangle$, that is

$$\mathcal{C}_i = \{\cdots \subset P_{i_3} \subset P_{i_2} \subset P_{i_1} = \langle x^{2^i}, y\rangle \mid P_{i_j} \in \mathcal{LN}(H(n))\}.$$

Suppose that $\mathcal{C}_i'$ denote the set obtained by removing $\langle x^{2^i}, y\rangle$ from each chain in $\mathcal{C}_i$. In other words, $\mathcal{C}_i'$ consists of the same chains as $\mathcal{C}_i$, but with $\langle x^{2^i}, y\rangle$ excluded from the end of each chain. Also, let $\mathcal{D}_{i_0}$ ($\mathcal{D}_{i_1}$, respectively) be the set of chains of subgroups of $\langle x^{2^i}\rangle \cong \mathbb{Z}_{2^{n-i-2}}$ ($\langle x^{2^i} y\rangle \cong \mathbb{Z}_{2^{n-i-2}}$, respectively) terminating in $\langle x^{2^i}\rangle$ ($\langle x^{2^i} y\rangle$, respectively). Then, every chain of $\mathcal{C}_i$ is obtained in one of the following ways:

(1) Appending $\langle x^{2^i}, y\rangle$ to the end of an arbitrary chain of $\mathcal{D}_{i_0} \cup \mathcal{D}_{i_1}$;

(2) Appending $\langle x^{2^i}, y\rangle$ to the end of an arbitrary chain of $\mathcal{C}_{i+1} \cup \mathcal{C}_{i+1}'$.

Consequently,

$$\begin{aligned}
|\mathcal{C}_i| &= |\mathcal{D}_{i_0}| + |\mathcal{D}_{i_1}| + |\mathcal{C}_{i+1}| + \left|\mathcal{C}_{i+1}'\right| = \mathcal{N}_{cn}(\langle x^{2^i}\rangle) + \mathcal{N}_{cn}(\langle x^{2^i} y\rangle) + 2\,|\mathcal{C}_{i+1}| \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-2}}) + 2\left(\mathcal{N}_{cn}(\langle x^{2^{i+1}}\rangle) + \mathcal{N}_{cn}(\langle x^{i+1} y\rangle) + 2\,|\mathcal{C}_{i+2}|\right) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-2}}) + 2\left(\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-3}}) + \mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-3}}) + 2\,|\mathcal{C}_{i+2}|\right) \\
&= \cdots \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-2}}) + 2^2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-3}}) + 2^3\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-4}}) + 2^4\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-5}}) \\
&\quad + \cdots + 2^{n-i-3}\mathcal{N}_{cn}(\mathbb{Z}_{2^2}) + 2^{n-i-3}\,|\mathcal{C}_{n-3}|\,.
\end{aligned}$$

On the other hand, $|\mathcal{C}_{n-3}| = 8$, because $\mathcal{C}_{n-3}$ contains the following chains:

$$\begin{array}{ll}
\langle x^{2^{n-3}}\rangle \subset \langle x^{2^{n-3}}, y\rangle; & \{e\} \subset \langle x^{2^{n-3}}\rangle \subset \langle x^{2^{n-3}}, y\rangle; \\
\langle x^{2^{n-3}} y\rangle \subset \langle x^{2^{n-3}}, y\rangle; & \{e\} \subset \langle x^{2^{n-3}} y\rangle \subset \langle x^{2^{n-3}}, y\rangle; \\
\langle y\rangle \subset \langle x^{2^{n-3}}, y\rangle; & \{e\} \subset \langle y\rangle \subset \langle x^{2^{n-3}}, y\rangle; \\
\langle x^{2^{n-3}}, y\rangle; & \{e\} \subset \langle x^{2^{n-3}}, y\rangle.
\end{array}$$

This leads to

$$|\mathcal{C}_i| = 2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-2}}) + 2^2\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-3}}) + 2^3\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-4}}) + 2^4\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-i-5}})$$

$$
\begin{aligned}
&+\cdots+2^{n-i-3}\mathcal{N}_{cn}(\mathbb{Z}_{2^2})+2^{n-i-3}(8)\\
&=2(2^{n-i-2})+2^2(2^{n-i-3})+2^3(2^{n-i-4})+\cdots+2^{n-i-3}(2^2)+2^{n-i}\\
&=(n-i-1)2^{n-i-1}.
\end{aligned}
$$

Consequently,

$$|\mathcal{C}_i|=\left|\mathcal{C}_i'\right|=(n-i-1)2^{n-i-1}. \tag{1}$$

Now, we proceed to determining $\mathcal{N}_{cn}(H(n))$. For $2\leqslant i\leqslant n-2$, we define two chains E_i and F_i as follows:

$$E_i\colon \langle x^{2^{i-1}}z,y\rangle\subset\langle x^{2^{i-1}},y,z\rangle\subset\langle x^{2^{i-2}},y,z\rangle\subset\cdots\langle x^{2^2},y,z\rangle\subset\langle x^2,y,z\rangle\subset H(n);$$
$$F_i\colon \langle x^{2^i},y,z\rangle\subset\langle x^{2^{i-1}},y,z\rangle\subset\langle x^{2^{i-2}},y,z\rangle\subset\cdots\langle x^{2^2},y,z\rangle\subset\langle x^2,y,z\rangle\subset H(n).$$

A chain of normal subgroups of $H(n)$ terminating in $H(n)$ can be constructed in one of the following ways:

(1) Appending one of the elements of $\mathcal{A}$ to the end of an arbitrary chain of $\mathcal{D}_{1_0}\cup\mathcal{D}_{1_1}\cup\mathcal{C}_2\cup\mathcal{C}_2'$, where

$$\mathcal{A}=\{\langle x^2,y\rangle\subset M_i\subset H(n);\ M_i\subset H(n);\ \langle x^2,y\rangle\subset H(n);\ H(n)\mid 1\leqslant i\leqslant 3\};$$

(2) Appending one of the elements of $E_i'\cup F_i'$, $2\leqslant i\leqslant n-2$, to the end of an arbitrary chain of $\mathcal{C}_i\cup\mathcal{C}_i'$, where the sets E_i' and F_i' consist of all subchains of E_i and F_i, respectively, such that each subchain starts with $\langle x^{2^{i-1}}z,y\rangle$ and $\langle x^{2^i},y,z\rangle$, respectively, and terminates in $H(n)$.

Since

$$\left|E_i'\right|=\left|F_i'\right|=\binom{i-1}{0}+\binom{i-1}{1}+\cdots+\binom{i-1}{i-1}=2^{i-1},$$

and $|\mathcal{A}|=8$, hence

$$\mathcal{N}_{cn}(H(n))=8\left(|\mathcal{D}_{1_0}|+|\mathcal{D}_{1_1}|+|\mathcal{C}_2|+\left|\mathcal{C}_2'\right|\right)+2\sum_{i=2}^{n-2}2^{i-1}\left|\mathcal{C}_i\right|+2\sum_{i=2}^{n-2}2^{i-1}\left|\mathcal{C}_i'\right|.$$

On the other hand, $|\mathcal{D}_{1_0}|=|\mathcal{D}_{1_1}|=\mathcal{N}_{cn}(\mathbb{Z}_{2^{n-3}})$. Therefore, by (1) and Theorem 1, we deduce that

$$\mathcal{N}_{cn}(H(n))=8\left(2^{n-2}+2(n-3)2^{n-3}\right)+4\sum_{i=2}^{n-2}2^{i-1}(n-i-1)2^{n-i-1}$$

$$= 2^{n+1}(n-2) + 2^n \sum_{i=2}^{n-2}(n-i-1)$$
$$= 2^{n+1}(n-2) + 2^n(n-3)(n-1) - 2^n \sum_{i=2}^{n-2} i$$
$$= 2^{n-1}(n^2-n-2). \qquad \square$$

2.2. Semi-dihedral Group SD_{8n}

The semi-dihedral group SD_{8n} (where $n \geqslant 2$) is generated by two elements and has the presentation $SD_{8n} = \langle x, y \mid x^{4n} = y^2 = e,\ yxy = x^{2n-1}\rangle$. Note that $SD_8 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Every element of $SD_{8n} \setminus \langle x\rangle$ is of the form $x^i y$, where $1 \leqslant i \leqslant 4n-1$. According to the presentation of SD_{8n} ($n \geqslant 2$), it follows that $x^i y = yx^{i(2n-1)}$, $o(x^{2i}y) = 2$, and $o(x^{2i+1}y) = 4$. The maximal subgroups of SD_{8n} are $\langle x\rangle$, $\langle x^2, y\rangle$, $\langle x^2, xy\rangle$, and $\langle x^p, x^{2i}y\rangle$, where p denotes an odd prime number. The normal subgroup lattice of SD_{8n} has a well-defined structure, which plays a fundamental role in the proof of the following theorem.

Theorem 2.2. *The number of chains of normal subgroups of SD_{8n} terminating in SD_{8n}, where $n \geqslant 2$, that is $\mathcal{N}_{cn}(SD_{8n})$ is given by the equality:*

$$\mathcal{N}_{cn}(SD_{8n}) = \begin{cases} 2\mathcal{N}_{cn}(\mathbb{Z}_{4n}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2n}) + 8\mathcal{N}_{cn}(\mathbb{Z}_n), & n \text{ is odd}, \\ 2\mathcal{N}_{cn}(\mathbb{Z}_{4n}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2n}), & n \text{ is even}. \end{cases}$$

In particular, $\mathcal{N}_{cn}(SD_{2^{m+3}}) = 2^{m+4}$, $\mathcal{N}_{cn}(SD_{8p^m}) = 2^m(m^2+11m+24)$, and $\mathcal{N}_{cn}(SD_{2^{m+3}p}) = 2^{m+3}(2m+7)$.

Proof. The group SD_{8n} contains a maximal cyclic normal subgroup $M_1 = \langle x\rangle \cong \mathbb{Z}_{4n}$. Furthermore, SD_{8n} possesses two additional maximal normal subgroups, namely $M_2 = \langle x^2, y\rangle \cong D_{4n}$ and $M_3 = \langle x^2, xy\rangle \cong T_{4n}$. It is clear that, all subgroups of the maximal subgroup M_1 are normal in SD_{8n}.

If n is odd, then according to the structure of the normal subgroup lattice of dihedral and dicyclic groups described in Section , we obtain

$$\mathcal{LN}(M_2) = \mathcal{L}(\langle x^2\rangle) \cup \{\langle x^4, y\rangle, \langle x^4, x^2y\rangle, M_2\} \text{ and } \mathcal{LN}(M_3) = \mathcal{L}(\langle x^2\rangle) \cup \{M_3\}.$$

All elements of $\mathcal{LN}(M_2)$ and $\mathcal{LN}(M_3)$ are normal in SD_{8n} which leads to $\mathcal{LN}(SD_{8n}) = \mathcal{L}(M_1) \cup \{\langle x^4, y\rangle, \langle x^4, x^2y\rangle, M_2, M_3, SD_{8n}\}$. Then, a maximal

chain of normal subgroups for SD_{8n} can be described by one of the following types [9, 11]:

$\{e\} \subset H_1 \subset \cdots \subset H_r \subset M_1 \subset SD_{8n}$,
$\{e\} \subset K_1 \subset \cdots \subset K_s = \langle x^2 \rangle \subset M_i \subset SD_{8n}$, where $i = 2, 3$,
$\{e\} \subset L_1 \subset \cdots \subset L_t = \langle x^4 \rangle \subset \langle x^4, x^{2i}y \rangle \subset M_2 \subset SD_{8n}$, where $i = 0, 1$.

Let $\mathcal{C}_j$ denote the set of chains of subgroups of $\langle x^j \rangle$ terminating in $\langle x^j \rangle$, where $j = 1, 2, 4$. Set $\mathcal{A} = \{\langle x^4, x^{2i}y \rangle \subset M_2 \subset SD_{8n};\ \langle x^4, x^{2i}y \rangle \subset SD_{8n} \mid i = 0, 1\}$. A chain of normal subgroups of SD_{8n} terminating in SD_{8n} can be constructed in one of the following ways:

(1) Appending SD_{8n} to the end of an arbitrary chain of $\mathcal{C}_1$;

(2) Replacing $\langle x \rangle$ with SD_{8n} in an arbitrary chain of $\mathcal{C}_1$;

(3) Appending the inclusion $M_i \subset SD_{8n}$, that is, both M_i and SD_{8n}, to the end of an arbitrary chain of $\mathcal{C}_2$, where $i = 1, 2$;

(4) Replacing $\langle x^2 \rangle$ with the inclusion $M_i \subset SD_{8n}$ in an arbitrary chain of $\mathcal{C}_2$, where $i = 1, 2$;

(5) Appending one of the elements of $\mathcal{A}$ to the end of an arbitrary chain of $\mathcal{C}_4$.

(6) Replacing $\langle x^4 \rangle$ with one of the elements of $\mathcal{A}$ in an arbitrary chain of $\mathcal{C}_4$.

Since $|\mathcal{C}_1| = \mathcal{N}_{cn}(\mathbb{Z}_{4n})$, $|\mathcal{C}_2| = \mathcal{N}_{cn}(\mathbb{Z}_{2n})$, and $|\mathcal{C}_4| = \mathcal{N}_{cn}(\mathbb{Z}_n)$ we obtain

$$\mathcal{N}_{cn}(SD_{8n}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{4n}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2n}) + 8\mathcal{N}_{cn}(\mathbb{Z}_n).$$

If n is even, then by the normal subgroup lattices D_{4n} and T_{4n} described in the previous section, we have

$$\mathcal{LN}(M_2) = \mathcal{L}(\langle x^2 \rangle) \cup \{\langle x^4, y \rangle, \langle x^4, x^2y \rangle, M_2\},$$

and

$$\mathcal{LN}(M_3) = \mathcal{L}(\langle x^2 \rangle) \cup \{\langle x^4, xy \rangle, \langle x^4, x^3y \rangle, M_3\}.$$

Clearly, $\langle x^4, y \rangle$, $\langle x^4, x^2y \rangle$, $\langle x^4, xy \rangle$, and $\langle x^4, x^3y \rangle$ are not normal in SD_{8n}. Consequently, every nontrivial subgroup of M_2 and M_3 that is normal in SD_{8n} is of the form $\langle x^{2i} \rangle$. Therefore, for even n, the set of chains of normal

subgroups of SD_{8n} terminating in SD_{8n} can be constructed in the same way as in the odd case, with the exception that cases (5) and (6) do not occur. This leads to

$$\mathcal{N}_{cn}(SD_{8n}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{4n}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2n}).$$

Substituting the values of $\mathcal{N}_{cn}(\mathbb{Z}_{4n})$, $\mathcal{N}_{cn}(\mathbb{Z}_{2n})$ and $\mathcal{N}_{cn}(\mathbb{Z}_n)$ from Theorem 1.1, the statement of the final part follows. □

2.3. Group U_{6n}

The group U_{6n} was first introduced in [7] with the presentation

$$U_{6n} = \langle x, y \mid x^{2n} = y^3 = e,\ yxy = x\rangle, \quad n \geqslant 2.$$

In fact, when $n = 1$, the group U_6 is isomorphic to D_6 and consequently $\mathcal{N}_{cn}(U_6) = 4$. It follows from the presentation of U_{6n}, $x^{2i}y = yx^{2i}$ and $x^{2i+1}y = y^2x^{2i+1}$. It has been shown that all maximal normal subgroups of U_{6n} have prime index and are of the form $\langle x^p, y\rangle$, where p is a prime divisor of $2n$ [11].

Let $n = 2^{\alpha_1}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the decomposition of n as a product of prime factors. Maximal normal subgroups of U_{6n} can be described as follows [12]:

$$M_1 = \langle x^2, y\rangle = \{x^{2t}, x^{2t}y, x^{2t}y^2 \mid 1 \leqslant t \leqslant n\};$$
$$M_{i+1} = \langle x^{p_i}, y\rangle = \{x^{p_i t}, x^{p_i t}y, x^{p_i t}y^2 \mid 1 \leqslant t \leqslant \frac{2n}{p_i}\} \cong U_{\frac{6n}{p_i}},\ \text{where } 1 \leqslant i \leqslant k.$$

The above maximal normal subgroups play a significant role in the normal subgroup lattice of U_{6n} and lead to the following theorem.

Theorem 2.3. *Let $n = 2^{\alpha_1}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$, where $2 < p_1 < p_2 < \cdots < p_k$, be the prime factorization of n. Moreover, let $m = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ be the odd part of n. The number of chains of normal subgroups of U_{6n} terminating in U_{6n}, where $n \geqslant 2$, is determined by*

$$\mathcal{N}_{cn}(U_{6n}) = \begin{cases} 2\mathcal{N}_{cn}(\mathbb{Z}_{3n}) + \sum\limits_{\substack{s|m \\ s\neq 1}} \mathcal{N}_{cn}(U_{6n/s}), & p_1 \neq 3, \\ 2\mathcal{N}_{cn}(\mathbb{Z}_{3n'}) + \sum\limits_{\substack{s|m \\ s\neq 1}} \mathcal{N}_{cn}(U_{6n/s}), & p_1 = 3, \end{cases}$$

where $n' = 2^{\alpha_1}q^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ and $q \neq 2, 3, p_2, \cdots, p_k$ is a prime number.

Proof. The normal subgroups of U_{6n} are of the following types:

$$H_r = \langle x^{2r} \rangle \cong \mathbb{Z}_{n/r},\ K_r = \langle x^{2r}, y \rangle,\ \text{and}\ L_s = \langle x^s, y \rangle \cong U_{6n/s},$$

where r, s are respectively divisors of n and m. It is easy to verify that, $H_r \leqslant K_r \leqslant M_1$. Therefore, the normal subgroup lattice of U_{6n} can be described as

$$\mathcal{LN}(U_{6n}) = \mathcal{L}(H_1) \cup \{K_r, L_s;\ \text{where}\ r \mid n\ \text{and}\ s \mid m\}.$$

In this case, a chain of normal subgroups of U_{6n} terminating in U_{6n} can be constructed in one of the following three ways:

(1) Appending U_{6n} to the end of an arbitrary chain of $\mathcal{C}_0$, where

$$\mathcal{C}_0 = \{\cdots \subset P_3 \subset P_2 \subset P_1 = M_1 = \langle x^2, y \rangle \ \mid\ P_i \in \mathcal{LN}(U_{6n})\};$$

(2) Replacing $\langle x^2, y \rangle$ with U_{6n} in an arbitrary chain of $\mathcal{C}_0$;

(3) For all divisor s of m, $s \neq 1$, appending U_{6n} to the end of an arbitrary chain of normal subgroups of $\langle x^s, y \rangle$ terminating in $\langle x^s, y \rangle$.

Therefore,

$$\mathcal{N}_{cn}(U_{6n}) = 2\,|\mathcal{C}_0| + \sum_{\substack{s|m \\ s\neq 1}} \mathcal{N}_{cn}(U_{6n/s}). \tag{2}$$

In what follows, we consider the two distinct cases $3 \nmid n$ and $3 \mid n$ separately.

First, suppose that $3 \nmid n$, i.e., $p_1 \neq 3$. In this case $M_1 = \langle x^2, y \rangle = \langle x^2 y \rangle \cong \mathbb{Z}_{3n}$ and $K_r = \langle x^{2r}, y \rangle = \langle x^{2r} y \rangle \cong \mathbb{Z}_{3n/r}$. Set

$$\mathcal{D} = \{P \leqslant M_1 \mid P \in \mathcal{LN}(U_{6n})\} = \{\langle x^{2r} \rangle, \langle x^{2r} y \rangle;\ \text{where}\ r \mid n\},$$

then $\mathcal{D} = \mathcal{L}(\langle x^2 y \rangle) \cong \mathcal{L}(\mathbb{Z}_{3n})$ and consequently $|\mathcal{C}_0| = \mathcal{N}_{cn}(\mathbb{Z}_{3n})$. Therefore, by Relation (2), we get

$$\mathcal{N}_{cn}(U_{6n}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{3n}) + \sum_{\substack{s|m \\ s\neq 1}} \mathcal{N}_{cn}(U_{6n/s}).$$

Now, suppose that $3 \mid n$, i.e., $p_1 = 3$. Since $x^2y = yx^2$, the subgroup $M_1 = \langle x^2, y\rangle$ is abelian. Moreover, $o(x^2) = n$, $o(y) = 3$, and $\langle x^2\rangle \cap \langle y\rangle = \{e\}$. Hence, $M_1 = \langle x^2, y\rangle = \langle x^2\rangle \times \langle y\rangle \cong \mathbb{Z}_n \times \mathbb{Z}_3$. Furthermore,

$$K_r = \begin{cases} \langle x^{2r}, y\rangle \cong \mathbb{Z}_{n/r} \times \mathbb{Z}_3, & 3 \mid \frac{n}{r}, \\ \langle x^{2r}, y\rangle = \langle x^{2r}y\rangle \cong \mathbb{Z}_{3n/r}, & 3 \nmid \frac{n}{r}. \end{cases}$$

Put

$$\mathcal{F} = \{P \leqslant M_1 \mid P \in \mathcal{LN}(U_{6n})\} = \{\langle x^{2r}\rangle, \langle x^{2r}, y\rangle;\ \text{where } r \mid n\}.$$

Comparing the sets $\mathcal{F}$ and $\mathcal{D}$ (defined above), we observe that they play exactly the same role in the normal subgroup lattice. Indeed, when $3 \nmid n$, the normal subgroup corresponding to a divisor $r \mid n$ is $\langle x^{2r}y\rangle$, whereas when $3 \mid n$ and $3 \mid \frac{n}{r}$, this role is played by $\langle x^{2r}, y\rangle$. Thus, when $3 \mid n$, the subgroup $\langle x^{2r}y\rangle$ is simply replaced by $\langle x^{2r}, y\rangle$. This replacement changes only the way the normal subgroup is represented and does not affect the inclusion relations among the normal subgroups. Consequently, the part of the normal subgroup lattice involved in the computation of $|\mathcal{C}_0|$ has the same inclusion structure and hence contains the same number of chains of normal subgroups.

Therefore, the computation of $|\mathcal{C}_0|$ is identical to that in the case $3 \nmid n$. However, since the expression obtained from Relation (2) involves $\mathcal{N}_{cn}(\mathbb{Z}_{3n})$, while the prime 3 already appears in the factorization of n, we replace the factor 3^{m_1} in n by q^{m_1}, where $q \neq 2, 3, p_2, \ldots, p_k$ is an arbitrary prime, and define $n' = 2^{\alpha_1} q^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. By Theorem 1.1, the value of $\mathcal{N}_{cn}(\mathbb{Z}_m)$ depends only on the exponents in the prime factorization of m. Hence, $|\mathcal{C}_0| = \mathcal{N}_{cn}(\mathbb{Z}_{3n'})$. Consequently, by Relation (2),

$$\mathcal{N}_{cn}(U_{6n}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{3n'}) + \sum_{\substack{s \mid m \\ s \neq 1}} \mathcal{N}_{cn}(U_{6n/s}). \qquad \square$$

We will now concentrate on calculating the exact value of $\mathcal{N}_{cn}(U_{6n})$ in two situations $n = p^m$ and $n = pq$.

Corollary 2.4. *The number of chains of normal subgroups of U_{6p^m} terminating in U_{6p^m}, where p is a prime number, is given by the following equality:*

$$\mathcal{N}_{cn}(U_{6\cdot p^m}) = \begin{cases} 2^{m+1}(m+2), & p = 2, \\ 2^{m-1}(m^2 + 7m + 8), & p \neq 2. \end{cases}$$

Proof. Suppose that $n = 2^m$, then by Theorem 2.3 and Theorem 1.1, we have

$$\mathcal{N}_{cn}(U_{6\cdot 2^m}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{3\cdot 2^m}) = 2^{m+1}(m+2).$$

If $p \neq 2, 3$, then by Theorem 2.3,

$$\begin{aligned}
\mathcal{N}_{cn}(U_{6p^m}) &= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + \sum_{\substack{s|p^m \\ s\neq 1}} \mathcal{N}_{cn}(U_{6p^m/s}) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + \mathcal{N}_{cn}(U_{6p^{m-1}}) + \mathcal{N}_{cn}(U_{6p^{m-2}}) \\
&\qquad + \cdots + \mathcal{N}_{cn}(U_{6p^2}) + \mathcal{N}_{cn}(U_{6p}) + \mathcal{N}_{cn}(U_6) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-1}}) \\
&\qquad + 2\left(\mathcal{N}_{cn}(U_{6p^{m-2}}) + \cdots + \mathcal{N}_{cn}(U_{6p}) + \mathcal{N}_{cn}(U_6)\right) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-1}}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-2}}) \\
&\qquad + 4\left(\mathcal{N}_{cn}(U_{6p^{m-3}}) + \cdots + \mathcal{N}_{cn}(U_{6p}) + \mathcal{N}_{cn}(U_6)\right) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-1}}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-2}}) + 8\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-3}}) \\
&\qquad + 8\left(\mathcal{N}_{cn}(U_{6p^{m-4}}) + \cdots + \mathcal{N}_{cn}(U_{6p}) + \mathcal{N}_{cn}(U_6)\right) \\
&= \cdots \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-1}}) + 2^2\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-2}}) + 2^3\mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-3}}) \\
&\qquad + \cdots + 2^{m-2}\mathcal{N}_{cn}(\mathbb{Z}_{3p^2}) + 2^{m-1}\mathcal{N}_{cn}(\mathbb{Z}_{3p}) + 2^{m-1}\mathcal{N}_{cn}(U_6) \\
&= 2\mathcal{N}_{cn}(\mathbb{Z}_{3p^m}) + 2^{m-1}\mathcal{N}_{cn}(U_6) + \sum_{i=1}^{m-1} 2^i \mathcal{N}_{cn}(\mathbb{Z}_{3p^{m-i}}) \\
&= 2^{m+1}(m+2) + 2^{m+1} + 2^m \sum_{i=1}^{m-1}(m+2-i).
\end{aligned}$$

Consequently,

$$\mathcal{N}_{cn}(U_{6p^m}) = 2^{m-1}(m^2+7m+8). \tag{3}$$

In a manner similar to Relation (3), by applying Theorem 2.3, we obtain

$$\mathcal{N}_{cn}(U_{6\times 3^m}) = 2^{m-1}(m^2+7m+8). \qquad \square$$

Corollary 2.5. *The number of chains of normal subgroups of U_{6pq} terminating in U_{6pq}, where p, q are distinct prime numbers, is given by the following equality:*

$$\mathcal{N}_{cn}(U_{6pq}) = \begin{cases} 64, & p = 2, \\ 88, & p, q \neq 2. \end{cases}$$

Proof. Assume that p, q are distinct prime numbers and $p, q \neq 3$. If $p = 2$, then by Theorem 2.3 and Corollary 2.4, we obtain

$$\mathcal{N}_{cn}(U_{12q}) = 2\mathcal{N}_{cn}(\mathbb{Z}_{6q}) + \mathcal{N}_{cn}(U_{12}) = 2 \times 26 + 12 = 64. \tag{4}$$

If $p \neq 2$, then by Theorem 2.3 and Corollary 2.4, we deduce that

$$\begin{aligned} \mathcal{N}_{cn}(U_{6pq}) &= 2\mathcal{N}_{cn}(\mathbb{Z}_{3pq}) + \sum_{\substack{s|pq \\ s\neq 1}} \mathcal{N}_{cn}(U_{6n/s}) \\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{3pq}) + \mathcal{N}_{cn}(U_{6p}) + \mathcal{N}_{cn}(U_{6q}) + \mathcal{N}_{cn}(U_6) \\ &= 2 \times 26 + 16 + 16 + 4 = 88. \end{aligned} \tag{5}$$

Similar to Relation (4), if $p = 2$ and $q = 3$, then $\mathcal{N}_{cn}(U_{36}) = 64$. Also, similar to Relation (5), if $p \neq 2$ and $q = 3$, we obtain $\mathcal{N}_{cn}(U_{18p}) = 88$. □

2.4. Group V_{8n}

The group V_{8n} was first introduced for odd n in [7]. For a detailed discussion of the properties of V_{8n} for even n, see [5]. The group V_{8n} has the following presentation:

$$V_{8n} = \langle x, y \mid x^{2n} = y^4 = e,\ xyx = y^{-1},\ xy^{-1}x = y \rangle, \quad n \geqslant 2.$$

In the case $n = 1$, the group V_8 is isomorphic to the dihedral group D_8. According to the presentation of V_{8n}, it follows that $x^i y^2 = y^2 x^i$ and

$$\begin{cases} x^i y = yx^{-i},\ x^i y^3 = y^3 x^{-i}, & i \text{ is even;} \\ x^i y = y^3 x^{-i},\ x^i y^3 = yx^{-i}, & i \text{ is odd.} \end{cases}$$

The maximal normal subgroups of V_{8n} are given in [11] as $M_1 = \langle x, y^2 \rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$, $M_2 = \langle x^2, y^2, y \rangle = \langle x^2 y^2, y \rangle$, and $M_3 = \langle x^2, y^2, xy \rangle$.

In this subsection, we will determine $\mathcal{N}_{cn}(V_{8n})$ by considering two distinct cases according to whether n is odd or even.

Theorem 2.6. *The number of chains of normal subgroups of V_{8n} terminating in V_{8n}, where $n > 1$ is an odd natural number, is given by the following equality:*

$$\mathcal{N}_{cn}(V_{8n}) = 8\mathcal{N}_{cn}(\mathbb{Z}_{2n}) + 2\sum_{\substack{s|n \\ s\neq n}} |\mathcal{C}_s|\,,$$

where $|\mathcal{C}_s|$ is the number of chains of subgroups of $\langle x^{n/s}, y^2\rangle \cap \mathcal{LN}(V_{8n})$ terminating in $\langle x^{n/s}, y^2\rangle$.

Proof. Let $n > 1$ be an odd natural number. The maximal normal subgroups of V_{8n} are then described as follows:

$$M_1 = \langle x, y^2\rangle = \{x^t, x^t y^2 \mid 1 \leqslant t \leqslant 2n\} \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2,$$
$$M_2 = \langle x^2y^2, y\rangle = \{x^{2t}, x^{2t}y, x^{2t}y^2, x^{2t}y^3 \mid 1 \leqslant t \leqslant n\} \cong T_{4n},$$
$$M_3 = \langle x^2, y^2, xy\rangle = \langle x^2y^2, xy\rangle = \{x^{2t}, x^{2t}y^2, x^{2t+1}y, x^{2t+1}y^3 \mid 1 \leqslant t \leqslant n\} \cong D_{4n}.$$

The other normal subgroups of V_{8n} are $H_r = \langle x^{2n/r}\rangle \cong \mathbb{Z}_r$, $K_r = \langle x^{2n/r}y^2\rangle \cong \mathbb{Z}_{2r}$, and $L_s = \langle x^{n/s}, y^2\rangle \cong \mathbb{Z}_{2s} \times \mathbb{Z}_2$, where r, s are divisors of n and $s \neq n$. Obviously, $H_r, K_r \leqslant M_i$ and $L_s \leqslant M_1$, where $1 \leqslant i \leqslant 3$. Since $H_r, K_r \leqslant K_n = \langle x^2y^2\rangle$, hence the normal subgroup lattice of V_{8n} can be written as

$$\mathcal{LN}(V_{8n}) = \mathcal{L}(K_n) \cup \{V_{8n}, M_1, M_2, M_3, L_s;\ \text{where } s \mid n \text{ and } s \neq n\}.$$

Therefore, if $n = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ is the decomposition of n as a product of odd prime factors, then maximal chains of normal subgroups of V_{8n} are one of the following types

$$\{e\} \subset \cdots \subset \langle x^2y^2\rangle \subset M_i \subset V_{8n}, \text{where } 1 \leqslant i \leqslant 3;$$
$$\{e\} \subset \cdots \subset \langle x^{p_j}, y^2\rangle \subset M_1 \subset V_{8n}, \text{where } 1 \leqslant j \leqslant k.$$

Let $\mathcal{C}_0$ be the set of chains of subgroups of $\langle x^2y^2\rangle$ terminating in $\langle x^2y^2\rangle$. Also, assume that for all divisor s of n, $s \neq n$, $\mathcal{C}_s$ is the set of chains of subgroups $P_{s_i} \in \mathcal{LN}(V_{8n})$ of $\langle x^{n/s}, y^2\rangle$ terminating in $\langle x^{n/s}, y^2\rangle$, that is

$$\mathcal{C}_s = \{\cdots \subset P_{s_3} \subset P_{s_2} \subset P_{s_1} = \langle x^{n/s}, y^2\rangle \ \mid\ P_{s_i} \in \mathcal{LN}(V_{8n})\}.$$

In this case, a chain of normal subgroups of V_{8n} terminating in V_{8n} can be constructed in one of the following six ways:

(1) Appending V_{8n} to the end of an arbitrary chain of $\mathcal{C}_0$;

(2) Replacing $\langle x^2y^2\rangle$ with V_{8n} in an arbitrary chain of $\mathcal{C}_0$;

(3) Appending the inclusion $M_i \subset V_{8n}$, that is, both M_i and V_{8n}, to the end of an arbitrary chain of $\mathcal{C}_0$, where $1 \leqslant i \leqslant 3$;

(4) Replacing $\langle x^2y^2\rangle$ with the inclusion $M_i \subset V_{8n}$ in an arbitrary chain of $\mathcal{C}_0$, where $1 \leqslant i \leqslant 3$;

(5) For all divisor s of n, $s \neq n$, appending V_{8n} to the end of an arbitrary chain of $\mathcal{C}_s$;

(6) For all divisor s of n, $s \neq n$, appending the inclusion $M_1 \subset V_{8n}$, that is, both M_1 and V_{8n}, to the end of an arbitrary chain of $\mathcal{C}_s$.

Consequently,

$$\mathcal{N}_{cn}(V_{8n}) = 8\,|\mathcal{C}_0| + 2\sum_{\substack{s|n\\ s\neq n}} |\mathcal{C}_s|\,.$$

Since $|\mathcal{C}_0| = \mathcal{N}_{cn}(\mathbb{Z}_{2n})$, the statement of the theorem is verified. □

The following corollaries present the exact values of $\mathcal{N}_{cn}(V_{8n})$ for $n = p^m$ and $n = pq$, where p and q are odd primes.

Corollary 2.7. *Let p be an odd prime and $m \in \mathbb{N}$. Then the number of chains of normal subgroups of V_{8p^m} terminating in V_{8p^m} is*

$$\mathcal{N}_{cn}(V_{8p^m}) = 2^m(m^2 + 11m + 16).$$

Proof. Assume that $n = p^m$, then by Theorem 2.6,

$$\mathcal{N}_{cn}(V_{8p^m}) = 8\mathcal{N}_{cn}(\mathbb{Z}_{2p^m}) + 2\sum_{s|p^{m-1}} |\mathcal{C}_s| = 2^{m+3}(m+2) + 2\sum_{k=0}^{m-1} \left|\mathcal{C}_{p^k}\right|. \quad (6)$$

Now, we treat to determining the size of the set

$$\mathcal{C}_{p^i} = \{\cdots \subset P_{i_3} \subset P_{i_2} \subset P_{i_1} = \langle x^{p^{m-i}}, y^2\rangle \;\mid\; P_{i_j} \in \mathcal{LN}(V_{8p^m})\},$$

where $0 \leqslant i \leqslant m-1$. Exactly, we determine the size of the set of chains of subgroups $P_{i_j} \in \mathcal{LN}(V_{8n})$ of $\langle x^{p^{m-i}}, y^2\rangle \cong \mathbb{Z}_{2p^i} \times \mathbb{Z}_2$ terminating in $\langle x^{p^{m-i}}, y^2\rangle$. For this purpose, suppose that $\mathcal{D}$ is the set of chains of subgroups $\langle x^{2p^{m-i}}y^2\rangle \cong \mathbb{Z}_{2p^i}$ terminating in $\langle x^{2p^{m-i}}y^2\rangle$, then every chain of $\mathcal{C}_{p^i}$ can be constructed in one of the following three ways:

(1) Appending $\langle x^{p^{m-i}}, y^2\rangle$ to the end of an arbitrary chain of $\mathcal{D}$;

(2) Replacing $\langle x^{2p^{m-i}}y^2\rangle$ with $\langle x^{p^{m-i}}, y^2\rangle$ in an arbitrary chain of $\mathcal{D}$;

(3) For $0 \leqslant j \leqslant i-1$, appending $\langle x^{p^{m-i}}, y^2\rangle$ to the end of an arbitrary chain of $\mathcal{C}_{p^j}$.

Therefore, for $i \geqslant 1$,

$$\begin{aligned}\left|\mathcal{C}_{p^i}\right| &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^i}) + \sum_{j=0}^{i-1}\left|\mathcal{C}_{p^j}\right| \qquad (7)\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^i}) + \left|\mathcal{C}_{p^{i-1}}\right| + \left|\mathcal{C}_{p^{i-2}}\right| + \left|\mathcal{C}_{p^{i-3}}\right| + \cdots + \left|\mathcal{C}_{p^2}\right| + |\mathcal{C}_p| + |\mathcal{C}_1|,\end{aligned}$$

which leads to

$$\begin{aligned}\sum_{k=0}^{m-1}\left|\mathcal{C}_{p^k}\right| &= \left|\mathcal{C}_{p^{m-1}}\right| + \left|\mathcal{C}_{p^{m-2}}\right| + \left|\mathcal{C}_{p^{m-3}}\right| + \cdots + \left|\mathcal{C}_{p^2}\right| + |\mathcal{C}_p| + |\mathcal{C}_1|\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-1}}) + 2\left(\left|\mathcal{C}_{p^{m-2}}\right| + \left|\mathcal{C}_{p^{m-3}}\right| + \cdots + \left|\mathcal{C}_{p^2}\right| + |\mathcal{C}_p| + |\mathcal{C}_1|\right)\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-1}}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-2}})\\ &\quad + 4\left(\left|\mathcal{C}_{p^{m-3}}\right| + \left|\mathcal{C}_{p^{m-4}}\right| + \cdots + \left|\mathcal{C}_{p^2}\right| + |\mathcal{C}_p| + |\mathcal{C}_1|\right)\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-1}}) + 4\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-2}})\\ &\quad + 8\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-3}}) + 8\left(\left|\mathcal{C}_{p^{m-4}}\right| + \cdots + \left|\mathcal{C}_{p^2}\right| + |\mathcal{C}_p| + |\mathcal{C}_1|\right)\\ &= \cdots\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-1}}) + 2^2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-2}}) + 2^3\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-3}}) + \cdots\\ &\qquad + 2^{m-2}\mathcal{N}_{cn}(\mathbb{Z}_{2p^2}) + 2^{m-2}\left(|\mathcal{C}_p| + |\mathcal{C}_1|\right)\\ &= 2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-1}}) + 2^2\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-2}}) + 2^3\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-3}}) + \cdots\\ &\qquad + 2^{m-2}\mathcal{N}_{cn}(\mathbb{Z}_{2p^2}) + 2^{m-1}\mathcal{N}_{cn}(\mathbb{Z}_{2p}) + 2^{m-1}|\mathcal{C}_1|\\ &= 2^{m-1}|\mathcal{C}_1| + \sum_{k=1}^{m-1}2^k\mathcal{N}_{cn}(\mathbb{Z}_{2p^{m-k}}) = 2^{m-1}|\mathcal{C}_1| + 2^m\sum_{k=1}^{m-1}(m-k+2)\\ &= 2^{m-1}|\mathcal{C}_1| + 2^{m-1}(m^2+3m-4).\end{aligned}$$

Obviously, $|\mathcal{C}_1| = 4$ because $\mathcal{C}_1$ consists of the following four chains:

$$\begin{aligned}&\langle x^{p^m}, y^2\rangle \subset V_{8p^m}; \qquad && \langle y^2\rangle \subset \langle x^{p^m}, y^2\rangle \subset V_{8p^m};\\ &\{e\} \subset \langle x^{p^m}, y^2\rangle \subset V_{8p^m}; && \{e\} \subset \langle y^2\rangle \subset \langle x^{p^m}, y^2\rangle \subset V_{8p^m}.\end{aligned}$$

Then, Relation (6) can be rewritten as

$$\mathcal{N}_{cn}(V_{8p^m}) = 2^{m+3}(m+2) + 2^{m+2} + 2^m(m^2+3m-4) = 2^m(m^2+11m+16).$$

□

Corollary 2.8. *Let p and q be odd primes. Then the number of chains of normal subgroups of V_{8pq} terminating in V_{8pq} is* 280.

Proof. . Assume that $n = pq$, where p and q are odd primes. Then, by Theorem 2.6, we deduce that

$$\mathcal{N}_{cn}(V_{8pq}) = 8\mathcal{N}_{cn}(\mathbb{Z}_{2pq}) + 2\sum_{\substack{s|pq \\ s\neq pq}} |\mathcal{C}_s| = 8\mathcal{N}_{cn}(\mathbb{Z}_{2pq}) + 2\left(|\mathcal{C}_1| + |\mathcal{C}_p| + |\mathcal{C}_q|\right),$$

where $\mathcal{C}_1$, $\mathcal{C}_p$, and $\mathcal{C}_q$ are defined as Corollary 2.7. By Relation (7) from the proof of Corollary 2.7, it follows that $|\mathcal{C}_p| = |\mathcal{C}_q| = 16$ and $|\mathcal{C}_1| = 4$. Consequently,

$$\mathcal{N}_{cn}(V_{8pq}) = 8 \times 26 + 2 \times 36 = 280.$$ □

We now turn to the case where n is even. As previously mentioned, the maximal normal subgroups of V_{8n} in the even case can be described analogously to the case of odd n as follows: $M_1 = \langle x, y^2\rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$, $M_2 = \langle x^2, y^2, y\rangle = \langle x^2y^2, y\rangle$, and $M_3 = \langle x^2, y^2, xy\rangle$. However, unlike the odd case, M_2 and M_3 are no longer isomorphic to a dicyclic group and a dihedral group, respectively. This difference causes the normal subgroup lattice to become more complex due to a significant increase in the number of normal subgroups, resulting in more intricate computations. Therefore, we restrict our analysis to two particular cases, $n = 2^m$ and $n = 2p$. Figure 2 illustrates the normal subgroup lattices of $V_{2^{m+3}}$ (for $m = 3$) and V_{16p}.

Theorem 2.9. *The number of chains of normal subgroups of $V_{2^{m+3}}$ terminating in $V_{2^{m+3}}$, that is $\mathcal{N}_{cn}(V_{2^{m+3}})$, where $m \geqslant 1$, is $2^{m+3}(2m+3)$.*

Proof. Normal subgroups of $V_{2^{m+3}}$ are of type $N_1 = \langle x^2y^2, xy\rangle \cong D_{2^{m+1}}$, $N_2 = \langle x^2y^2, xy^3\rangle \cong D_{2^{m+1}}$, $H_i = \langle x^{2^i}\rangle \cong \mathbb{Z}_{2^{m+1-i}}$, $K_i = \langle x^{2^i}y^2\rangle \cong \mathbb{Z}_{2^{m+1-i}}$, $L_i = \langle x^{2^i}, y^2\rangle \cong \mathbb{Z}_{2^{m+1-i}} \times \mathbb{Z}_2$, $H_{m+1} = \{e\}$, and $K_{m+1} = L_{m+1} = \langle y^2\rangle \cong \mathbb{Z}_2$, where $1 \leqslant i \leqslant m$. Obviously, $N_1, N_2 \leqslant M_3$ and $H_i, K_i, L_i \leqslant M_1, M_2, M_3$, where M_1, M_2, M_3 are maximal normal subgroups of $V_{2^{m+3}}$.

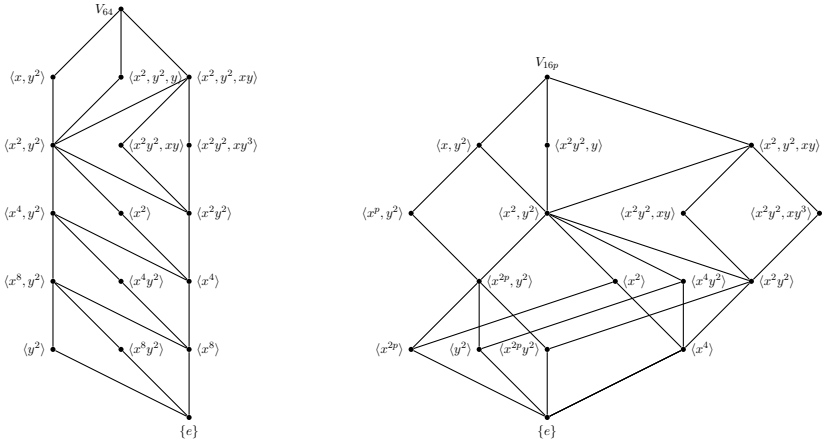


Figure 2: Normal subgroup lattices of V_{32} and V_{16p}.

The maximal chains of normal subgroups of $V_{2^{m+3}}$ are as follows, where $1 \leqslant i \leqslant 3$:

$$\{e\} \subset \cdots \subset \langle x^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{2^{m+3}};$$
$$\{e\} \subset \cdots \subset \langle x^2y^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{2^{m+3}};$$
$$\{e\} \subset \cdots \subset \langle x^4, y^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{2^{m+3}};$$
$$\{e\} \subset \cdots \subset \langle x^2y^2\rangle \subset N_1 = \langle x^2y^2, xy\rangle \subset M_3 = \langle x^2, y^2, xy\rangle \subset V_{2^{m+3}};$$
$$\{e\} \subset \cdots \subset \langle x^2y^2\rangle \subset N_2 = \langle x^2y^2, xy^3\rangle \subset M_3 = \langle x^2, y^2, xy\rangle \subset V_{2^{m+3}}.$$

Let $\mathcal{C}_1$ be the set of chains of subgroups of $\langle x^2y^2\rangle \cong \mathbb{Z}_{2^m}$ terminating in $\langle x^2y^2\rangle$, and for all $2 \leqslant i \leqslant m+1$,

$$\mathcal{C}_i = \{\cdots \subset P_{i_2} \subset P_{i_1} \subset L_i = \langle x^{2^i}, y^2\rangle \ \mid \ P_{i_j} \in \mathcal{LN}(V_{2^{m+3}})\}.$$

Also, assume that

$$\mathcal{A} = \{V_{2^{m+3}};\ M_i \subset V_{2^{m+3}};\ \langle x^2, y^2\rangle \subset V_{2^{m+3}};\ \langle x^2, y^2\rangle \subset M_i \subset V_{2^{m+3}} \ \mid \ 1 \leqslant i \leqslant 3\},$$

and

$$\mathcal{B} = \{N_i \subset V_{2^{m+3}};\ N_i \subset M_3 \subset V_{2^{m+3}} \ \mid \ i = 1, 2\}.$$

In this case, a chain of normal subgroups of $V_{2^{m+3}}$ terminating in $V_{2^{m+3}}$ can be constructed in one of the following five ways:

(1) Appending an arbitrary element of $\mathcal{A}$ to the end of a chain of subgroups of $\langle x^2\rangle \cong \mathbb{Z}_{2^m}$ terminating in $\langle x^2\rangle$;

(2) Appending an arbitrary element of $\mathcal{A}\cup\mathcal{B}$ to the end of an arbitrary chain of $\mathcal{C}_1$;

(3) Putting an arbitrary element of $\mathcal{B}$ instead of $\langle x^2y^2\rangle$ in an arbitrary chain of $\mathcal{C}_1$;

(4) Appending an arbitrary element of $\mathcal{A}$ to the end of an arbitrary chain of $\mathcal{C}_2$;

(5) Putting an arbitrary element of $\mathcal{A}$ instead of $\langle x^4, y^2\rangle$ in an arbitrary chain of $\mathcal{C}_2$.

Therefore,

$$\begin{aligned}\mathcal{N}_{cn}(V_{2^{m+3}}) &= 8\mathcal{N}_{cn}(\mathbb{Z}_{2^m}) + 16\,|\mathcal{C}_1| + 16\,|\mathcal{C}_2| \\ &= 8\mathcal{N}_{cn}(\mathbb{Z}_{2^m}) + 16\mathcal{N}_{cn}(\mathbb{Z}_{2^m}) + 16\,|\mathcal{C}_2| = 24\times 2^m + 16\,|\mathcal{C}_2|\,.\end{aligned} \tag{8}$$

Every chain of $|\mathcal{C}_2|$ is obtained in one of the following ways:

(1) Appending $\langle x^4, y^2\rangle$ to the end of a chain of subgroups of $\langle x^4\rangle \cong \mathbb{Z}_{2^{m-1}}$ terminating in $\langle x^4\rangle$;

(2) Appending $\langle x^4, y^2\rangle$ to the end of a chain of subgroups of $\langle x^4y^2\rangle \cong \mathbb{Z}_{2^{m-1}}$ terminating in $\langle x^4y^2\rangle$;

(3) Appending $\langle x^4, y^2\rangle$ to the end of an arbitrary chain of $\mathcal{C}_3$;

(4) Putting $\langle x^4, y^2\rangle$ instead of $\langle x^8, y^2\rangle$ in an arbitrary chain of $\mathcal{C}_3$.

Thus, $|\mathcal{C}_2| = 2\mathcal{N}_{cn}(\mathbb{Z}_{2^{m-1}}) + 2\,|\mathcal{C}_3| = 2^m + 2\,|\mathcal{C}_3|$. By continuing the above pattern, we find that

$$\begin{aligned}|\mathcal{C}_2| &= 2^m + 2\,|\mathcal{C}_3| = 2^m + 2\left(2^{m-1} + 2\,|\mathcal{C}_4|\right) = 2\times 2^m + 2^2\,|\mathcal{C}_4| \\ &= 2\times 2^m + 2^2\left(2^{m-2} + 2\,|\mathcal{C}_5|\right) = 3\times 2^m + 2^3\,|\mathcal{C}_5| \\ &= \cdots = (m-1)2^m + 2^{m-1}\,|\mathcal{C}_{m+1}| = (m-1)2^m + 2^{m-1}\times 2 = 2^m m.\end{aligned}$$

Substituting the above value into Equation (8) yields the statement of the theorem. □

Theorem 2.10. *Let $n = 2p$, where p is an odd prime. The number of chains of normal subgroups of V_{16p} terminating in V_{16p} is* 368.

Proof. Normal subgroups of V_{16p} are of type $N_1 = \langle x^2y^2, xy\rangle \cong D_{4p}$, $N_2 = \langle x^2y^2, xy^3\rangle \cong D_{4p}$, $K_1 = \langle x^2y^2\rangle \cong \mathbb{Z}_{2p}$, $K_2 = \langle x^4y^2\rangle \cong \mathbb{Z}_{2p}$, $K_p = \langle x^{2p}y^2\rangle \cong \mathbb{Z}_2$, $K_{2p} = \langle y^2\rangle \cong \mathbb{Z}_2$, $H_r = \langle x^{2r}\rangle \cong \mathbb{Z}_{2p/r}$, $L_r = \langle x^{2p/r}, y^2\rangle \cong \mathbb{Z}_{2r} \times \mathbb{Z}_2$ and $H_{2p} = \{e\}$,where $r = 1, 2, p$. Obviously, $N_1, N_2 \leqslant M_3$ and $H_r, K_r, L_r \leqslant M_1, M_2, M_3$, where M_1, M_2, M_3 are maximal normal subgroups of V_{16p}. The maximal chains of normal subgroups of V_{16p} are as follows, where $1 \leqslant i \leqslant 3$:

$\{e\} \subset P \subset \langle x^{2p}, y^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{16p}$; where $P = \langle x^{2p}\rangle, \langle x^{2p}y^2\rangle, \langle y^2\rangle$;
$\{e\} \subset P \subset \langle x^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{16p}$; where $P = \langle x^4\rangle, \langle x^{2p}\rangle$;
$\{e\} \subset P \subset \langle x^2y^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{16p}$; where $P = \langle x^4\rangle, \langle x^{2p}y^2\rangle$;
$\{e\} \subset P \subset \langle x^4y^2\rangle \subset \langle x^2, y^2\rangle \subset M_i \subset V_{16p}$; where $P = \langle x^4\rangle, \langle y^2\rangle$;
$\{e\} \subset P \subset \langle x^2y^2\rangle \subset N_1 = \langle x^2y^2, xy\rangle \subset M_3 \subset V_{16p}$; where $P = \langle x^4\rangle, \langle x^{2p}y^2\rangle$;
$\{e\} \subset P \subset \langle x^2y^2\rangle \subset N_2 = \langle x^2y^2, xy^3\rangle \subset M_3 \subset V_{16p}$; where $P = \langle x^4\rangle, \langle x^{2p}y^2\rangle$;
$\{e\} \subset P \subset \langle x^{2p}, y^2\rangle \subset \langle x^p, y^2\rangle \subset M_1 \subset V_{16p}$; where $P = \langle x^{2p}\rangle, \langle x^{2p}y^2\rangle, \langle y^2\rangle$.

Suppose that

$$\mathcal{A} = \{V_{16p};\ M_i \subset V_{16p};\ \langle x^2, y^2\rangle \subset V_{16p};\ \langle x^2, y^2\rangle \subset M_i \subset V_{16p} \mid 1 \leqslant i \leqslant 3\},$$

$$\mathcal{B} = \{N_i \subset V_{16p};\ N_i \subset M_3 \subset V_{16p} \mid i = 1, 2\},$$

$$\mathcal{D} = \{\langle x^p, y^2\rangle \subset V_{16p};\ \langle x^p, y^2\rangle \subset M_1 \subset V_{16p}\},$$

and

$$\mathcal{C}_0 = \{\cdots \subset P_2 \subset P_1 \subset L_1 = \langle x^{2p}, y^2\rangle \mid P_i \in \mathcal{LN}(V_{16p})\}.$$

The members of $\mathcal{C}_0$ are

$\langle x^{2p}\rangle \subset \langle x^{2p}, y^2\rangle$; $\{e\} \subset \langle x^{2p}\rangle \subset \langle x^{2p}, y^2\rangle$; $\langle x^{2p}y^2\rangle \subset \langle x^{2p}, y^2\rangle$;
$\{e\} \subset \langle x^{2p}y^2\rangle \subset \langle x^{2p}, y^2\rangle$; $\langle y^2\rangle \subset \langle x^{2p}, y^2\rangle$; $\{e\} \subset \langle y^2\rangle \subset \langle x^{2p}, y^2\rangle$;
$\langle x^{2p}, y^2\rangle$; $\{e\} \subset \langle x^{2p}, y^2\rangle$.

Also, assume that $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are the set of chains of subgroups of $\langle x^2\rangle \cong \mathbb{Z}_{2p}$, $\langle x^4y^2\rangle \cong \mathbb{Z}_{2p}$ and $\langle x^2y^2\rangle \cong \mathbb{Z}_{2p}$ terminating in $\langle x^2\rangle$, $\langle x^4y^2\rangle$, $\langle x^2y^2\rangle$, respectively.

In this case, a chain of normal subgroups of V_{16p} terminating in V_{16p} can be constructed in one of the following ways:

(1) Appending an arbitrary element of $\mathcal{A}$ to the end of an arbitrary chain of $\bigcup_{i=0}^{3} \mathcal{C}_i$;

(2) Putting an arbitrary element of $\mathcal{A}$ instead of $\langle x^{2p}, y^2 \rangle$ in an arbitrary chain of $\mathcal{C}_0$;

(3) Appending an arbitrary element of $\mathcal{A}$ to the end of the chains $\langle x^4 \rangle$ and $\{e\} \subset \langle x^4 \rangle$;

(4) Appending an arbitrary element of $\mathcal{B}$ to the end of an arbitrary chain of $\mathcal{C}_3$;

(5) Putting an arbitrary element of $\mathcal{B}$ instead of $\langle x^2 y^2 \rangle$ in an arbitrary chain of $\mathcal{C}_3$;

(6) Appending an arbitrary element of $\mathcal{D}$ to the end of an arbitrary chain of $\mathcal{C}_0$;

(7) Putting an arbitrary element of $\mathcal{D}$ instead of $\langle x^p, y^2 \rangle$ in an arbitrary chain of $\mathcal{C}_0$.

Since, $\mathcal{A} = 8$, $\mathcal{B} = 4$, $\mathcal{D} = 2$, $\mathcal{C}_0 = 8$, and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = \mathcal{N}_{cn}(\mathbb{Z}_{2p}) = 6$ we find that

$$\mathcal{N}_{cn}(V_{16p}) = 8(8+6+6+6)+8(8)+8(2)+4(6)+4(6)+2(8)+2(8) = 368. \quad \square$$

Conclusion

In this study, we investigated the number of chains of normal subgroups that terminate at the group itself in several families of finite groups, namely the semi-dihedral group SD_{8n}, U_{6n}, V_{8n}, and $H(n)$. By applying recursive techniques and combinatorial reasoning, we obtained explicit formulas for counting such chains in each case. These results lead to a refined understanding of the normal subgroup lattice structure for these groups. A brief summary of the main results is presented in Table 1.

It should be noted that, for each of these groups, the number of such normal subgroup chains terminating in the group coincides with the number of distinct fuzzy normal subgroups under the natural equivalence relation $\sim$. For the family V_{8n}, our analysis in the even case was restricted to the two particular instances $n = 2^m$ and $n = 2p$, where p is an odd prime, while the general case of even n remains an open problem for future research.

Table 2: Values of $\mathcal{N}_{cn}(G)$ for Certain Groups

Group G	$\mathcal{N}_{cn}(G)$	Reference
$H(n)$, $n \geqslant 5$	$2^{n-1}(n^2-n-2)$	Theorem 2.1
$SD_{2^{m+3}}$	2^{m+4}	Theorem 2.2
SD_{8p^m}, $p \neq 2$	$2^m(m^2+11m+24)$	Theorem 2.2
$SD_{2^{m+3}p}$, $p \neq 2$	$2^{m+3}(2m+7)$	Theorem 2.2
$U_{6\cdot 2^m}$	$2^{m+1}(m+2)$	Corollary 2.4
U_{6p^m}, $p \neq 2$	$2^{m-1}(m^2+7m+8)$	Corollary 2.4
U_{12p}, $p \neq 2$	64	Corollary 2.5
U_{6pq}, $p, q \neq 2$	88	Corollary 2.5
V_{8p^m}, $p \neq 2$	$2^m(m^2+11m+16)$	Corollary 2.7
V_{8pq}, $p, q \neq 2$	280	Corollary 2.8
$V_{2^{m+3}}$, $m \geqslant 1$	$2^{m+3}(2m+3)$	Theorem 2.9
V_{16p}, $p \neq 2$	368	Theorem 2.10

Moreover, the methods introduced in this work may be extended to other finite groups whose normal subgroup lattice is well known. Further investigations may also consider related enumeration problems involving fuzzy subgroups or chains that remain invariant under group automorphisms.

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