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Characterization of the finite simple groups $L_7(2)$, $U_7(2)$, and ${}^2D_{p+1}(2)$ by their order and the largest element order

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Abstract. Following the classification of finite simple groups, one of the central problems in group theory is to characterize groups by using specific properties. Such properties include the set of element orders, the number of elements of the same order, the largest element order, the second-largest element order, conjugacy class sizes, and the average element order, among others.

In this paper, we show that the finite simple groups $L_7(2)$, $U_7(2)$, and ${}^2D_{p+1}(2)$, where $2^{p+1}+1$ is a prime number, can be uniquely determined (up to isomorphism) by their order and the largest element order. In particular, if G is a finite group such that $|G| = |M|$ and $k(G) = k(M)$ for $M \in \{L_7(2), U_7(2), {}^2D_{p+1}(2)\}$, then $G \cong M$.

1. Introduction

Throughout this paper, G denotes a finite group. The set of prime divisors of $|G|$ is denoted by $\pi(G)$, and the set of element orders in G is denoted by $\pi_e(G)$. The largest order of elements in G is denoted by $k(G)$. Moreover, a Sylow p-subgroup of G is denoted by G_p.

The *prime graph* $\Gamma(G)$ of the group G is defined as a graph whose vertex set is $\pi(G)$. Two distinct vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. We assume that $\Gamma(G)$ has $t(G)$ connected components, denoted by π_i for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is even, we always assume that $2 \in \pi_1$.

One of the important problems in group theory is the *characterization of groups by given properties*, that is, determining whether there exists only one group (up to isomorphism) possessing certain specified properties. There are

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various types of group characterizations for instance, by the set of element orders, by the prime graph, or by the number of elements of the same order.

One method of group characterization is by the *order of the group and the largest element order*. In other words, we say that a group G is characterizable by its order and the largest element order if, for any group H with $|H| = |G|$ and $k(H) = k(G)$, it follows that $G \cong H$.

Recently, He and Chen studied the characterization of groups by the largest element order. In [7], they proved that the groups $PSL_2(q)$ for $q < 125$ are characterizable by their order and the largest and second-largest element orders. Later, the same authors showed that $PSL_3(q)$ and $PSU_3(q)$ (for some values of q) are characterizable by their order and the first, second, and third largest element orders [8].

Furthermore, in works such as [3], [9], [18], [10], [11], [12], and [13], it has been shown that several groups such as $PGL(2, q)$, the Suzuki groups $Sz(q)$ where $q - 1$ or $q \pm \sqrt{2q} + 1$ is prime, $PSU_3(3^n)$, $PSp(8, q)$, and the simple groups $C_4(q)$, $PSL(5, 2)$, and $PSL(4, 5)$ can be characterized by their largest element order.

In this paper, we prove that the simple groups $L_7(2)$, $U_7(2)$, and ${}^2D_{p+1}(2)$ (where $2^{p+1} + 1$ is a prime number and p is an odd prime) are characterizable by their order and the largest element order. In fact, we prove the following main theorem:

Main Theorem. *Let G be a group and $M \in \{L_7(2), U_7(2), {}^2D_{p+1}(2)\}$ with $|G| = |M|$ and $k(G) = k(M)$. Then $G \cong M$.*

More precisely, we have the following theorems.

Theorem A. *Let G be a group with $|G| = |{}^2D_{p+1}(2)|$, where $2^{p+1} + 1$ is a prime number and $k(G) = k({}^2D_{p+1}(2))$. Then $G \cong {}^2D_{p+1}(2)$.*

Theorem B. *Let G be a group and $M \in \{L_7(2), U_7(2)\}$ such that $|G| = |M|$ and $k(G) = k(M)$. Then $G \cong M$.*

It is also important to mention here that no two of the nonabelian simple groups considered here have the same size.

We first prove Theorem A, and along the way, we show that Theorem B can also be established.

2. Notation and preliminaries

In this section, we present several useful lemmas which will be used in the proof of the main Theorem.

Lemma 2.1. [6] *Let G be a Frobenius group of even order with kernel K and complement H. Then:*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3. [2] *Let G be a 2-Frobenius group of even order. Then:*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.4. [20] *Let G be a finite group with $t(G) \geqslant 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Lemma 2.5. [14] *Let G be a simple K_3-group. Then G is isomorphic to one of the following groups: $A_5, A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), \mathrm{PSL}_2(17), \mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, or $\mathrm{PSU}_4(2)$.*

Lemma 2.6. [1] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

1. $\mathrm{PSL}_2(q)$ *where* q *is a prime power satisfying*

$$q(q^2-1) = (2, q-1)2^a \cdot 3^b \cdot p^c \cdot r^d$$

with $p, r > 3$ *distinct primes and* a, b, c, d *natural numbers;*

2. A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $\mathrm{PSL}_2(16)$, $\mathrm{PSL}_2(25)$, $\mathrm{PSL}_2(49)$, $\mathrm{PSL}_2(81)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(5)$, $\mathrm{PSL}_3(7)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSL}_3(17)$, $\mathrm{PSL}_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $\mathrm{PSU}_3(4)$, $\mathrm{PSU}_3(5)$, $\mathrm{PSU}_3(7)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_3(9)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_5(2)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, ${}^3D_4(2)$, ${}^2F_4(2)$.

Lemma 2.7. [15] *Let* G *be a simple* K_5*-group. Then* G *is isomorphic to one of the following groups:*

1. $O_5(q)$ *where* q *satisfies* $|\pi(q^4-1)| = 4$;
2. $\mathrm{PSL}_2(q)$ *where* q *satisfies* $|\pi(q^2-1)| = 4$;
3. $\mathrm{PSL}_3(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3-1))| = 4$;
4. $\mathrm{PSU}_3(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3+1))| = 4$;
5. $\mathrm{Sz}(q)$ *where* $q = 2^{2m+1} \geqslant 8$ *satisfies* $|\pi((q-1)(q+1))| = 4$;
6. $\mathrm{R}(q)$ *where* q *is an odd power of* 3 *and*

$$|\pi(q^2-1)| = 3 \quad \textit{and} \quad |\pi(q^2-q+1)| = 1;$$

7. $A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, \mathrm{PSL}_4(4), \mathrm{PSL}_4(5), \mathrm{PSL}_4(7)$, $\mathrm{PSL}_5(2)$, $\mathrm{PSL}_5(3)$, $\mathrm{PSL}_6(2)$, $O_7(3)$, $O_9(2)$, $\mathrm{PSp}_6(3)$, $\mathrm{PSp}_8(2)$, $\mathrm{PSU}_4(4)$, $\mathrm{PSU}_4(5)$, $\mathrm{PSU}_4(7)$, $\mathrm{PSU}_4(9), \mathrm{PSU}_5(3), \mathrm{PSU}_6(2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5), G_2(7), G_2(9)$.

Lemma 2.8. [15] *Let* G *be a simple* K_6*-group. Then* G *is isomorphic to one of the following groups:*

1. $O_5(q)$ *where* q *satisfies* $|\pi(q^4-1)| = 5$;
2. $\mathrm{PSL}_2(q)$ *where* q *satisfies* $|\pi(q^2-1)| = 5$;
3. $\mathrm{PSL}_3(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3-1))| = 5$;
4. $\mathrm{PSU}_3(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3+1))| = 5$;

5. $\mathrm{PSL}_4(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3-1)(q^4-1))| = 5$;
6. $\mathrm{Sz}(q)$ *where* $q = 2^{2m+1}$ *satisfies* $|\pi((q-1)(q^2+1))| = 5$;
7. $G_2(q)$ *where* q *satisfies* $|\pi(q^6-1)| = 5$;
8. $\mathrm{PSU}_4(q)$ *where* q *satisfies* $|\pi((q^2-1)(q^3+1)(q^4-1))| = 5$;
9. $\mathrm{R}(q)$ *where* q *is an odd power of* 3 *and*

$$|\pi(q^2-1)| = 5, \quad |\pi(q^3+1)| = 1;$$

10. $A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Ru, Co_2, Co_3, \mathrm{Fi}_{22}, HN, \mathrm{PSL}_5(7)$, $\mathrm{PSL}_6(3)$, $\mathrm{PSL}_7(2)$, $O_7(4)$, $O_7(5)$, $O_7(7), O_9(3), \mathrm{PSp}_6(4), \mathrm{PSp}_6(5), \mathrm{PSp}_6(7)$, $\mathrm{PSp}_8(3)$, $\mathrm{PSU}_5(4)$, $\mathrm{PSU}_5(5)$, $\mathrm{PSU}_5(9)$, $\mathrm{PSU}_6(3)$, $\mathrm{PSU}_7(2)$, $F_4(2)$, $O_8^+(4)$, $O_8^+(5)$, $O_8^+(7)$, $O_{10}^+(2)$, $O_8^-(3)$, $O_{10}^-(2), {}^3D_4(4)$.

3. Proof of Theorem A

In this section, we prove theorem A. From now on, we denote the groups ${}^2D_{P+1}(2)$ by D. In the way, also we denote $2^{p+1}+1$ by r . We recall that G is a group with $|G| = |D| = 2^{p(p+1)}(2^{p+1}+1)\prod_{i=1}^{p}(2^{2i}-1)$ and $k(G) = k(D) = 2^{p+2}-1$. To prove of the main theorem, we prove the following cases.

Lemma 3.1. *r is an isolated vertex of $\Gamma(G)$.*

Proof. We prove by contradiction that r is an isolated vertex. Suppose r is not an isolated vertex. Then there is prime number $t \in \pi(G) - r$, so that $tr \in \pi_e(G)$. As a result we have $tr \geqslant 2r = 2(2^{p+1}+1) > (2^{p+2}-1)$, as a result $k(G) > (2^{p+2}-1)$, which is a contradiction. Hence $t(G) \geqslant 2$. So Lemma 2.4 implies that G satisfies one of the following cases:

Case 1. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since r is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/r$ and $|K| = r$, or (ii) $|H| = r$ and $|K| = |G|/r$. Assume that $|H| = |G|/r$ and $|K| = r$. Then Lemma 2.1 implies that $|G|/r$ divides $r-1$, which is imposible. So the case $|H| = r$ and $|K| = |G|/r$ can occur. Lemma 2.1 implies that r divides $\frac{|G|}{r} - 1$, in the other words $2^{p+1}+1 \mid$

$\frac{2^{p(p+1)}(2^{p+1}+1)\prod_{i=1}^{p}(2^{2i}-1))}{2^{p+1}+1} - 1$. Therefore $2^{p+1}+1 \mid 2^{p(p+1)}\prod_{i=1}^{p}(2^{2i}-1)-1$. As a result $2^{p+1}+1$ must be divided $2^{p(p+1)}$, that is a contradiction. Therefore, we deduce that G is not a Frobenius group.

Case 2. Let G be a 2-Frobenius group. Then by Lemma 2.3, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Since r is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{r\}$ and $|K/H| = r$. Hence $|\mathrm{Aut}(K/H)| = r - 1$. As $|G/K|$ divides $|\mathrm{Aut}(K/H)|$, we deduce that $|G/K||2^{p+1}$. Now since $(2^{p+1}, t) = 1$, where t be prime divisor of $2^{p+1}+1$ so $t \mid |H|$. Now $K/H \rtimes H_t$ be a Frobenius group by kernel H_t, so $2^{p+1}+1 \mid |H_t|-1$. As a result $2^{p+1}+1 \mid 2^{p+1}$, which is a contradiction. So G is not 2-Frobenius group.

Case 3. By Lemma 2.4, and Cases 1,2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathrm{Out}(K/H)|$. Since r is an isolated vertex of $\Gamma(G)$. In conclude according to the classification of the finite simple groups we know that the possibilities for K/H are alternating group A_m, $m \geqslant 5$, 26 sporadic groups, simple groups Lie type. First, since K/H is a non-abelian simple group, so K/H is isomorphic one of the following groups, for this purpose, first we suppose K/H be isomorphic alternating groups, in the other words, we have the following isomorphic:

Step 1. Let $K/H \cong A_m$, where $m \geqslant 5$ and $m = r'$,$r'+1$, $r'+2$. Then by [16] $k(A_m) = m$. So $m \geqslant q^{2^{p+1}+1}$. Since $2^{p+1}+1 \in \pi(K/H)$, so we suppose z be a prime number between 2^p+1 and $2^{p+1}+1$, then we have $z \in \pi(A_m) - \pi(G)$, which is a contradiction.

Step 2. If K/H be isomorphic sporadic groups, then by [16], $k(S) = \{11, 13, 17, 19, 23, 29, 31, 43, 47, 67, 71\}$, where S be a sporadic groups and also this number are the largest elements order of sporadic groups. Since the odd order components of a sporadic simple group are prime less than 71, it follows that so we consider $2^{p+1}+1 < 71$, as a result $p = 3, 5$. But $|S| \nmid |^2D_{p+1}(2)|$, which is a contradiction.

Step 3. In this case, we consider K/H is isomorphic to a group of Lie-type.

3.1. If $K/H \cong {}^3D_4(q')$, then by [16], $k({}^3D_4(q')) = (q'^3-1)(q'+1)$. Also we know $|{}^3D_4(q')| \mid |G|$. Now we consider $2^{p+2}-1 = (q'^3-1)(q'+1)$, then

$2^p = \frac{(q'^3-1)(q'+1)}{4} < (q'^3-1)(q'+1$. Hence $2^p < q'^4$. Thus $q'^{12} \geqslant 2^{3p}$. On the other hand , we know $q'^{12} \mid |G|$. But $2^{3p} \nmid |G|$, which is a contradiction.

3.2. $K/H \cong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For example if $K/H \cong E_8(q')$, then by[16] $k(E_8(q') = (q'+1)(q'^2+q'+1)(q'^5-1)$. On the other hand, $|E_8(q')| = q'^{120}(q'^{30}-1)(q'^{24}-1)(q'^{20}-1)(q'^{18}-1)(q'^{14}-1)(q'^{12}-1)(q'^8-1)(q'^2-1) \mid |G|$. Hence, we consider $(2^{p+2}-1) = (q'+1)(q'^2+q'+1)(q'^5-1)$, as a result $2^p = \frac{q'^8+2q'^7+2q'^6+q'^5-q'^3-2q'^2-2q'}{4} < q'^8+2q'^7+2q'^6+q'^5-q'^3-2q'^2-2q$. Hence $2^p < q'^8$. Thus $q'^{120} > 2^{15p}$. On the other hand, we know $q'^{120} \mid |G|$. But $2^{15p} \nmid |G|$, which is a contradiction.

For $K/H \ncong E_6(q')$,$E_7(q')$, $F_4(q')$,we have a contradiction, similarily.

3.3. If $K/H \cong^2 E_6(q')$, then by [16] $k(^2E_6(q') = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. On the other hand, $|^2E_6(q')| = \frac{q'^{36}(q'^2-1)(q'^5+1)(q'^6-1)}{(3,q'+1)} \mid |G|$. Now we consider $(2^{p+2}-1) = \frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. First if $(3, q'+1) = 1$, then $2^{p+2}-1 = (q'+1)(q'^2+1)(q'^3-1)$. As a result $2^p = \frac{q'^6+q'^5+q'^4-q'^2-q'}{4} < q'^6+q'^5+q'^4-q'^2-q$. Hence $2^p < q'^6$. Thus $q'^{36} > 2^{6p}$. On the other hand , we know $q'^{36} \mid |G|$. But $2^{6p} \nmid |G|$, which is a contradiction.

3.4. If $K/H \cong {}^2G_2(3^{2m+1})$, where $m \geqslant 1$ then by [16], $k({}^2G_2(3^{2m+1}))) = 3^{2m+1} + 3^{m+1} + 1$. Now we consider $2^{p+2} - 1 = 3^{2m+1} + 3^{m+1} + 1$, so $2^{p+2} - 2 = 3^{m+1}(3^m + 1)$. As a result $2(2^{p+1} - 1) = 3^{m+1}(3^m + 1)$. Since $(3^{m+1} + 1, 3^m + 1) = 1$, so $2 = 3^m + 1$ and $2^{p+1} - 1 = 3^{m+1}$. First if $2 = 3^m + 1$, then we deduce $m = 0$,which is a contradiction. Now if $2^{p+1} - 1 = 3^{m+1}$, then we deduce $m = 1$, $p = 2$. Since $|^2G_2(27| \nmid |^2D_3(2)|$, which is a contradiction.

3.5. If $K/H \cong {}^2F_4(q')$, where $q'^{2m+1} > 2$ then by[16], $k({}^2F_4(q'^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. So we consider $2^{p+2} - 1 = 2^{4m+2)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. As a result, $2^{p+2} - 2 = 2^{4m+2)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1}$. So $2(2^{p+1} - 1) = 2^{m+1}(2^{3m+1} + 2^{2m+1} + 2^m + 1)$. Hence $2 = 2^{m+1}$ and $2^{p+1} - 1 = 2^{3m+1} + 2^{2m+1} + 2^m + 1$. First if $2 = 2^{m+1}$, then $m = 0$, which is acontradiction. The other case is impossible.

3.6. If $K/H \cong {}^2B_2(2^{2m+1})$, where $m \geqslant 1$ then by [16], $k({}^2B_2(2^{2m+1}))) = 2^{2m+1} + 2^{m+1} + 1$, also $|^2B_2(2^{2m+1})| \mid |G|$. For this purpose, we consider $2^{p+2} - 1 = 2^{2m+1} + 2^{m+1} + 1$. As a result $2^{p+2} - 2 = 2^{m+1}(2^m + 1)$, so $2(2^{p+1} = 2^{m+1}(2^m + 1)$. It follows $2 = 2^m + 1$ and $2^{p+1} = 2^{m+1}$. First if $2 = 2^m + 1$, then $m = 0$, which is a contradiction. The other case is impossible.

3.7. If $K/H \cong G_2(q')$, then by [16], $k(G_2(q') = q'^2 + q' + 1$ and also $|G_2(q')| = q'^6(q'^6-1)(q'^2-1) \mid |G|$. For this purpose, we consider $2^{p+2}-1 = q'^2 + q' + 1$. As a result, $2^{p+2} - 2 = q'(q' + 1)$. As a result $2(2^{p+1} - 1) = q'(q'+1)$, so $q' = 2$ and $2^{p+1} - 1 = q' + 1$. Hence $4 = 2^{p+1}$, we deduce $p = 1$, which is a contradiction.

3.8. If $K/H \cong {}^2A_n(q')$, where $n > 1$. Then by [16], $k({}^2A_n(q')) = \frac{q'^{2n}-1}{(n+1,q'+1)}$. Also we know $|{}^2A_n(q')| \mid |G|$. Hence $\frac{1}{(n+1,q'+1)}q'^{n(n+1)/2}\prod_{i=2}^{n+1}(q'^i - (-1)^i$ divides $|G|$. For this purpose, we consider $2^{p+2} - 1) = \frac{q'^{2n}-1}{(n+1,q'+1)} < q'^{2n} - 1$. As a result, $2^{p+2} < q'^{2n}$. Since $q'^{2n} \mid |G|$, but $2^{p+2} \nmid |G|$, which is a contradiction.

3.9. If $K/H \cong D_n(q')$, $C_n(q')$ where $n \geqslant 4$, $n \geqslant 2$, respectively. Then we have a contradiction proof 3.8. similarily.

3.10. If $K/H \cong L_{n+1}(q')$, where $n \geqslant 1$. For this purpose, first we assume $n = 1$, so we have $K/H \cong L_2(q')$. For this purpose, by [16], $k(L_2(q') = q' + 1, q'$, where q' be even, odd respectively. Now we consider $2^{p+2} - 1) = q', q' + 1$. Since $|L_2(q')| \nmid |G|$, which is a contradiction. For $n > 1$, $K/H \ncong L_{n+1}(q')$, similarily.

Hence $K/H \cong D$. It follows that $|K/H| = |D|$. On the other hand, we know that $H \trianglelefteq K \trianglelefteq G$, where r is an isolated vertex of $\Gamma(G)$. Also by assumtion we have $k(K/H) \mid k(G)$. Hence $2^{p+2} - 1 = 2^{p'+2} - 1$. As a result $p = p'$. Now since $|K/H| = |D|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong D$. □

4. Proof of Theorem B

In this section, we prove theorem B. For this purpose, we state two lemmas whose proofs will yield the main result. Let G be a group such that $|G| = |S|$ and $k(G) = k(S)$, where S is the simple group under consideration.

Lemma A. *If G is a group and $S = L_7(2)$ such that $|G| = |S|$ and $k(G) = k(S)$, then $G \cong S$.*

Lemma B. *If G is a group and $S = U_7(2)$ such that $|G| = |S|$ and $k(G) = k(S)$, then $G \cong S$.*

Proof of Lemma A. First, note that $|L_7(2)| = 2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$, and $k(L_7(2)) = 127$. Hence, $|G| = 2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$, $k(G) = 127$. Therefore, 127 is an isolated vertex in the graph $\Gamma(G)$, which implies $t(G) \geqslant 2$.

By Lemma 2.4, the group G falls into one of the following cases:

Case 1. Suppose G is a Frobenius group with kernel K and complement H. According to Lemma 2.1, $t(G) = 2$, and $\pi(H)$ and $\pi(K)$ are the sets of vertices of the connected components of the graph $\Gamma(G)$. Since 127 is an isolated vertices, we have:

$$(i) \quad |H| = \frac{|G|}{127}, \quad |K| = 127, \quad \text{or} \quad (ii) \quad |H| = 127, \quad |K| = \frac{|G|}{127}.$$

Assume the first case holds. By Lemma 2.1, $\frac{|G|}{127} \mid 127 - 1$, which is a contradiction. However, again by Lemma 2.1, $127 \mid \frac{|G|}{127} - 1$, which is also a contradiction. Thus, G cannot be a Frobenius group.

Case 2. Suppose G is a 2-Frobenius group. According to Lemma 2.3, there exists a normal series

$$1 \triangleleft H \triangleleft K \triangleleft G,$$

such that G/H and K are Frobenius groups with respective kernels and complements. Moreover, $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$, and $|G/K| \mid |\mathrm{Aut}(K/H)|$.

Since 127 is an isolated vertex, it follows that $\pi_2 = \{127\}$, $|K/H| = 127$. Therefore, $|\mathrm{Aut}(K/H)| = 126$. Since $|G/K| \mid |\mathrm{Aut}(K/H)|$, we get $|G/K| \mid 30$, and $|H| = \frac{|G|}{|G/K|} \mid 2^{20} \cdot 3^4 \cdot 5 \cdot 7 \cdot 31$.

Because H is a nilpotent group, H must decompose into Sylow subgroups corresponding to these primes. Thus G must be have the element of order 2.3.5.7.31. This contradicts the condition $k(G) = 127$. Hence this case is also impossible.

Case 3. Let G has a normal series

$$1 \triangleleft H \triangleleft K \triangleleft G,$$

where H and G/K are π_1-groups, K/H is a non-abelian simple group, H is nilpotent, and $|G/K| \mid |\mathrm{Out}(K/H)|$.

Since 127 is an isolated vertex, $127 \mid |K/H|$. Also, $|\pi(G)| = 6$, so K/H is a simple group K_n with $n < 7$.

Step 1. Assume K/H is a simple group K_3. Since $31 \notin \pi(K/H)$, by Lemma 2.5 we get a contradiction.

Step 2. If K/H is $K_4 - group$, then by Lemma 2.6 $K/H \cong PSL_2(q)$ with $k(K/H) = q$ or $q + 1$, where q is a prime power greater than 3. Because 127 divides $|K/H|$, q must be 126 or 127, both leading to contradictions.

Step 3. If K/H is $K_5-group$, then by Lemma 2.7 K/H is one of the groups $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$, $O_5(q)$, $R(q)$, $Sz(q)$. Similar analysis shows none of these can have 127 as an isolated vertex.

Step 4. If K/H is $K_6 - group$, then by Lemma 2.8 K/H is one of the groups $PSL_2(q)$, $PSL_3(q)$, $PSL_4(q)$, $PSU_3(q)$, $PSU_4(q)$, $O_5(q)$, $G_2(q)$, $R(q)$, $Sz(q)$, $L_7(2)$. Checking each case as before leads to contradictions. Therefore, the only possible case is that $G \cong S$, which completes the proof of Lemma A. □

Proof of Lemma B. Let G be a group and $S = U_7(2)$, with $|G| = |S|$ and $k(G) = k(S)$. Then $G \cong S$.

First, since $|U_7(2)| = 2^{21} \cdot 3^7 \cdot 5 \cdot 7 \cdot 31 \cdot 43$ and $k(U_7(2)) = 66$, we conclude that $|G| = 2^{21} \cdot 3^7 \cdot 5 \cdot 7 \cdot 31 \cdot 43$, $k(G) = 66$. Thus 43 is an isolated vertex of $\Gamma(G)$ and hence $t(G) \geqslant 2$. So Lemma 2.4 implies that G satisfies one of the following cases:

Case 1. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$, and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since 43 is an isolated vertex of $\Gamma(G)$, we have either:

(i) $|H| = \frac{|G|}{43}$ and $|K| = 43$, or

(ii) $|H| = 43$ and $|K| = \frac{|G|}{43}$.

Assume that $|H| = \frac{|G|}{43}$ and $|K| = 43$. Then Lemma 2.1 implies that $\frac{|G|}{43}$ divides $43-1$, which is a contradiction. So the case $|H| = 43$ and $|K| = \frac{|G|}{43}$ is left. Lemma 2.1 implies that 43 divides $\frac{|G|}{43} - 1$, which is a contradiction. Therefore, G is not a Frobenius group.

Case 2. Let G be a 2-Frobenius group. Then by Lemma 2.3, there exists a normal series

$$1 \triangleleft H \triangleleft K \triangleleft G,$$

such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Also, $t(G) = 2$,

$$\pi(G/K) \cup \pi(H) = \pi_1, \quad \pi(K/H) = \pi_2,$$

and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Since 43 is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{43\}$ and $|K/H| = 43$. Hence $|\mathrm{Aut}(K/H)| = 42$.

As $|G/K|$ divides $|\mathrm{Aut}(K/H)|$, we have $|G/K| \mid 42$, so $|H| = \frac{|G|}{|G/K|} \geqslant 2^{21} \cdot 3^7 \cdot 5 \cdot 11$. Since H is nilpotent, it must have an element of order $2 \cdot 3 \cdot 5 \cdot 11$, which contradicts $k(G) = 66$. Thus Case 2 cannot occur.

Case 3. Suppose G has a normal series

$$1 \lhd H \lhd K \lhd G,$$

such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is nilpotent, and $|G/K|$ divides $|\mathrm{Out}(K/H)|$. Since 43 is an isolated vertex of $\Gamma(G)$, we deduce $43 \mid |K/H|$. On the other hand, $|\pi(G)| = 6$, thus K/H is a simple K_n-group where $n < 7$. Now we analyze possible cases:

Step 1. Let K/H be a simple K_3-group. Since $43 \mid |K/H|$, K/H is a K_3-group, which is a contradiction by Lemma 2.5.

Step 2. Let K/H be a simple K_4-group. Since $43 \mid |K/H|$, by Lemma 2.6 $K/H \cong PSL_2(q)$ where $q(q^2-1) = 2^a \cdot 3^b \cdot p^c$, $a, b, c, d \geqslant q1$, and p, r are primes larger than 3, or $K/H \cong U_3(7)$. Then $k(K/H) = q+1$ or q.

Since 43 divides $|K/H|$, we consider values of q such that $43 \mid q$ or $43 \mid q+1$ and $k(K/H) \leq 66$. Possible q are $42, 43, 47, 49, 53, 59, 61$, all leading to contradictions. If $K/H \cong U_3(7)$, since $|U_3(7)| \nmid |K/H|$, this also is a contradiction.

Step 3. Let K/H be a simple K_5-group. Since $43 \mid |K/H|$, by Lemma 2.7 possible groups are $U_4(7)$, $G_2(7)$, $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$, $O_5(q)$, $R(q)$, $Sz(q)$.

(i) If $K/H \cong PSL_2(q)$, by [10], $k(K/H) = q+1$ or q. Considering 43 divides $|K/H|$, we have a contradiction similar to Step 2.

(ii) If $K/H \cong PSL_3(q)$, by [10], $k(K/H) = q^2+q+1$ or $\frac{q^2+q+1}{3}$. Considering $43 \leq k(K/H) \leq 66$, $q = 7, 11, 13$, contradicting $|PSL_3(q)| \mid |K/H|$.

(iii) If $K/H \cong PSU_3(q)$, by [10], $k(K/H) = q^2+q$ or $\frac{q^2+q}{3}$. Now, assume $q^2+q = 66$ so $q^2+q-66 = 0$ where leads to contradictions. Now, if $\frac{q^2+q}{3} = 66$ then $q^2+q-198 = 0$, where this equation has not any solution.

(iv) If $K/H \cong Sz(q)$ where $q = 2^{2m+1} > 2$, by [10], $k(K/H) = q+\sqrt{2q}+1$. Considering divisibility by 43 yields no valid m. In the way, we have $q+\sqrt{2q}+1 = 66$, where this equation has not any solution.

(v) If $K/H \cong O_5(q)$, by [10], $k(K/H) = \frac{q^2+1}{2}$ or q^2+q. Possible $q = 7, 11$ contradicts $|O_5(q)| \mid |K/H|$.

(vi) If $K/H \cong R(q)$, where $q = 3^{2m+1}$, by [10], $k(K/H) = q+\sqrt{3q}+1$. No m satisfies divisibility by 43. In other words, we consider $q+\sqrt{3q}+1 = 66$ as $3^{2m+1} + 3^{m+1} + 1 = 66$ it follows that $3^{2m+1} + 3^{m+1} - 65 = 0$, where this equation has not any solution.

Step 4. Let K/H be a simple K_6-group. Since $43 \mid |K/H|$, by Lemma 2.8 possible groups are $PSL_2(q)$, $PSL_3(q)$, $PSL_4(q)$, $PSU_3(q)$, $PSU_4(q)$, $O_5(q)$, $G_2(q)$, $Sz(q)$, $R(q)$.

By previous steps, K/H is not isomorphic to $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$, $Sz(q)$, $R(q)$, $O_5(q)$. Also,

(i) $K/H \cong PSL_4(q)$: $k(K/H) = \frac{q^4-1}{q-1}$ or $\frac{q^3+q^2+q+1}{4}$. Considering 43 divisibility, no solution exists. In the way, we consider $\frac{q^4-1}{q-1} = 66$ so $q^3 + q^2 + q - 65 = 0$. As a result, this equation has not any solution in natural number. Now, if $\frac{q^3+q^2+q+1}{4} = 66$ it follows that $q^3 + q^2 + q - 263 = 0$, where this equation has not any solution in natural number.

(ii) $K/H \cong PSU_4(q)$: $k(K/H) = q^3 + 1$ or $\frac{q^3+1}{4}$. For this purpose, we consider $q^3 + 1 = 66$ so $q^3 - 65 = 0$ has not any solution. Now, if $\frac{q^3+1}{4} = 66$ then $q^3 - 263 = 0$ has not any solution in natural number.

(iii) $K/H \cong G_2(q)$: $k(K/H) = q^2+q+1$. Now, we consider $q^2+q+1 = 66$ so $q^2 + q - 65 = 0$ has not any solution in natural number.

(iv) $K/H \cong PSL_3(q)$: $k(K/H) = \frac{q^3-1}{(q-1)(3,q-1)}$. Now, we consider $q^3 - 1 = 66$ so $q^3 - 65 = 0$ has not any solution in natural number. Now, if $\frac{q^2+q+1}{3} = 66$ then $q^2 + q - 197 = 0$ has not any solution in natural number. For $K/H \ncong PSL_2(q)$, we have similar proof.

(v) $K/H \cong PSU_3(q)$: $k(K/H) = \frac{q^2-1}{(3,q+1)}$. Now, we consider $\frac{q^2-1}{(3,q+1)} = 66$ so we can see easily $q^2-199 = 0$ and $q^2-67 = 0$ have not any solution in natural number.

(vi) $K/H \cong R(q)$: $k(K/H) = q + \sqrt{3q} + 1$, $q = 3^{2m+1} > 3$. Now, we consider $q+\sqrt{3q}+1 = 66$ so we can see easily $3^{2m+1} + 3^{m+1} - 65 = 0$ has not any solution in natural number.

(vii) $K/H \cong Sz(q)$: $k(K/H) = q + \sqrt{2q} + 1$, $q = 2^{2m+1}$. Now, we consider $q + \sqrt{2q} + 1 = 66$ so we can see easily $2^{2m+1} + 2^{m+1} - 65 = 0$ has not any solution in natural number. For $K/H \ncong O_5(q)$, we have similar proof.

From the above cases, we conclude

$$K/H \cong U_7(2),$$

and $k(G) = k(U_7(2)) = 66$. By Lemma 2.5, $G \cong U_7(2)$. The proof is complete. □

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