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Quasigroups that are a disjoint union of 4-cycles

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Abstract. This article explores the conjecture "An idempotent, alterable quasigroup Q that is a disjoint union of a finite, admissible number of four-cycles based on $x \in Q$ is a quadratical quasigroup." The conjecture is proved when the number of four-cycles is either one, two or three.

1. Introduction

A *magma* $(M, \cdot)$ consists of a set M and a binary operation denoted by juxtaposition, i.e. $x \cdot y = xy$. The *dual magma* of $(M, \cdot)$, denoted by M^*, is defined as $(M, *)$, where $x * y = yx$. A magma is called *idempotent* if it satisfies the identity $xx = x$ and *alterable* if it satisfies the implication

$$(alt) \quad xy = zw \Rightarrow yz = wx$$

(which implies $xy = zw \Leftrightarrow yz = wx$). We use idempotency and alterability throughout this paper without mention.

A magma Q satisfying the identity

$$(qm) \quad zx \cdot yz = xy \cdot x$$

is called *quadratical* [5].

Such a quasigroup has an interesting geometric interpretation, which is a motivation to study them. For example, in the Euclidean plane, a square quasigroup can be constructed by defining the action ab as the center of a square with positive orientation whose neighboring vertices are the points a and b (for details see [6], [7] and [8])

Quadratical quasigroups are idempotent and alterable (cf. [3]). They also satisfy the following identities (for proofs see [5], [3] and [1]).

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(el)	$xy \cdot x = x \cdot yx,$	$(elasticity,\ or\ flexibility)$
(se)	$x \cdot yx = xy \cdot x = yx \cdot y,$	$(strong\ elasticity)$
(bk)	$yx \cdot xy = x,$	$(bookend\ =\ the\ third\ Stein's\ law)$
(ld)	$x \cdot yz = xy \cdot xz,$	$(left\ distributivity)$
(rd)	$xy \cdot z = xz \cdot yz,$	$(right\ distributivity)$
(m)	$xy \cdot zw = xz \cdot yw,$	$(mediality)$

We will write i instead of a_i and ij instead of a_ia_j. When needed we write $\underline{11}$ or $\underline{12}$ and so on, to distinguish those elements from $1 \cdot 1$, $1 \cdot 2$, et cetera.

For $t, n \in \{1, 2, \ldots\}$, by $|t|_n$ we denote the value of t modulo n. We also assume that $0 = |n|_n = n$. So $\mathbb{Z}_n = \{1, 2, \ldots, n\}$.

The number n is called *admissible* if $4n + 1$ is the order of a quadratical quasigroup.

2. 4-cycles

In a magma M an *n-cycle based on an element $x \in M$* is an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ of distinct elements of M such that $x = a_1a_2 = a_2a_3 = \ldots = a_{n-1}a_n = a_na_1$, $x \notin \{a_1, a_2, \ldots, a_n\}$, and $n \geqslant 2$. A 4-cycle $(1, 2, 3, 4)$ based on $x \in M$ is called *full* if $14 \in \{2, 3\}$. It is called *complete* if $\{x, 1, 2, 3, 4\}$ is a sub-magma of M. The elements $14, 21, 32$ and 43 in the Cayley table of $\{1, 2, 3, 4\} \cdot \{1, 2, 3, 4\}$ are called the *indefinite elements of the 4-cycle* $(1, 2, 3, 4)$ based on $x \in M$.

For a quasigroup Q that is a disjoint union of 4-cycles based on $x \in Q$, for $b \in Q$, $b \neq x$, we define b', b'', b''' by the equalities $x = b'b = bb'' = b'''b'$. If $(1, 2, 3, 4)$ is a 4-cycle based on x then $x = 12 = 23 = 34 = 41$, so $x = b'b = 41$ implies $1' = 4$. Hence $1'' = 2$ and $1''' = 3$. Similarly, $2' = 1$, $2'' = 3$, $2''' = 4$, $3' = 2$, $3'' = 4$, $3''' = 1$, $4' = 3$, $4'' = 1$ and $4''' = 2$. Note also that if Q is an idempotent, alterable quasigroup and a disjoint union of 4-cycles based on $x \in Q$ then $bb''' = b'$, $b''b''' = b'b = x$, $b''b' = b$ and $b'''b = b''$. So, (b, b'', b''', b') is a 4-cycle based on x.

Theorem 2.1. *In an alterable left or right cancellative magma M, with $n \geqslant 3$, n-cycles, if they exist, must be 4-cycles.*

Proof. Suppose $(1, 2, \ldots, n)$ is an n-cycle based on $x \in M$. Since M is alterable, we have $x = n1 = 2(n-1) = 23 = (n-2)(n-1)$. It follows that

if M is right or left cancellative then either $2 = n-2$ or $n-1 = 3$. In either case, $n = 4$. □

Lemma 2.2. *In any magma M, if $(1,2,3,4)$ is a 4-cycle based on $x \in M$ then so are $(2,3,4,1)$, $(3,4,1,2)$, and $(4,1,2,3)$.*

Proof. Clearly, the 4-tuples $(2,3,4,1)$, $(3,4,1,2)$, and $(4,1,2,3)$ consist of 4 distinct elements. It follows from the identity $x = 12 = 23 = 34 = 41$ that $x = 23 = 34 = 41 = 12$, $x = 34 = 41 = 12 = 23$ and $x = 41 = 12 = 23 = 24$. By definition then, $(2,3,4,1)$, $(3,4,1,2)$, and $(4,1,2,3)$ are 4-cycles based on $x \in M$. □

Lemma 2.3. *In an alterable left or right cancellative magma M, 4-cycles based on $x \in M$ are either disjoint or equal as sets.*

Proof. Suppose that $(1,2,3,4)$ and $(5,6,7,8)$ are 4-cycles based on $x \in M$. Using Lemma 2.2, we need only prove that $1 = 5$ implies $\{1,2,3,4\} = \{5,6,7,8\}$. Suppose that $1 = 5$. Then, $x = 12 = 52 = 56$ and so $25 \overset{(alt)}{=} 65$. Hence, $2 = 6$ by left or right cancellativity. Then, $x = 23 = 63 = 67$ and so $36 \overset{(alt)}{=} 76$. Hence, $3 = 7$ by left or right cancellativity. Finally, $x = 34 = 74 = 78$ and so $47 \overset{(alt)}{=} 87$. Hence, $4 = 8$ and $\{1,2,3,4\} = \{5,6,7,8\}$. □

Corollary 2.4. *A non-degenerative idempotent alterable quasigroup Q consists of disjoint 4-cycles based on any element $x \in Q$.*

Proof. Let $1, x \in Q$ with $1 \neq x$. Since Q is a quasigroup, there exist elements $2,3,4 \in Q$ such that $x = 12 = 23 = 34$. Alterability implies $41 = 23 = x$. Idempotency and cancellability imply $2 \notin \{1,x\}$, $3 \notin \{1,2,x\}$ and $4 \notin \{1,2,3,x\}$. Similarly, any other element $5 \in Q - \{x,1,2,3,4\}$ is an element of a 4-cycle $(5,6,7,8)$. The Corollary then follows from Lemma 2.3. □

It is known that a non-degenerative quadratical quasigroup Q consists of disjoint 4-cycles based on the element $ab{\cdot}a$, where $a, b \in Q$, $a \neq b$ (cf. [3]).

The 4-cycle conjecture. *For any admissible n in $\{1,2,3,\ldots\}$ an idempotent, alterable quasigroup Q consisting of n disjoint 4-cycles based on an element $x \in Q$ is a quadratical quasigroup.*

Lemma 2.5. *If $(1,2,3,4)$ is a 4-cycle based on an element x in an alterable, idempotent quasigroup Q then the Cayley table of $\{1,2,3,4\} \cdot \{1,2,3,4\}$ is*

	1	2	3	4
1	1	x	4	c
2	c''	2	x	1
3	2	c'''	3	x
4	x	3	c'	4

where $c \in Q$, $c \neq x$.

Proof. By definition $x \notin \{1,2,3,4\}$. Since $x = 12 = 23 = 34 = 41$, this, together with alterability and idempotency implies $13 = 4$, $24 = 1$, $31 = 2$ and $42 = 3$. Suppose that $14 = c$. Since by definition $x = c'c = cc'' = c'''c'$, alterability and idempotency implies $c''c''' = c'c = x$, $c = c''c'$, $cc''' = c'$ $c'c'' = c'''$ and $c'''c = c''$.

Then $4c = c1 = 2c' = c''3$ implies $32 = c'c'' = c'''$ implies $2c''' = c'''3 = 4c'' = c'1$ implies $3c' = 1c''' = c'4 = c2$ implies $43 = c'$ and $21 = c'''c = c''$. Since Q is a quasigroup, $c = 14 \neq 12 = x$. □

Corollary 2.6. *If* $(1,2,3,4)$ *is a* 4*-cycle based on an element* x *in an alterable, idempotent quasigroup* Q *then* $(14,21,32,43)$ *is a* 4*-cycle based on the same element* x.

Recall that if Q is a quasigroup then its dual magma Q^* is also quasigroup. A quasigroup induced by the additive group $\mathbb{Z}_m$ is quadratical if and only if its dual quasigroup is quadratical (cf. [3]). The quadratical quasigroups induced by $\mathbb{Z}_m$ have the form $xy = |ax + (1-a)y|_m$ or $x * y = |(1-a)x + ay|_m$, where $a \in \mathbb{Z}_m$ and $|2a^2 - 2a + 1|_m = 0$ (cf. [3]). For $m = 5$ we have only two such quasigroups: one with the operation $xy = |\mathbf{2}x + \mathbf{4}y|_5$ and the other with the operation $x * y = |\mathbf{4}x + \mathbf{2}y|_5$, where $\mathbb{Z}_5 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

Theorem 2.7. *An alterable, idempotent quasigroup* Q *consisting only of a* 4*-cycle* $(1,2,3,4)$ *based on an element* $w \in Q$ *is isomorphic to a quadratical quasigroup induced by the additive group* $\mathbb{Z}_5$ *or to its dual quasigroup.*

Proof. We have, $w = 12 = 23 = 34 = 41$. Using Lemma 2.5, the Cayley table of Q is:

Q	w	1	2	3	4
w	w				
1		1	w	4	c
2		c''	2	w	1
3		2	c'''	3	w
4		w	3	c'	4

Since Q is a quasigroup, $c \in \{2,3\}$. If $c = 2$ then $c'' = 3$, $c''' = 4$ and $c' = 1$. If $c = 3$ then $c'' = 4$, $c''' = 1$ and $c' = 2$. So, the completed Cayley tables in each case are:

Q_1	w	1	2	3	4
w	w	4	1	2	3
1	3	1	w	4	2
2	4	3	2	w	1
3	1	2	4	3	w
4	2	w	3	1	4

Q_2	w	1	2	3	4
w	w	3	4	1	2
1	2	1	w	4	3
2	3	4	2	w	1
3	4	2	1	3	w
4	1	w	3	2	4

We re-write the Cayley tables of the quasigroups determined by the products $xy = |\mathbf{2}x + \mathbf{4}y|_5$ and $x * y = |\mathbf{4}x + \mathbf{2}y|_5$ as follows:

$\cdot$	**5**	**1**	**2**	**4**	**3**
5	**5**	**4**	**3**	**1**	**2**
1	**2**	**1**	**5**	**3**	**4**
2	**4**	**3**	**2**	**5**	**1**
4	**3**	**2**	**1**	**4**	**5**
3	**1**	**5**	**4**	**2**	**3**

$*$	**5**	**1**	**3**	**4**	**2**
5	**5**	**2**	**1**	**3**	**4**
1	**4**	**1**	**5**	**2**	**3**
3	**2**	**4**	**3**	**5**	**1**
4	**1**	**3**	**2**	**4**	**5**
2	**3**	**5**	**4**	**1**	**2**

Then the first table is the same as the table of Q_2, with the elements re-named as $w \to \mathbf{5}$, $1 \to \mathbf{1}$, $2 \to \mathbf{2}$, $3 \to \mathbf{4}$ and $4 \to \mathbf{3}$. Also, the second table is the same as the table of Q_1, with elements re-named as $w \to \mathbf{5}$, $1 \to \mathbf{1}$, $2 \to \mathbf{3}$, $3 \to \mathbf{4}$ and $4 \to \mathbf{2}$. This completes the proof. □

Lemma 2.8. *If* $(1,2,3,4)$ *and* $(5,6,7,8)$ *are disjoint* 4*-cycles based on the same element* x *in an alterable, idempotent quasigroup* Q *then the Cayley table of* $\{1,2,3,4\} \cdot \{5,6,7,8\}$ *is as follows:*

	5	6	7	8
1	c	r	s	t
2	t''	c''	r''	s''
3	s'''	t'''	c'''	r'''
4	r'	s'	t'	c'

Proof. We have $x = 12 = 23 = 34 = 41 = 56 = 67 = 78 = 85$ and $x = c'c = cc'' = c'''c'$. Thus, using idempotency, alterability, Lemma 2.2 and Corollary 2.4, we obtain the following implications: $15 = c = cc \Rightarrow 5c = c1 = 2c' = c''8 \Rightarrow 82 = 37 = c'c'' = c''' \Rightarrow 2c''' = c'''8 = 5c'' = c'1 \Rightarrow 8c' = 1c''' = c'4 = c7 \Rightarrow 48 = c'$ and $26 = 71 = c'''c = c''$.

Similarly, $16 = r = 74 \Rightarrow 27 = r'' = 81$, $38 = r''' = 52$ and $45 = r' = 63$, $17 = s = 84 \Rightarrow 28 = s'' = 51$, $35 = s''' = 62$ and $46 = s' = 73$, and finally, $18 = t = 54 \Rightarrow 25 = t'' = 61$, $36 = t''' = 72$ and $47 = t' = 83$. □

As a consequence of Lemma 2.8 and alterability we obtain

Corollary 2.9. *If* $(1,2,3,4)$ *and* $(5,6,7,8)$ *are disjoint* 4*-cycles based on the same element* x *in an alterable, idempotent quasigroup* Q *then the Cayley table of* $\{1.2.3.4\} \cdot \{5,6,7,8\}$ *and of* $\{5,6,7,8\} \cdot \{1,2,3,4\}$ *are as follows:*

Q	1	2	3	4	5	6	7	8
1					c	r	s	t
2					t''	c''	r''	s''
3					s'''	t'''	c'''	r'''
4					r'	s'	t'	c'
5	s''	r'''	c'	t				
6	t''	s'''	r'	c				
7	c''	t'''	s'	r				
8	r''	c'''	t'	s				

Lemma 2.10. *If* $(1,2,3,4)$ *and* $(5,6,7,8)$ *are disjoint* 4*-cycles based on the same element* x *in an alterable, idempotent quasigroup* Q *then the Cayley table of* $\{1,2,3,4\} \cdot \{5,6,7,8\}$ (*and* $\{5,6,7,8\} \cdot \{1,2,3,4\}$) *cannot contain four equal entries.*

Proof. We refer to the Cayley table defined in Corollary 2.9. Clearly, $x \notin \{1,2,3,4\} \cdot \{5,6,7,8\}$. Therefore, for $q \in \{c,r,s,t\}$, $\{q,q',q'',q'''\}$ contains four distinct elements. It follows from Lemma 2.2, that we need only prove that $\{c,c'',c''',c'\} = \{r,r'',r''',r'\} = \{s,s'',s''',s'\} = \{t,t'',t''',t'\}$ leads to a contradiction.

Since Q is a quasigroup, we can easily see from the Cayley table in Corollary 2.9 that $\{c,c'',c''',c'\} = \{r,r'',r''',r'\} = \{s,s'',s''',s'\} = \{t,t'',t''',t'\} \Rightarrow c \in \{r'',r'''\} = \{s',s''\} = \{t',t'''\}$. Therefore, there are eight possible cases.

(1) $c = r'' = s' = t'$,

(2) $c = r'' = s' = t'''$,

(3) $c = r'' = s'' = t'$,

(4) $c = r'' = s'' = t'''$,

(5) $c = r''' = s' = t'$,

(6) $c = r''' = s' = t'''$,

(7) $c = r''' = s'' = t'$,

(8) $c = r''' = s'' = t'''$.

Now cases (1), (3), (4), (5), (6) and (8) all lead to a contradiction. This is because, as seen in the Cayley table of the quasigroup Q in Lemma 2.8, the sets $\{c', r', s', t'\}$, $\{c'', r'', s'', t''\}$ and $\{c''', r''', s''', t'''\}$ each contain four distinct elements.

In case (2), $c = t''' = s'$ is a contradiction because $36 = t''' = s' = 46$, contradicting the fact that Q is a quasigroup.

In case (7), $c = r''' = s''$ implies $s'' = 28 = r''' = 38$, also contradicting that Q is a quasigroup.

We have proved that $\{1,2,3,4\} \cdot \{5,6,7,8\}$ cannot contain four equal elements. By alterability, $\{1,2,3,4\} \cdot \{5,6,7,8\} = \{5,6,7,8\} \cdot \{1,2,3,4\}$. This completes the proof of Lemma 2.10. □

Lemma 2.11. *If $(1,2,3,4)$ is a 4-cycle based on x in an alterable, idempotent quasigroup Q then $x1 = 2x$, $x2 = 3x$, $x3 = 4x$ and $x4 = 1x$. Also, if $x1 = b \in Q$ then $x2 = b''$, $x3 = b'''$, $x4 = b'$ and $(x1, x2, x3, x4) = (b, b'', b''', b')$ and $(1x, 2x, 3x, 4x) = (b', b, b'', b''')$ are 4-cycles based on x. Finally, $(1,2,3,4)$ is complete if and only if $x4 \in \{2,3\}$, if and only if $x3 \in \{1,2\}$, if and only if $x2 \in \{1,4\}$, if and only if $x1 \in \{3,4\}$.*

Proof. The fact that $x1 = 2x$, $x2 = 3x$, $x3 = 4x$ follows from the fact that $(1,2,3,4)$ is a 4-cycle based on x in the alterable, idempotent quasigroup Q. If $x1 = b$ then $x1 = b = 2x \Rightarrow 1b = bx = b''4 = xb' \Rightarrow 4x = b'b'' = b''' = x3$. Also, $x1 = b = 2x \Rightarrow b2 = xb = b''x = 3b' \Rightarrow b3 = b'x = 4b' = xb''' \Rightarrow 3x = b'''b = b''$ and $x4 = b' = 1x$. Since by definition $x = b'b = bb'' = b'''b' = b''b'''$, $(x1, x2, x3, x4)$ and $(1x, 2x, 3x, 4x)$ are 4-cycles based on x. Using Lemma 2.5, the fact that $(1,2,3,4)$ is complete if and only if $x4 \in \{2,3\}$, follows from the following Cayley table of $\{x, 1, 2, 3, 4\}$.

	x	1	2	3	4
x	x	b	b''	b'''	b'
1	b'	1	x	4	c
2	b	c''	2	x	1
3	b''	2	c'''	3	x
4	b'''	x	3	c'	4

From this table we see that if $x4 \in \{2,3\}$ then the remainder of the table can be filled in. So, $(1,2,3,4)$ is complete. Also, clearly the table implies

that if $(1,2,3,4)$ is complete then $x4 \in \{2,3\}$. The proofs that $(1,2,3,4)$ is complete if and only if $x3 \in \{1,2\}$, if and only if $x2 \in \{1,4\}$, if and only if $x1 \in \{3,4\}$ are similar. □

Theorem 2.12. *Suppose that an alterable, idempotent quasigroup Q consists only of two disjoint 4-cycles $(1,2,3,4)$ and $(5,6,7,8)$ based on $x \in Q$. Then Q is isomorphic to the quadratical quasigroup of order nine.*

Proof. We prove firstly that neither $(1,2,3,4)$ nor $(5,6,7,8)$ is a complete 4-cycle. If one of these 4-cycles is complete then we can assume that it is $(1,2,3,4)$, because if necessary, we can just re-name the elements.

Now $(1,2,3,4)$ a complete 4-cycle, by definition, implies that in this case $\{x,1,2,3,4\} \cdot \{x,1,2,3,4\} = \{x,1,2,3,4\}$ and that every column of this table contains the elements $x,1,2,3,4$. This leads to a contradiction, because $\{5\} \cdot \{x,1,2,3,4\}$ cannot contain any element of $\{x,1,2,3,4\}$ or of $\{5,8\}$. This implies that $\{6,7\} = \{5\} \cdot \{x,1,2,3,4,8\}$, contradicting the fact that Q is a quasigroup.

Now since $(1,2,3,4)$ is not a complete 4-cycle, by Lemma 2.11, we get $x4 \in \{5,6,7,8\}$. Then, by Lemma 2.2, we can assume that $x4 = 5$. It follows from Lemma 2.11 that $x1 = 6 = 2x$, $x2 = 7 = 3x$, $x3 = 8 = 4x$ and $x4 = 5 = 1x$. Using Corollary 2.9, we have the following Cayley table, where $a \in \{1,2,3,4\}$.

Q	x	1	2	3	4	5	6	7	8
x	x	6	7	8	5	a''	a'''	a'	a
1	5	1	x	4		c	r	s	t
2	6		2	x	1	t''	c''	r''	s''
3	7	2		3	x	s'''	t'''	c'''	r'''
4	8	x	3		4	r'	s'	t'	c'
5	a	s''	r'''	c'	t	5	x	8	
6	a''	t''	s'''	r'	c		6	x	5
7	a'''	c''	t'''	s'	r	6		7	x
8	a'	r''	c'''	t'	s	x	7		8

We proceed to show that $14 \notin \{1,2,3,4\}$. That is, that $14 \notin \{2,3\}$.

Suppose that $14 = 3$. Since $x4 = 5$, we have $63 \overset{(alt)}{=} 45 \overset{(alt)}{=} 5x \overset{(alt)}{=} x8$. So, $45 = x8 \overset{(alt)}{\Rightarrow} 84 = 5x = a$. But $14 = 3$, $24 = 1$ and $44 = 4$ and so $84 = a = 2$. By the Cayley table, $74 = 8$, $54 \in \{6,7\}$ and $64 \in \{7,8\}$. Hence, $64 = 7$ and $54 = 6$. Then, $54 = 6 = t \Rightarrow 7 = t'' = 61$, $8 = t''' = 72$,

$5 = t' = 83$. Also, $64 = 7 = c \Rightarrow 8 = c'' = 71$, $5 = c''' = 82$, $6 = c' = 53$. Hence, $6 = 54 = 53$, a contradiction.

If $14 = 2$, then $84 = 5x = 3$ and, again, $74 = 8$, $54 \in \{6, 7\}$ and $64 \in \{7, 8\}$. Hence, $64 = 7$ and $54 = 6$. So, $64 = 7 = c$. But $74 = 8 = r \Rightarrow 5 = r''$, $6 = r'''$, $7 = r' = c$, which contradicts the 5-column of the Cayley table of Q. So $14 \in \{5, 6, 7, 8\}$. Using Lemma 2.2, without loss of generality we can assume that $14 = 5$ and using Lemma 2.5, the Cayley table looks like this.

Q	x	1	2	3	4	5	6	7	8
x	x	b	b''	b'''	b'	a	a''	a'''	a'
1	b'	1	x	4	5	c	r	s	t
2	b	6	2	x	1	t''	c''	r''	s''
3	b''	2	7	3	x	s'''	t'''	c'''	r'''
4	b'''	x	3	8	4	r'	s'	t'	c'
5	a'	s''	r'''	c'	t	5	x	8	q
6	a	t''	s'''	r'	c	q''	6	x	5
7	a''	c''	t'''	s'	r	6	q'''	7	x
8	a'''	r''	c'''	t'	s	x	7	q'	8

Now since $14 = 5$, $45 = 51$ and by the Cayley table, $r' = s''$. Then $x = r'r = s''r = s''s''' \Rightarrow r = s'''$. Since $(s''')''' = s$, $r = s'''$ implies $s = r'''$. Considering $s = r'''$ in the Cayley table, we see that $s \notin \{x, 1, 2, 3, 4, 5, 7, 8\}$, so $s = r''' = 6$, $s'' = 7 = r'$, $s''' = 8 = r$ and $s' = 5 = r''$.

Now $6 = r''' = 52 \Rightarrow 26 = 65 \Rightarrow c'' = q'' \Rightarrow c = q \notin \{x, 1, 4, 5, s, s'', 8\} = \{x, 1, 4, 5, 6, 7, 8\}$ and so $c = q \in \{2, 3\}$. But $t \notin \{x, 1, 4, 5, 8, s, s'', c\} = \{x, 1, 4, 5, 8, 6, 7, c\}$ so $t \in \{2, 3\}$, $c = q \in \{2, 3\}$ and $t \neq c$. Note that in the 4-column of the Cayley table, $r = 8$, $s = 6$ and $\{t, c\} = \{2, 3\}$. This implies $b' = 7$, $b = 8$, $b'' = 5$ and $b''' = 6$.

Now if $t = 2$, $t'' = 3$, $t''' = 4$ and $t' = 1$. Since $t \neq c \in \{2, 3\}$, $c = 3$ and $c' = 2 = t$, which contradicts the 5-row of the Cayley table. Hence, $t = 3$, $t'' = 4$, $t''' = 1$, $t' = 2$, $c = q = 2$, $c' = q'' = 3$, $c''' = q''' = 4$ and $c' = q' = 1$. In the 6-row of the Cayley table, $t'' = 4$, $c = 2$ and $q'' = 3$. This implies that $a = 1$, $a'' = 2$, $a''' = 3$ and $a' = 4$. This determines all the values of the entries in the Cayley table as follows:

Q	x	1	2	3	4	5	6	7	8
x	x	8	5	6	7	1	2	3	4
1	7	1	x	4	5	2	8	6	3
2	8	6	2	x	1	4	3	5	7
3	5	2	7	3	x	8	1	4	6
4	6	x	3	8	4	7	5	2	1
5	4	7	6	1	3	5	x	8	2
6	1	4	8	7	2	3	6	x	5
7	2	3	1	5	8	6	4	7	x
8	3	5	4	2	6	x	7	1	8

It only remains to show that Q is quadratical. This can be done by showing that Q satisfies the identity $zw \cdot qz = wq \cdot w$ but we choose a different proof. We have $13 \cdot 1 = 41 = x = ab \cdot a$, where $a = 1$ and $b = 3$. Then $ba = 31 = 2$, $ab = 13 = 4$, $a \cdot ab = 14 = 5$, $ba \cdot a = 21 = 6$, $b \cdot ba = 32 = 7$ and $ab \cdot b = 43 = 8$. The Cayley table then becomes

Q	x	a	ba	b	ab	$a{\cdot}ab$	$ba{\cdot}a$	$b{\cdot}ba$	$ab{\cdot}b$
x	x	$ab{\cdot}b$	$a{\cdot}ab$	$ba{\cdot}a$	$b{\cdot}ba$	a	ba	b	ab
a	$b{\cdot}ba$	a	x	ab	$a{\cdot}ab$	ba	$ab{\cdot}b$	$ba{\cdot}a$	b
ba	$ab{\cdot}b$	$ba{\cdot}a$	ba	x	a	ab	b	$a{\cdot}ab$	$b{\cdot}ba$
b	$a{\cdot}ab$	ba	$b{\cdot}ba$	b	x	$ab{\cdot}b$	a	ab	$ba{\cdot}a$
ab	$ba{\cdot}a$	x	b	$ab{\cdot}b$	ab	$b{\cdot}ba$	$a{\cdot}ab$	ba	a
$a{\cdot}ab$	ab	$b{\cdot}ba$	$ba{\cdot}a$	a	b	$a{\cdot}ab$	x	$ab{\cdot}b$	ba
$ba{\cdot}a$	a	ab	$ab{\cdot}b$	$b{\cdot}ba$	ba	b	$ba{\cdot}a$	x	$a{\cdot}ab$
$b{\cdot}ba$	ba	b	a	$a{\cdot}ab$	$ab{\cdot}b$	$ba{\cdot}a$	ab	$b{\cdot}ba$	x
$ba{\cdot}a$	b	$a{\cdot}ab$	ab	ba	$ba{\cdot}a$	x	$b{\cdot}ba$	a	$ab{\cdot}b$

It is known that, up to isomorphism, there is only one quadratical quasigroup of order nine [1]. The Cayley table for this quasigroup, called $Q2$ in [4], has exactly the same Cayley table as Q and so Q is isomorphic to $Q2$, as required. □

Quadratical quasigroups of order nine are isomorphic to the quasigroup $(\mathbb{Z}_3 \times \mathbb{Z}_3, \circ)$ with $(x, y) \circ (z, u) = (y + z + 2u, x + y + 2z)$ (cf. [2]) and can be identified with the quasigroup $Q2$ in [4]. Quadratical quasigroups of order thirteen are isomorphic to the quasigroup $(\mathbb{Z}_{13}, \cdot)$ with $xy = |3x + 11y|_{13}$ or to its dual quasigroup $(\mathbb{Z}_{13}, *)$ where $x * y = |11x + 3y|_{13}$ (cf. [3]). Their tables can be identified, respectively, with the tables of quasigroups $Q3$ and $(Q3)^*$ presented in [4].

Consider the following Cayley table.

Q	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	4	1	2	3								
1	3	1	x	4	2								
2	4	3	2	x	1								
3	1	2	4	3	x								
4	2	x	3	1	4								
5						5	x	8					
6							6	x	5				
7						6		7	x				
8						x	7		8				
9										9	x	$\underline{12}$	
$\underline{10}$											$\underline{10}$	x	9
$\underline{11}$										$\underline{10}$		$\underline{11}$	x
$\underline{12}$										x	$\underline{11}$		$\underline{12}$

Theorem 2.13. *Suppose that an alterable, idempotent quasigroup Q consists only of three disjoint 4-cycles based on $x \in Q$. Then Q is isomorphic to the quadratical quasigroup $Q3$ of order thirteen or its dual quasigroup $(Q3)^*$.*

Proof. As a visual aid, let's consider the Cayley table for Q above, assuming that an arbitrarily chosen 4-cycle, call it $(1, 2, 3, 4)$, is complete and that $14 = 2$. (The arguments that follow hold even if $(1, 2, 3, 4)$ is complete and $14 = 3$.) We call the other two 4-cycles $(5, 6, 7, 8)$ and $(9, \underline{10}, \underline{11}, \underline{12})$.

By definition then, $\{x, 1, 2, 3, 4\} \cdot \{x, 1, 2, 3, 4\} = \{x, 1, 2, 3, 4\}$. Since Q is a quasigroup, this implies that the entries in each row and column of $\{x, 1, 2, 3, 4\} \cdot \{x, 1, 2, 3, 4\}$ in the Cayley table of Q are $x, 1, 2, 3, 4$. This implies that $\{5\} \cdot \{x, 1, 2, 3, 4\} \cap \{x, 1, 2, 3, 4\} = \emptyset$ and so $\{1, 2, 3, 4\} \subseteq \{58, 59, 5\,\underline{10}, 5\,\underline{11}, 5\,\underline{12}\}$.

Now if $58 \notin \{1, 2, 3, 4\}$ then $\{1, 2, 3, 4\} = \{59, 5\,\underline{10}, 5\,\underline{11}, 5\,\underline{12}\}$ and by Lemma 2.8, every row and column of $\{5, 6, 7, 8\} \cdot \{9, \underline{10}, \underline{11}, \underline{12}\}$ contains the entries $1, 2, 3, 4$. This contradicts Lemma 2.10, and so $58 = a \in \{1, 2, 3, 4\}$. By Lemma 2.5 we have, $65 = a''$, $76 = a'''$ and $87 = a'$. This implies $\{6, 7\} \cap (\{5\} \cdot \{x, 1, 2, 3, 4\}) \neq \emptyset$. However, $5x = 6 \Rightarrow 65 = a'' = x6$, a contradiction because $\{x\} \cdot \{1, 2, 3, 4\} = \{1, 2, 3, 4\}$.

Also, $5x = 7 \Rightarrow 6 = 75 = x7 \Rightarrow 6x = 76 = a'''$, a contradiction because $\{1, 2, 3, 4\} \cdot \{x\} = \{1, 2, 3, 4\}$. For all $t \in \{1, 2, 3, 4\}$, $5t = 6 \Rightarrow t6 = 65 = a$, a contradiction because $\{t\} \cdot \{x, 1, 2, 3, 4\} = \{x, 1, 2, 3, 4\}$. Also, $5t = 7 \Rightarrow t7 = 75 = 6 \Rightarrow 6t = 76 = a$, a contradiction because $\{x, 1, 2, 3, 4\} \cdot \{t\} = \{x, 1, 2, 3, 4\}$. Hence, $(1, 2, 3, 4)$ cannot be complete.

Since $(1,2,3,4)$ was an arbitrary chosen 4-cycle, $(5,6,7,8)$ and $(9,\underline{10},\underline{11},\underline{12})$ are not complete.

Let $A=\{1,2,3,4\}$, $B=\{5,6,7,8\}$ and $C=\{9,\underline{10},\underline{11},\underline{12}\}$. By Lemma 2.11, $\{x\}\cdot\{1,2,3,4\}\neq A$, $\{x\}\cdot\{5,6,7,8\}\neq B$ and $\{x\}\cdot\{9,\underline{10},\underline{11},\underline{12}\}\neq C$. Therefore, we get $\{x\}\cdot\{1,2,3,4\}\in\{B,C\}$, $\{x\}\cdot\{5,6,7,8\}\in\{A,C\}$ and $\{x\}\cdot\{9,\underline{10},\underline{11},\underline{12}\}\in\{A,B\}$. Without loss of generality, we can assume that $\{x\}\cdot\{1,2,3,4\}=C$. Therefore, $\{x\}\cdot\{5,6,7,8\}=A$ and $\{x\}\cdot\{9,\underline{10},\underline{11},\underline{12}\}=B$.

Specifically, we can assume that $x1=c$, $x2=c''$, $x3=c'''$, $x4=c'$, $x5=a$, $x6=a''$, $x7=a'''$, $x8=a'$, $x9=b$, $x\underline{10}=b''$, $x\underline{11}=b'''$ and $x\underline{12}=b'$ with $a\in A$, $b\in B$ and $c\in C$. Using Corollary 2.9 and table below (next page), we now prove that $(1,2,3,4)$ is not full. That is, that $14\notin\{2,3\}$. If $14=2\in\{a,a'',a''',a'\}$ then by Lemma 2.5, $21=3$, $32=4$ and $43=1$. If $2=a'$ then, $x5=a=2=6x\Rightarrow 2x=52=x1\Rightarrow 2x=15=52\Rightarrow 21=5$, contradiction. If $2=a''$ then, $x5=a=1\Rightarrow 1x=51=x4\Rightarrow 45=51\Rightarrow 14=5$, contradiction. If $2=a'''$ then, $x5=a=4\Rightarrow 4x=54=x3=35\Rightarrow 43=5$, contradiction. If $a=a'$ then, $x5=a=3\Rightarrow 3x=53=x2=25\Rightarrow 32=5$, contradiction. Hence, $14\neq 2$.

If $14=3\in\{a,a'',a''',a'\}$, by Lemma 2.5, $21=4$, $32=1$ and $43=2$. If $3=a$ then $x5=a=3\Rightarrow 3x=53=x2=25\Rightarrow 32=5$, contradiction. If $3=a''$ then $x5=a=2\Rightarrow 2x=52=x1=15\Rightarrow 21=5$, contradiction. If $3=a'''$ then $x5=a=1\Rightarrow 1x=51=x4=45\Rightarrow 14=5$, contradiction. If $3=a'$ then $x5=a=4\Rightarrow 4x=54=x3=35\Rightarrow 43=5$, contradiction. Hence, $14\notin\{2,3\}$ and $(1,2,3,4)$ is not full. Similarly, using the following large table

Q	x	1	2	3	4	5	6	7	8	9	10	11	12
x	x	c	c''	c'''	c'	a	a''	a'''	a'	b	b''	b'''	b'
1	c'	1	x	4		f''	e'''	d'	g	q''	n'''	s'	r
2	c		2	x	1	g''	f'''	e'	d	r''	q'''	n'	s
3	c''	2		3	x	d''	g'''	f'	e	s''	r'''	q'	n
4	c'''	x	3		4	e''	d'''	g'	f	n''	s'''	r'	q
5	a'	d	e	f	g	5	x	8		i	j	k	l
6	a	g''	d''	e''	f''		6	x	5	l''	i''	j''	k''
7	a''	f'''	g'''	d'''	e'''	6		7	x	k'''	l'''	i'''	j'''
8	a'''	e'	f'	g'	d'	x	7		8	j'	k'	l'	i'
9	b'	s	n	q	r	k''	j'''	i'	l	9	x	12	
10	b	r''	s''	n''	q''	l''	k'''	j'	i		10	x	9
11	b''	q'''	r'''	s'''	n'''	i''	l'''	k'	j	10		11	x
12	b'''	n'	q'	r'	s'	j''	i'''	l'	k	x	11		12

we can prove that $(5,6,7,8)$ and $(9,\underline{10},\underline{11},\underline{12})$ are not full. This, together with Corollary 2.9, gives the Cayley table for Q.

Since $(1,2,3,4)$ is not full, $14 \in B \cup C$. Using Lemma 2.2, without loss of generality we can assume that $14 = 5$. We will use the following notation, where $c, d, e, f \in Q$.

c	c''	c'''	c'
d'	d	d''	d'''
e''	e'''	e'	e
f'''	f'	f	f''

Let $\{i,j,n\} \in \{1,2,3,4\}$. It is straightforward to prove that if $i \neq j$ and the entries in the i^{th} and j^{th} rows, n^{th} column, are equal, then the entries in the i^{th} and j^{th} rows, k^{th} column are equal for all $k \in \{1,2,3,4\}$.

Now, by Lemma 2.5, $14 = 5 \Rightarrow 21 = 6$, $32 = 7$ and $43 = 8$. Also, $14 = 5 \in \{b, b'', b''', b'\}$.

Case 1: $14 = 5 = b$. Then, by the above table, $14 = 5 = b = \underline{10}\, x = x9$, $51 = 45 \Rightarrow d = e''$, $5\,\underline{10} = x5 \Rightarrow j = a$, $4x = 91 \Rightarrow c''' = s$ and $5x = 95 \Rightarrow a' = k''$. Hence,

c	c''	c'''	c'	$-$	d	d''	d'''	d'	$-$	a	a''	a'''	a'
s'''	s'	s	s''	$-$	e''	e'''	e'	e	$-$	j	j''	j'''	j'
$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	k'''	k'	k	k''

We now prove that $a \in \{2,3,4\}$ contradicts the fact that $21 = 6$.

If $2 = a = x5 = 6x \Rightarrow 2x = 52$ and $26 = x2$, , which by the above large table implies $c = e$ and $f''' = c''$. So,

c	c''	c'''	c'
e	e''	e'''	e'
f''	f'''	f'	f

Then, by the large table, $e = f'' \Rightarrow 52 = 15 \Rightarrow 21 = 5$, contradiction.

If $3 = a = x5 = 6x \Rightarrow 3x = 53$ and $36 = x3$, which by the large table implies $c'' = f$ and $g''' = c'''$. So,

c	c''	c'''	c'
f'	f	f''	f'''
g	g''	g'''	g'

Then, $f' = 82 = g = 18 \Rightarrow 21 = 8$, contradiction.

If $4 = a = x5 = 6x \Rightarrow 4x = 54$ and $46 = x4$, so by the large table, $c''' = g$ and $d''' = c'$. Hence,

c	c''	c'''	c'
g'''	g'	g	g''
d'	d	d''	d'''

Then, $g''' = 72 = d' = 17 \Rightarrow 21 = 7$, contradiction.

We have proved that $14 = 5 = b$ implies $1 = a$. Since $a = j = k'''$ we have $1 = a = x5 = 6x \overset{a=j}{=} 5\,\underline{10} = \underline{11}\,8 \overset{a=k'''}{=} 79 = \underline{10}\,6$. So, $1x = 51$, $16 = x1$, $15 = \underline{10}\,1$ and $1\,\underline{10} = 61$ which by the above large table implies $c' = d$, $e''' = c$, $f'' = r''$ and $n''' = g''$. So,

c	c''	c'''	c'
d''	d'''	d'	d
s'''	s'	s	s''
e'''	e'	e	e''

Note that, by the large table, $c = s'''$gives $c = 4\,\underline{10} = \underline{11}\,3$, which implies $c = 9$ (Case 1.1)or $c = \underline{12}$ (Case 1.2). We will see that specifying whether $c = 9$ or $c = \underline{12}$ allows us to find the values of x, x', x'' and x''' for $x \in \{a, b, c, d, e, f, g, s, n, q, r, i, j, k, l\}$ and the indefinite entries of $(1, 2, 3, 4)$, $(5, 6, 7, 8)$ and $(9, \underline{10}, \underline{11}, \underline{12})$. This gives two different Cayley tables, one of which we will prove is a quadratical quasigroup Q_{13} of order thirteen and the other of which we will prove is its dual quasigroup $(Q_{13})^*$. For $c = 9$ we call the quasigroup $Q_{1.1}$ and for $c = \underline{12}$ we call the quasigroup $Q_{1.2}$.

Indeed, if $c = 9$ then $c = 9 = 2x = x1 \overset{s'''=c}{=} 4\,\underline{10} = \underline{11}\,3 \overset{d''=c}{=} 35 = 62 \overset{e'''=c}{=} 16 = 74$ and the following implications are a consequence of this: $9x = 19 \Rightarrow b' = q''$, $94 = \underline{10}\,9 \Rightarrow r = \underline{10}\,9$, $93 = 59 \Rightarrow q = i$, $96 = 29 \Rightarrow j''' = r''$, $91 = 69 \Rightarrow s = l''$, $97 = 49 \Rightarrow i' = n''$, $\underline{10}\,3 = 54 \Rightarrow n'' = g$, $\underline{10}\,1 = 64 \Rightarrow r'' = f''$. Hence,

1	2	3	4	–	5	6	7	8	–	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
j	j''	j'''	j'	–	q'''	q'	q	q''	–	d''	d'''	d'	d
k'''	k'	k	k''	–	n	n''	n'''	n'	–	s'''	s'	s	s''
r'	r	r''	r'''	–	i'''	i'	i	i''	–	e'''	e'	e	e''
f'	f	f''	f'''	–	g'	g	g''	g'''	–	l'	l	l''	l'''

This means we can fill in the large table of $Q_{1.1}$ as below. The only thing remaining is to prove that $Q_{1.1}$ is a quadratical quasigroup. We can do this by proving that the quadratical identity $zx \cdot yz = xy \cdot x$ holds in $Q_{1.1}$; for example, $x7 \cdot 3x = 3 \cdot \underline{10} = 4 = \underline{10}\,7 = 73 \cdot 7$. We leave the remaining calculations to he reader. Here is the Cayley table of $Q_{1.1}$:

$Q_{1.1}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	9	$\underline{10}$	$\underline{11}$	$\underline{12}$	1	2	3	4	5	6	7	8
1	$\underline{12}$	1	x	4	5	3	9	$\underline{11}$	6	8	7	$\underline{10}$	2
2	9	6	2	x	1	7	4	$\underline{10}$	$\underline{12}$	3	5	8	$\underline{11}$
3	$\underline{10}$	2	7	3	x	9	8	1	$\underline{11}$	$\underline{12}$	4	6	5
4	$\underline{11}$	x	3	8	4	$\underline{12}$	$\underline{10}$	5	2	6	9	1	7
5	4	$\underline{12}$	$\underline{11}$	2	6	5	x	8	9	7	1	3	$\underline{10}$
6	1	7	9	$\underline{12}$	3	$\underline{10}$	6	x	5	$\underline{11}$	8	2	4
7	2	4	8	$\underline{10}$	9	6	$\underline{11}$	7	x	1	$\underline{12}$	5	3
8	3	$\underline{10}$	1	5	$\underline{11}$	x	7	$\underline{12}$	8	4	2	9	6
9	8	$\underline{11}$	5	7	2	4	3	6	$\underline{10}$	9	x	$\underline{12}$	1
$\underline{10}$	5	3	$\underline{12}$	6	8	$\underline{11}$	1	4	7	2	$\underline{10}$	x	9
$\underline{11}$	6	5	4	9	7	8	$\underline{12}$	2	1	$\underline{10}$	3	$\underline{11}$	x
$\underline{12}$	7	8	6	1	$\underline{10}$	2	5	9	3	x	$\underline{11}$	4	$\underline{12}$

When $c = \underline{12}$, using the large table again, we can prove that

1	2	3	4	–	5	6	7	8	–	$\underline{12}$	9	$\underline{10}$	$\underline{11}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
j	j''	j'''	j'	–	q	q''	q'''	q'	–	d''	d'''	d'	d
k'''	k'	k	k''	–	r'''	r'	r	r''	–	s'''	s'	s	s''
n	n''	n'''	n'	–	f'''	f'	f	f''	–	e'''	e'	e	e''
g'	g	g''	g'''	–	l'	l	l''	l'''	–	i	i''	i'''	i'

So, the Cayley table of $Q_{1.2}$ is as follows:

$Q_{1.2}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{12}$	9	$\underline{10}$	$\underline{11}$	1	2	3	4	5	6	7	8
1	$\underline{11}$	1	x	4	5	8	$\underline{12}$	$\underline{10}$	2	6	3	9	7
2	$\underline{12}$	6	2	x	1	3	5	9	$\underline{11}$	8	7	4	$\underline{10}$
3	9	2	7	3	x	$\underline{12}$	4	6	$\underline{10}$	$\underline{11}$	5	8	1
4	$\underline{10}$	x	3	8	4	$\underline{11}$	9	1	7	2	$\underline{12}$	6	5
5	4	$\underline{11}$	$\underline{10}$	7	2	5	x	8	9	$\underline{12}$	1	3	6
6	1	3	$\underline{12}$	$\underline{11}$	8	$\underline{10}$	6	x	5	7	9	2	4
7	2	5	4	9	$\underline{12}$	6	$\underline{11}$	7	x	1	8	$\underline{10}$	3
8	3	9	6	1	$\underline{10}$	x	7	$\underline{12}$	8	4	2	5	$\underline{11}$
9	8	$\underline{10}$	1	5	7	4	3	$\underline{11}$	6	9	x	$\underline{12}$	2
$\underline{10}$	5	8	$\underline{11}$	2	6	7	1	4	$\underline{12}$	3	$\underline{10}$	x	9
$\underline{11}$	6	7	5	$\underline{12}$	3	9	8	2	1	$\underline{10}$	4	$\underline{11}$	x
$\underline{12}$	7	4	8	6	9	2	$\underline{10}$	5	3	x	$\underline{11}$	1	$\underline{12}$

We will find the isomorphism $\varphi : (Q_{1.1})^* \to Q_{1.2}$. Since $(Q_{1.1})^*$ is dual to $Q_{1.1}$, $1 * 3 = 3 \cdot 1 = 2$ and $1 \cdot 3 = 4$. Therefore we can put $\varphi 2 = 4$.

Analogously $2 * 3 = 3 \cdot 2 = 7 = (1 * 3) * 3$ and $(1 \cdot 3) \cdot 3 = 8$ imply $\varphi 7 = 8$. Similarly, from $6 * 7 = 7 \cdot 6 = \underline{11} = \big(1 * (1 * 3)\big) * \big((1 * 3) * 3\big)$ and $\Big(1 \cdot (1 \cdot 3)\Big) \cdot \Big((1 \cdot 3) \cdot 3\Big) = (1 \cdot 4) \cdot (4 \cdot 3) = 5 \cdot 8 = 9$, hence, we choose $\varphi\underline{11} = 9$. Proceeding similarly with all the elements of $(Q_{1.1})^*$, we obtain the desired mapping φ. It can be considered as a permutation of the form $\varphi = (2,4)(5,6)(7,8)(9,\underline{11})$. It is straightforward to check that such defined φ is an isomorphism between $(Q_{1.1})^*$ and $Q_{1.2}$.

Cases 2, 3 and 4 are when $1{\cdot}4 = 5 = b''$, $1{\cdot}4 = 5 = b'''$ and $1{\cdot}4 = 5 = b'$ respectively. As in case 1, in each of the cases 2, 3 and 4, $a = 1$. In case 2, that is, when $1 \cdot 4 = 5 = b''$, $c \in \{9, \underline{10}\}$. In case 3, that is, when $1 \cdot 4 = 5 = b'''$, $c \in \{\underline{10}, \underline{11}\}$. In case 4, that is, when $1 \cdot 4 = 5 = b'$, $c \in \{\underline{11}, \underline{12}\}$.

We will not give the details, as the proofs of the following are similar to the proof in case 1, when $1 \cdot 4 = 5 = b$ and $c \in \{9, \underline{12}\}$.

Case 2.1: $1 \cdot 4 = 5 = b''$ and $c = 9$.

1	2	3	4	–	8	5	6	7	–	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
g'	g	g''	g'''	–	f''	f'''	f'	f	–	d''	d'''	d'	d
k	k''	k'''	k'	–	i'''	i'	i	i''	–	e'''	e'	e	e''
l'''	l'	l	l''	–	n	n''	n'''	n'	–	j	j''	j'''	j'
s''	s'''	s'	s	–	q'''	q'	q	q''	–	r'	r	r''	r'''

Case 2.2: $1 \cdot 4 = 5 = b''$ and $c = \underline{10}$.

1	2	3	4	–	8	5	6	7	–	$\underline{10}$	$\underline{11}$	$\underline{12}$	9
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
f'	f	f''	f'''	–	g	g''	g'''	g'	–	d''	d'''	d'	d
k	k''	k'''	k'	–	s'''	s'	s	s''	–	e'''	e'	e	e''
l'''	l'	l	l''	–	n	n''	n'''	n'	–	i'	i	i''	i'''
q	q''	q'''	q'	–	j'	j	j''	j'''	–	r'	r	r''	r'''

Case 3.1: $1 \cdot 4 = 5 = b'''$ and $c = \underline{10}$.

1	2	3	4	–	7	8	5	6	–	$\underline{10}$	$\underline{11}$	$\underline{12}$	9
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
i'''	i'	i	i''	–	f	f''	f'''	f'	–	d''	d'''	d'	d
r'''	r'	r	r''	–	s	s''	s'''	s'	–	e'''	e'	e	e''
l	l''	l'''	l'	–	n'''	n'	n	n''	–	q	q''	q'''	q'
g'	g	g''	g'''	–	j''	j'''	j'	j	–	k	k''	k'''	k'

CASE 3.2: $1 \cdot 4 = 5 = b'''$ and $c = \underline{11}$.

1	2	3	4	–	7	8	5	6	–	$\underline{11}$	$\underline{12}$	9	$\underline{10}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
f'	f	f''	f'''	–	g''	g'''	g'	g	–	d''	d'''	d'	d
i'''	i'	i	i''	–	s'''	s'	s	s''	–	e'''	e'	e	e''
l	l''	l'''	l'	–	r	r''	r'''	r'	–	j'	j	j''	j'''
n''	n'''	n'	n	–	k	k''	k'''	k'	–	q	q''	q'''	q'

CASE 4.1: $1 \cdot 4 = 5 = b'$ and $c = \underline{11}$.

1	2	3	4	–	6	7	8	5	–	$\underline{11}$	$\underline{12}$	9	$\underline{10}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
j'''	j'	j	j''	–	f'	f	f''	f'''	–	d''	d'''	d'	d
i	i''	i'''	i'	–	s'''	s'	s	s''	–	e'''	e'	e	e''
q'	q	q''	q'''	–	r	r''	r'''	r'	–	n''	n'''	n'	n
g'	g	g''	g'''	–	k	k''	k'''	k'	–	l	l''	l'''	l'

CASE 4.2: $1 \cdot 4 = 5 = b'$ and $c = \underline{12}$.

1	2	3	4	–	6	7	8	5	–	$\underline{12}$	9	$\underline{10}$	$\underline{11}$
a	a''	a'''	a'	–	b	b''	b'''	b'	–	c	c''	c'''	c'
f'	f	f''	f'''	–	g	g''	g'''	g'	–	d''	d'''	d'	d
i	i''	i'''	i'	–	l'	l	l''	l'''	–	e'''	e'	e	e''
j'''	j'	j	j''	–	r'''	r'	r	r''	–	k'	k	k''	k'''
s'''	s'	s	s''	–	q	q''	q'''	q'	–	n''	n'''	n'	n

From these results, using the large table on page 278, the Cayley tables of all these quasigroups can be filled in. Here are results, with each quasigroup named after its corresponding Case. The quasigroups $Q_{2.1}$, $Q_{3.1}$ and $Q_{4.1}$ satisfy the identity $zu \cdot vz = uv \cdot u$ and are therefore quadratical quasigroups. Below we give isomorphism $\varphi_2 \colon (Q_{2.1})^* \to Q_{2.2}$, $\varphi_3 \colon (Q_{3.1})^* \to Q_{3.2}$ and $\varphi_4 \colon (Q_{4.1})^* \to Q_{4.2}$. Note that, when expressed as a product of cycles, $\varphi = \varphi_3$ and $\varphi_2 = \varphi_4$.

Also

$\psi_1 \colon Q_{1.1} \to Q_{2,2}$, where $\psi_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9, \underline{10}, \underline{11}, \underline{12})$,

$\psi_2 \colon Q_{1.1} \to Q_{3,2}$, where $\psi_2 = (1)(2)(3)(4)(5)(6)(7)(8)(9, \underline{11}), (\underline{10}, \underline{12})$,

$\psi_3 \colon Q_{1.1} \to Q_{4,2}$, where $\psi_3 = (1)(2)(3)(4)(5)(6)(7)(8)(9, \underline{12}, \underline{11}, \underline{10})$,

are isomorphisms. It follows that $Q_{1,2}$, $Q_{2,1}$, $Q_{3.1}$ and $Q_{4.1}$ are isomorphic quasigroups.

Case 2.1: $14 = 5 = b''$ and $c = 9$.

$Q_{2.1}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	9	$\underline{10}$	$\underline{11}$	$\underline{12}$	1	2	3	4	8	5	6	7
1	$\underline{12}$	1	x	4	5	8	9	$\underline{11}$	2	7	6	3	$\underline{10}$
2	9	6	2	x	1	3	5	$\underline{10}$	$\underline{12}$	$\underline{11}$	8	7	4
3	$\underline{10}$	2	7	3	x	9	4	6	$\underline{11}$	1	$\underline{12}$	5	8
4	$\underline{11}$	x	3	8	4	$\underline{12}$	$\underline{10}$	1	7	5	2	9	6
5	4	$\underline{12}$	$\underline{11}$	7	2	5	x	8	$\underline{10}$	6	9	1	3
6	1	3	9	$\underline{12}$	8	$\underline{11}$	6	x	5	4	7	$\underline{10}$	2
7	2	5	4	$\underline{10}$	9	6	$\underline{12}$	7	x	3	1	8	$\underline{11}$
8	3	$\underline{10}$	6	1	$\underline{11}$	x	7	9	8	$\underline{12}$	4	2	5
9	7	4	8	6	$\underline{10}$	2	$\underline{11}$	5	3	9	x	$\underline{12}$	1
$\underline{10}$	8	$\underline{11}$	1	5	7	4	3	$\underline{12}$	6	2	$\underline{10}$	x	9
$\underline{11}$	5	8	$\underline{12}$	2	6	7	1	4	9	$\underline{10}$	3	$\underline{11}$	x
$\underline{12}$	6	7	5	9	3	$\underline{10}$	8	2	1	x	$\underline{11}$	4	$\underline{12}$

Case 2.2: $14 = 5 = b''$ and $c = \underline{10}$

$Q_{2.2}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{10}$	$\underline{11}$	$\underline{12}$	9	1	2	3	4	8	5	6	7
1	9	1	x	4	5	3	$\underline{10}$	$\underline{12}$	6	2	8	7	$\underline{11}$
2	$\underline{10}$	6	2	x	1	7	4	$\underline{11}$	9	$\underline{12}$	3	5	8
3	$\underline{11}$	2	7	3	x	$\underline{10}$	8	1	$\underline{12}$	5	9	4	6
4	$\underline{12}$	x	3	8	4	9	$\underline{11}$	5	2	7	6	$\underline{10}$	1
5	4	9	$\underline{12}$	2	6	5	x	8	$\underline{10}$	$\underline{11}$	7	1	3
6	1	7	$\underline{10}$	9	3	$\underline{11}$	6	x	5	4	$\underline{12}$	8	2
7	2	4	8	$\underline{11}$	$\underline{10}$	6	$\underline{12}$	7	x	3	1	9	5
8	3	$\underline{11}$	1	5	$\underline{12}$	x	7	9	8	6	4	2	$\underline{10}$
9	7	8	6	1	$\underline{11}$	2	5	$\underline{10}$	3	9	x	$\underline{12}$	4
$\underline{10}$	8	$\underline{12}$	5	7	2	4	3	6	$\underline{11}$	1	$\underline{10}$	x	9
$\underline{11}$	5	3	9	6	8	$\underline{12}$	1	4	7	$\underline{10}$	2	$\underline{11}$	x
$\underline{12}$	6	5	4	$\underline{10}$	7	8	9	2	1	x	$\underline{11}$	3	$\underline{12}$

$\varphi_2 = (1)(24)(3)(56)(78)(9)(\underline{10}\,\underline{12})(11)$ is an isomorphism $(Q_{2.1})^* \to Q_{2.2}$. For example, $\varphi_2(6 * \underline{11}) = \varphi_2(\underline{11} \cdot 6) = \varphi_2 1 = 1 = 5\,\underline{11} = \varphi_2 6 \cdot \varphi_2 \underline{11}$ and $\varphi_2(\underline{10} * 4) = \varphi_2(4 \cdot \underline{11}) = \varphi_2 2 = 4 = \underline{12} \cdot 2 = \varphi_2 \underline{10} \cdot \varphi_2 4$. Analogously, $\varphi_2(1 * 8) = \varphi_2(8 \cdot 1) = \varphi_2 \underline{10} = \underline{12} = 17 = \varphi_2 1 \cdot \varphi_2 8$, $\varphi_2(\underline{12} * 6) = \varphi_2(6 \cdot \underline{12}) = \varphi_2 2 = 4 = \underline{10}\,5 = \varphi_2 \underline{12} \cdot \varphi_2 6$ and $\varphi_2(4 * \underline{11}) = \varphi_2(\underline{11} \cdot 4) = \varphi_2 6 = 5 = 2\,\underline{11} = \varphi_2 4 \cdot \varphi_2 \underline{11}$.

CASE 3.1: $14 = 5 = b'''$ and $c = \underline{10}$.

$Q_{3.1}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{10}$	$\underline{11}$	$\underline{12}$	9	1	2	3	4	7	8	5	6
1	9	1	x	4	5	8	$\underline{10}$	$\underline{12}$	2	$\underline{11}$	7	6	3
2	$\underline{10}$	6	2	x	1	3	5	$\underline{11}$	9	4	$\underline{12}$	8	7
3	$\underline{11}$	2	7	3	x	$\underline{10}$	4	6	$\underline{12}$	8	1	9	5
4	$\underline{12}$	x	3	8	4	9	$\underline{11}$	1	7	6	5	2	$\underline{10}$
5	4	9	$\underline{12}$	7	2	5	x	8	$\underline{11}$	3	6	$\underline{10}$	1
6	1	3	$\underline{10}$	9	8	$\underline{12}$	6	x	5	2	4	7	$\underline{11}$
7	2	5	4	$\underline{11}$	$\underline{10}$	6	9	7	x	$\underline{12}$	3	1	8
8	3	$\underline{11}$	6	1	$\underline{12}$	x	7	$\underline{10}$	8	5	9	4	2
9	6	7	5	$\underline{10}$	3	$\underline{11}$	8	2	1	9	x	$\underline{12}$	4
$\underline{10}$	7	4	8	6	$\underline{11}$	2	$\underline{12}$	5	3	1	$\underline{10}$	x	9
$\underline{11}$	8	$\underline{12}$	1	5	7	4	3	9	6	$\underline{10}$	2	$\underline{11}$	x
$\underline{12}$	5	8	9	2	6	7	1	4	$\underline{10}$	x	$\underline{11}$	3	$\underline{12}$

CASE 3.2: $14 = 5 = b'''$ and $c = \underline{11}$

$Q_{3.2}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{11}$	$\underline{12}$	9	$\underline{10}$	1	2	3	4	7	8	5	6
1	$\underline{10}$	1	x	4	5	3	$\underline{11}$	9	6	$\underline{12}$	2	8	7
2	$\underline{11}$	6	2	x	1	7	4	$\underline{12}$	$\underline{10}$	8	9	3	5
3	$\underline{12}$	2	7	3	x	$\underline{11}$	8	1	9	6	5	$\underline{10}$	4
4	9	x	3	8	4	$\underline{10}$	$\underline{12}$	5	2	1	7	6	$\underline{11}$
5	4	$\underline{10}$	9	2	6	5	x	8	$\underline{11}$	3	$\underline{12}$	7	1
6	1	7	$\underline{11}$	$\underline{10}$	3	$\underline{12}$	6	x	5	2	4	9	8
7	2	4	8	$\underline{12}$	$\underline{11}$	6	9	7	x	5	3	1	$\underline{10}$
8	3	$\underline{12}$	1	5	9	x	7	$\underline{10}$	8	$\underline{11}$	6	4	2
9	6	5	4	$\underline{11}$	7	8	$\underline{10}$	2	1	9	x	$\underline{12}$	3
$\underline{10}$	7	8	6	1	$\underline{12}$	2	5	$\underline{11}$	3	4	$\underline{10}$	x	9
$\underline{11}$	8	9	5	7	2	4	3	6	$\underline{12}$	$\underline{10}$	1	$\underline{11}$	x
$\underline{12}$	5	3	$\underline{10}$	6	8	9	1	4	7	x	$\underline{11}$	2	$\underline{12}$

$\varphi_3 = (1)(24)(3)(56)(78)(9\underline{11})(\underline{10})(\underline{12})$ is an isomorphism $(Q_{3.1})^* \to Q_{3.2}$.

For example, $\varphi_3(3 * 8) = \varphi_3(8 \cdot 3) = \varphi_3 1 = 1 = 37 = \varphi_3 3 \cdot \varphi_3 8$ and $\varphi_3(\underline{10} * 2) = \varphi_3(2 \cdot \underline{10}) = \varphi_3 \underline{12} = \underline{12} = \underline{10}\,4 = \varphi_3 \underline{10} \cdot \varphi_3 2$. Similarly, $\varphi_3(6 * \underline{11}) = \varphi_3(\underline{11} \cdot 6) = \varphi_3 3 = 3 = 5\varphi_3 6 \cdot \varphi_3 \underline{11}$, $\varphi_3(9 * 4) = \varphi_3(4 \cdot 9) = \varphi_6 = 5 = \underline{11}\,2 = \varphi_3 9 \cdot \varphi_3 4$ and $\varphi_3(2 * 8) = \varphi_3(8 \cdot 2) = \varphi_3 6 = 5 = 47 = \varphi_3 2 \cdot \varphi_3 8$.

Case 4.1: $14 = 5 = b'$ and $c = \underline{11}$.

$Q_{4.1}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{11}$	$\underline{12}$	8	$\underline{10}$	1	2	3	4	6	7	8	5
1	$\underline{10}$	1	x	4	5	8	$\underline{11}$	9	2	3	$\underline{12}$	7	6
2	$\underline{11}$	6	2	x	1	3	5	$\underline{12}$	$\underline{10}$	7	4	9	8
3	$\underline{12}$	2	7	3	x	$\underline{11}$	4	6	9	5	8	1	$\underline{10}$
4	9	x	3	8	4	$\underline{10}$	$\underline{12}$	1	7	$\underline{12}$	6	5	2
5	4	$\underline{10}$	9	7	2	5	x	8	$\underline{12}$	1	3	6	$\underline{11}$
6	1	3	$\underline{11}$	$\underline{10}$	8	9	6	x	5	$\underline{11}$	2	4	7
7	2	5	4	$\underline{12}$	$\underline{11}$	6	$\underline{10}$	7	x	8	9	3	1
8	3	$\underline{12}$	6	1	9	x	7	$\underline{11}$	8	2	5	$\underline{10}$	4
9	5	8	$\underline{10}$	2	6	7	1	4	$\underline{11}$	9	x	$\underline{12}$	3
$\underline{10}$	6	7	5	$\underline{11}$	3	$\underline{12}$	8	2	1	4	$\underline{10}$	x	9
$\underline{11}$	7	4	8	6	$\underline{12}$	2	9	5	3	$\underline{10}$	1	$\underline{11}$	x
$\underline{12}$	8	9	1	5	7	4	3	$\underline{10}$	6	x	$\underline{11}$	2	$\underline{12}$

Case 4.2: $14 = 5 = b'$ and $c = \underline{12}$

$Q_{4.2}$	x	1	2	3	4	5	6	7	8	9	$\underline{10}$	$\underline{11}$	$\underline{12}$
x	x	$\underline{12}$	9	$\underline{10}$	$\underline{11}$	1	2	3	4	6	7	8	5
1	$\underline{11}$	1	x	4	5	3	$\underline{12}$	$\underline{10}$	6	7	9	2	8
2	$\underline{12}$	6	2	x	1	7	4	9	$\underline{11}$	5	8	$\underline{10}$	3
3	9	2	7	3	x	$\underline{12}$	8	1	$\underline{10}$	4	6	5	$\underline{11}$
4	$\underline{10}$	x	3	8	4	$\underline{11}$	9	5	2	$\underline{12}$	1	7	6
5	4	$\underline{11}$	$\underline{10}$	2	6	5	x	8	$\underline{12}$	1	3	9	7
6	1	7	$\underline{12}$	$\underline{11}$	3	9	6	x	5	8	2	4	$\underline{10}$
7	2	4	8	9	$\underline{12}$	6	$\underline{10}$	7	x	$\underline{11}$	5	3	1
8	3	9	1	5	$\underline{10}$	x	7	$\underline{11}$	8	2	$\underline{12}$	6	4
9	5	3	$\underline{11}$	6	8	$\underline{10}$	1	4	7	9	x	$\underline{12}$	2
$\underline{10}$	6	5	4	$\underline{12}$	7	8	$\underline{11}$	2	1	3	$\underline{10}$	x	9
$\underline{11}$	7	8	6	1	9	2	5	$\underline{12}$	3	$\underline{10}$	4	$\underline{11}$	x
$\underline{12}$	8	$\underline{10}$	5	7	2	4	3	6	9	x	$\underline{11}$	1	$\underline{12}$

It is not difficult to see that $\varphi_4 = (1)(24)(3)(56)(78)(9)(\underline{10}\,\underline{12})(\underline{11})$ is an isomorphism $(Q_{4.1})^* \to Q_{4.2}$. Since the dual of a quadratical quasigroup is quadratical, $Q_{1.2}$, $Q_{2.2}$, $Q_{3.2}$ and $Q_{4.2}$ are quadratical quasigroups and we have proved Theorem 2.13. □

Just a note on proving the 4-cycle conjecture for $n = 4$. As in the case when $n = 2$ and $n = 3$, a vital part of the proof will be to show that no 4-cycles is complete or full. We give this proof in a future publication.

As is known, there are only three (up to isomorphism) quadratical quasigroups of order $k < 16$. They have 1, 2 and 3 4-cycles respectively. The quadratical quasigroup of order 5 has the form $(\mathbb{Z}_5, *)$, where $x * y = |2x + 4y|_5$ and is the union of the element $x = 1$ and the cycle $(2, 3, 5, 4)$. The quadratical quasigroup of order 9 has the form $(\mathbb{Z}_3 \times \mathbb{Z}_3, *)$, where $(x, y)*(z, u) = (y+z+2u, x+y+2z)$. It is the union of the element $x = (0, 0)$ and two cycles $\big((0,1),(1,2),(0,2),(2,1)\big)$, $\big((1,0),(1,1),(2,0),(2,2)\big)$. Furthermore, the quadratical quasigroup $(\mathbb{Z}_{13}, *)$, with $x * y = |3x + 11y|_{13}$, is the union of three cycles $(2, 9, 13, 6)$, $(3, 4, 12, 11)$, $(5, 7, 10, 8)$ and the element $x = 1$.

Since the dual of a quadratical quasigroup is also quadratical, there are in fact six quadratical quasigroups of order $k < 16$. However, we omit the dual quasigroups because their properties are analogous to those of the original quasigroups.

The assumption that the magma under consideration is alterable is significant. In a commutative quasigroup $(Q, \cdot)$ that is not alterable, for any $x, a \in Q$, there exists an element $b \in Q$ such that $x = ab = ba$. Thus, only 2-cycles exist in commutative quasigroups.

In the quasigroup

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	2	5	1
4	4	5	1	2	3
5	5	3	4	1	2

we have four 3-cycles: $(3, 5, 4)$, $(2, 4, 5)$, $(2, 5, 3)$, $(2, 3, 4)$ based on 1, 3, 4 and 5, respectively.

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