Quasigroups and Related Systems **34** (2026), 259 – 266

https://doi.org/10.56415/qrs.v34.19

Paramedial n-quasigroups

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Abstract. We will show that a paramedial n-quasigroup with an idempotent element is medial. Moreover, each paramedial n-quasigroup is uniquely determined by an abelian group and its automorphisms γ_i such that $\gamma_i\gamma_j = \gamma_{n-j+1}\gamma_{n-i+1}$ for all $i, j \in \{1, 2, \ldots, n\}$.

An n-quasigroup $Q(A)$ is called *paramedial* if it satisfies the identity

$$\begin{aligned} &A(A(x_{11}, \ldots, x_{n1}), A(x_{12}, \ldots, x_{n2}), \ldots, A(x_{1n}, \ldots, x_{nn})) \\ &= A(A(x_{nn}, \ldots, x_{n1}), \ldots, A(x_{2n}, \ldots, x_{21}), A(x_{1n}, \ldots, x_{11})). \end{aligned} \tag{1}$$

On the right side, the operation A is applied to rows $r_1, r_2, \ldots, r_n$ of the matrix $[x_{ij}]_{n\times n}$, on the left side to columns $c_n, c_{n-1}, \ldots, c_2, c_1$ with the rows in reverse order.

The following two fact are obvious.

Proposition 1. *A principal parastrophe of a paramedial n-quasigroup also is a paramedial n-quasigroup.*

Proposition 2. *If $Q(A)$ is a paramedial n-quasigroup, then for any fixed $a_1^n \in Q$ the operation*

$$B(x_1^n) = A(a_1^{i-1}, A(x_1^n), a_{i+1}^n)$$

is medial.

We will first describe n-loops and n-quasigroups containing at least one idempotent element.

Proposition 3. *A paramedial n-loop is commutative.*

2010 Mathematics Subject Classification: 20N05, 20N10, 20N15.
Keywords: Paramedial n-quasigroups, parastrophe, isotopy.

Proof. Let $Q(A)$ be a paramedial n-loop and e by its neutral element. Then for any $\sigma \in \mathbb{S}_n$ and $x_1^n \in Q$ substituting in (1) $x_{ik} = x_i$ for $k = \sigma(n+1-i)^{-1}$, $i \in N_n$, and $x_{ij} = e$ for the remaining ones we obtain $A(x_1^n) = A(x_{\sigma 1}^{\sigma n})$. □

Theorem 4. *A paramedial n-loop is a commutative n-group derived from a binary group.*

Proof. Let $Q(A)$ be a paramedial n-loop and e by its neutral element. Then $Q(+)$, where $x + y = A(x, y, \overset{(n-2)}{e})$, is a commutative loop. Moreover, for every $x, y, u, v \in Q$ we have

$$\begin{aligned}(x+y)+(u+v) &= A(A(x,y,\overset{(n-2)}{e}),A(u,v,\overset{(n-2)}{e}),A(\overset{(n)}{e}),\dots,A(\overset{(n)}{e}))\\ &= A(A(\overset{(n)}{e}),\dots,A(\overset{(n)}{e}),A(\overset{(n-2)}{e},v,y),A(\overset{(n-2)}{e},u,x))\\ &= A(A(v,y,\overset{(n-2)}{e}),A(u,x,\overset{(n-2)}{e}),A(\overset{(n)}{e}),\dots,A(\overset{(n)}{e})\\ &= (v+y)+(u+x).\end{aligned}$$

Using this fact we obtain

$$\begin{aligned}(x+y)+z &= (x+y)+(z+e) = (y+x)+(z+e) = (e+x)+(z+y)\\ &= x+(y+z).\end{aligned}$$

Hence $Q(+)$ is a commutative group.

Now we prove, by induction, that

$$A(x_1^n) = x_1 + x_2 + \ldots + x_n.$$

For $n = 2$ we have

$$A(x_1, x_2, \overset{(n-2)}{e}) = x_1 + x_2.$$

Suppose,

$$A(x_1, x_2, \ldots, x_i, \overset{(n-i)}{e}) = x_1 + x_2 + \ldots + x_i.$$

Then

$$\begin{aligned}&A(x_1,\dots,x_i,x_{i+1},\overset{(n-i)}{e})\\ &= A(A(\overset{(n-1)}{e},x_1),\dots,A(\overset{(n-1)}{e},x_i),A(\overset{(n-2)}{e},x_{i+1},e),A(\overset{(n)}{e}),\dots,A(\overset{(n)}{e}))\\ &\overset{(1)}{=} A(A(\overset{(n-i)}{e},x_i,\dots,x_2,x_1),A(\overset{(n-i-1)}{e},x_{i+1},\overset{(i)}{e}),A(\overset{(n)}{e}),\dots,A(\overset{(n)}{e}))\\ &= A(A(x_1,\dots,x_i,\overset{(n-i)}{e}),x_{i+1},\overset{(n-2)}{e}))\\ &= A((x_1+x_2+\ldots+x_i),x_{i+1},\overset{(n-2)}{e}) = x_1+x_2+\ldots+x_i+x_{i+1}.\end{aligned}$$

This completes the proof. □

Lemma 5. *If a is an idempotent element of a paramedial n-quasigroup $Q(A)$, then $T_{\overline{a}_i}x = A(\overset{(i-1)}{a}, x, \overset{(n-i)}{a})$ is an antiautomorphism of $Q(A)$ and $T_{\overline{a}_i}T_{\overline{a}_j} = T_{\overline{a}_j}T_{\overline{a}_i}$ for all $i, j \in N_n$.*

Proof. It is clear that $T_{\overline{a}_i}$ is a bijection. This is also antiautomorphism since

$$\begin{aligned}
T_{\overline{a}_i}A(x_1^n) &= A(\overset{(i-1)}{a}, A(x_1^n), \overset{(n-i)}{a}) \\
&= A(\overbrace{A(a,\dots,a)}^{(i-1)}, A(x_1^n), \overbrace{A(a,\dots,a)}^{(i-1)}) \\
&= A(A(\overset{(i-1)}{a}, x_n, \overset{(n-i)}{a}), A(\overset{(i-1)}{a}, x_{n-1}, \overset{(n-1)}{a}), \dots, A(\overset{(i-1)}{a}, x_1, \overset{(n-1)}{a})) \\
&= A(T_{\overline{a}_i}x_n, T_{\overline{a}_i}x_{n-1}, \dots, T_{\overline{a}_i}x_1).
\end{aligned}$$

Hence $T_{\overline{a}_i}$ is an antiautomorphism of $Q(A)$.

Moreover,

$$\begin{aligned}
&T_{\overline{a}_i}T_{\overline{a}_j}x = A(\overset{(i-1)}{a}, A(\overset{(j-1)}{a}, x, \overset{(n-j)}{a}), \overset{(n-i)}{a}) \\
&= A(\overbrace{A(a,\dots,a)}^{(i-1)}, A(\overset{(j-1)}{a}, x, \overset{(n-j)}{a}), \overbrace{A(a,\dots,a)}^{(n-i)}) \\
&\overset{(1)}{=} A(\overbrace{A(a,\dots,a)}^{(j-1)}, A(\overset{(i-1)}{a}, x, \overset{(n-i)}{a}), \overbrace{A(a,\dots,a)}^{(n-j)}) \\
&= A(\overset{(j-1)}{a}, T_{\overline{a}_i}x, \overset{(n-j)}{a}) = T_{\overline{a}_j}T_{\overline{a}_i}x
\end{aligned}$$

for all $x \in Q$ and $i, j \in N_n$. □

Proposition 6. *If $\alpha_1, \dots, \alpha_n$ are pairwise commuting antiautomorphism of a paramedial n-quasigroup $Q(A)$, then $Q(A^*)$, where*

$$A^*(x_1^n) = A(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n)$$

is a medial n-quasigroup and $\alpha_1, \dots, \alpha_n$ are automorphisms of $Q(A^*)$.

Proof. Indeed,

$$\begin{aligned}
&A^*(A^*(x_{11},\dots,x_{1n}),\dots,A^*(x_{n1},\dots,x_{nn})) \\
&= A(\alpha_1 A(\alpha_1 x_{11},\dots,\alpha_n x_{1n}),\dots,\alpha_n A(\alpha_1 x_{n1},\dots,\alpha_n x_{nn})) \\
&= A(A(\alpha_1\alpha_n x_{1n},\dots,\alpha_1\alpha_1 x_{11}),\dots,A(\alpha_n\alpha_n x_{nn},\dots,\alpha_n\alpha_1 x_{n1})) \\
&= A(A(\alpha_n\alpha_1 x_{1n},\dots,\alpha_1\alpha_1 x_{11}),\dots,A(\alpha_n\alpha_n x_{nn},\dots,\alpha_1\alpha_n x_{n1})) \\
&\overset{(1)}{=} A(A(\alpha_1\alpha_n x_{n1},\dots,\alpha_1\alpha_1 x_{11}),\dots,A(\alpha_n\alpha_n x_{nn},\dots,\alpha_n\alpha_1 x_{1n})) \\
&= A(\alpha_1 A(\alpha_1 x_{11},\dots,\alpha_n x_{n1}),\dots,\alpha_n A(\alpha_1 x_{1n},\dots,\alpha_n x_{nn}))
\end{aligned}$$

$$= A^*(A^*(x_{11}, \ldots, x_{n1}), \ldots, A^*(x_{1n}, \ldots, x_{nn})).$$

Thus $Q(A^*)$ is a medial n-quasigroup.

Any α_i is an automorphism of $Q(A^*)$ because

$$\begin{aligned}\alpha_i A^*(x_1^n) &= \alpha_i A(\alpha_1 x_1, \ldots, \alpha_n x_n)\\ &= A(\alpha_i\alpha_n x_n, \ldots, \alpha_i\alpha_1 x_1) = A(\alpha_n\alpha_i x_n, \ldots, \alpha_1\alpha_i x_1)\\ &= A^*(\alpha_i x_1, \ldots, \alpha_i x_n).\end{aligned}$$

The proof is complete. □

Proposition 7. *If a is an idempotent of a paramedial n-quasigroup $Q(A)$ and*

$$A'(x_1^n) = A(T_{\overline{a}_1}^{-1} x_1, T_{\overline{a}_2}^{-1} x_2, \ldots, T_{\overline{a}_n}^{-1} x_n),$$

then $Q(A')$ is a medial n-loop, a is its neutral element and all $T_{\overline{a}_i}^{-1}$ are its automorphisms.

Proof. By Lemma 5, translations $T_{\overline{a}_1}, \ldots, T_{\overline{a}_n}$ of $Q(A)$ are pairwise commuting antiautomorphisms of $Q(A)$. Hence $T_{\overline{a}_1}^{-1}, \ldots, T_{\overline{a}_n}^{-1}$ also are commuting antiautomorphisms. Thus, by Proposition 6, $Q(A')$ is a medial n-quasigroup. Also, $T_{\overline{a}_j} a = a$ for $j \in N_n$. Consequently

$$\begin{aligned}A'(\overset{(i-1)}{a}, x, \overset{(n-i)}{a}) &= A(T_{\overline{a}_1}^{-1} a, \ldots, T_{\overline{a}_{i-1}}^{-1} a, T_{\overline{a}_i}^{-1} x, T_{\overline{a}_{i+1}}^{-1} a, \ldots, T_{\overline{a}_n}^{-1} a)\\ &= A(\overset{(i-1)}{a}, T_{\overline{a}_i}^{-1} x, \overset{(n-i)}{a}) = T_{\overline{a}_i} T_{\overline{a}_i}^{-1} x = x,\end{aligned}$$

i.e. a is a neutral element of $Q(A')$ and $Q(A')$ is a medial n-loop. □

Theorem 8. *For a paramedial n-quasigroup $Q(A)$ with an idempotent element there exists a commutative group $Q(+)$ and its pairwise commuting automorphisms $\alpha_1, \ldots, \alpha_n$ such that*

$$A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n.$$

Proof. By Proposition 7 for a paramedial n-quasigroup $Q(A)$ with an idempotent element there exists a commutative group $Q(+)$ and its pairwise commuting antiautomorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$A'(x_1^n) = A(\alpha_1^{-1} x_1, \alpha_2^{-1} x_2, \ldots, \alpha_n^{-1} x_n)$$

and $Q(A')$ is a medial n-loop. Thus, by Theorem 4,

$$A'(x_1^n) = x_1 + x_2 + \ldots + x_n,$$

i.e.

$$A(\alpha_1^{-1}x_1, \alpha_2^{-1}x_2, \ldots, \alpha_n^{-1}x_n) = x_1 + x_2 + \ldots + x_n.$$

Therefore

$$A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are automorphisms of $Q(+)$. □

Remark 9. The same results holds for medial n-quasigroups with an idempotent element. The proof is very similar.

Theorem 10. *For a paramedial n-quasigroup $Q(A)$ there is a commutative group $Q(+)$, pairwise commuting automorphisms $\alpha_1, \ldots, \alpha_n$ of $Q(+)$ and a bijection δ of Q such that*

$$A(x_1^n) = \delta(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n).$$

Proof. In an n-quasigroup $Q(A)$ for an arbitrary element $a \in Q$ there is only one $z \in Q$ satisfying the equation

$$A(\overset{(i-1)}{a}, A(\overset{(n)}{a}), \overset{(j-i-1)}{a}, z, \overset{(n-j)}{a}) = a$$

and only one $w \in Q$ satisfying the equation

$$A(\overset{(i-1)}{a}, w, \overset{(j-i-1)}{a}, z, \overset{(n-j)}{a}) = b.$$

Then

$$B(x_1^n) = A(\overset{(i-1)}{a}, A(x_1^n), \overset{(j-i-1)}{a}, z, \overset{(n-j)}{a})$$

is an n-quasigroup operation and a is an idempotent of $Q(B)$. According to Proposition 2, $Q(B)$ is a medial n-quasigroup. Thus, by Remark 9,

$$B(x_1^n) = \alpha_1 x_1 + \alpha_2 + \ldots + \alpha_n x_n.$$

But $B(x_1^n) = \varphi A(x_1^n)$, where $\varphi x = A(\overset{(i-1)}{a}, x, \overset{(j-i-1)}{a}, z, \overset{(n-j)}{a})$ is a bijection of Q. So,

$$A(x_1^n) = \varphi^{-1}B(x_1^n) = \varphi^{-1}(\alpha_1 x_1 + \alpha_2 + \ldots + \alpha_n x_n),$$

which completes the proof. □

To prove the most important theorem of this section we need the following auxiliary lemma.

Lemma 11. *Let $Q(+)$ be a commutative group and $\gamma_1, \gamma_2, \ldots, \gamma_n$ be bijections on Q such that*

$$\begin{aligned} &\gamma_1(x_{11} + \ldots + x_{1n}) + \ldots + \gamma_n(x_{n1} + \ldots + x_{nn}) \\ &= \gamma_n(x_{nn} + \ldots + x_{1n}) + \ldots + \gamma_1(x_{n1} + \ldots + x_{11}), \end{aligned} \tag{2}$$

then there is an automorphism ψ of $Q(+)$ such that

$$\gamma_i x = \psi x + c_i$$

for some fixed $c_1^n \in Q$, all $i \in N_n$ and $x \in Q$.

Proof. Let e be a neutral element of the group $Q(+)$. If we substitute $x_{ij} = \gamma_i^{-1} e$ or $x_{ij} = e$ in the given equality except for two selected elements x_{1i} and x_{1i}, we get

$$\gamma_1 x_{1i} + \gamma_i x_{i1} = \gamma_i x_{1i} + \gamma_1 x_{i1},$$

which for $x_{1i} = \gamma_1^{-1} = e$ implies

$$\gamma_i x_{i1} = \gamma_i \gamma_1^{-1} e + \gamma_1 x_{i1}.$$

So,

$$\gamma_i x = \gamma_1 x + d_i. \tag{3}$$

Substituting this expression to (2) and reducing all d_i, we obtain

$$\begin{aligned}\gamma_1(x_{11} + \ldots + x_{1n}) + \ldots + \gamma_1(x_{n1} + \ldots + x_{nn}) \\ = \gamma_1(x_{nn} + \ldots + x_{1n}) + \ldots + \gamma_1(x_{n1} + \ldots + x_{11}).\end{aligned}$$

For $x_{ii} = \gamma_1^{-1} e$ and $x_{ij} = e$, where $x_{ij} \notin \{x_{11}, x_{12}\}$, this expression has the form

$$\gamma_1(x_{11} + x_{12}) + \gamma_1 e = \gamma_1 x_{12} + \gamma_1 x_{11}.$$

Let $\psi x = \gamma_1 x - \gamma_1 e$. Then, by the last equation, ψ is an endomorphism of $Q(+)$. Since γ_1 is a bijection, also ψ is bijection. Hence ψ is an automorphism of $Q(+)$. Thus, by (3), $\gamma_i x = \psi x + \gamma_1 e + d_i = \psi x + c_i$. □

Theorem 12. *An n-quasigroup $Q(A)$ is paramedial if and only if*

$$A(x_1^n) = \gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n + d,$$

where $Q(+)$ is a commutative group $Q(+)$, $d \in Q$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are automorphisms of $Q(+)$ such that $\gamma_i \gamma_j = \gamma_{n-j+1}\gamma_{n-i+1}$ for all $i, j \in N_n$.

Proof. Since $Q(A)$ is a paramedial n-quasigroup, the operation A is as in Theorem 10. Thus

$$A(A(x_{11}^{1n}), \ldots, A(x_{n1}^{nn})) = A(A(x_{nn}, \ldots, x_{n1}), \ldots, A(x_{1n}, \ldots, x_{11}))$$

means that

$$\begin{aligned}\delta(\alpha_1 \delta(\alpha_1 x_{11} + \ldots + \alpha_n x_{1n}) + \ldots + \alpha_n \delta(\alpha_1 x_{n1} + \ldots + \alpha_n x_{nn})) \\ = \delta(\alpha_1 \delta(\alpha_1 x_{nn} + \ldots + \alpha_n x_{1n}) + \ldots + \alpha_n \delta(\alpha_1 x_{n1} + \ldots + \alpha_n x_{11})).\end{aligned}$$

From this, reducing one δ and using the commutativity of $Q(+)$, we obtain

$$\begin{aligned}&\alpha_1\delta\alpha_n^{-1}(\alpha_n\alpha_1x_{11}+\ldots+\alpha_n\alpha_nx_{1n})+\ldots\\&\qquad\qquad+\alpha_n\delta\alpha_1^{-1}(\alpha_1\alpha_1x_{n1}+\ldots+\alpha_1\alpha_nx_{nn})\\&=\alpha_1\delta\alpha_n^{-1}(\alpha_n\alpha_1x_{nn}+\ldots+\alpha_n\alpha_nx_{1n})+\ldots\\&\qquad\qquad+\alpha_n\delta\alpha_1^{-1}(\alpha_1\alpha_1x_{n1}+\ldots+\alpha_1\alpha_nx_{11}),\end{aligned}$$

which for $\beta_i=\alpha_i\delta\alpha_{n-i+1}^{-1}$ and $y_{ij}=\alpha_{(n-i+1)}\alpha_jx_{ij}$ gives

$$\begin{aligned}&\beta_1(y_{11}+\ldots+y_{1n})+\ldots+\beta_n(y_{n1}+\ldots+y_{nn})\\&\qquad=\beta_1(y_{nn}+\ldots+y_{1n})+\ldots+\beta_n(y_{n1}+\ldots+y_{11}).\end{aligned}$$

Hence, by Lemma 11,

$$\alpha_i\delta\alpha_{n-i+1}^{-1}x=\beta_i=\psi x+c_i,$$

i.e.

$$\delta x=\alpha_i^{-1}\psi\alpha_{n-i+1}x+\alpha_i^{-1}\alpha_{n-i+1}c_i=\eta_ix+d_i,$$

where $d_i=\alpha_i^{-1}\alpha_{n-i+1}c_i$ is an element of Q and $\eta_i=\alpha_i^{-1}\psi\alpha_{n-i+1}$. But

$$\delta e=\alpha_i^{-1}\psi\alpha_{n-i+1}e+\alpha_i^{-1}\alpha_{n-i+1}c_i=\alpha_i^{-1}\alpha_{n-i+1}c_i=d_i$$

so, $d_i=\delta e=d$ for every $i\in N_n$. Then $\delta x=\eta_ix+d$. Consequently $\eta_1=\eta_2=\ldots=\eta_n=\eta$. Obviously η is an automorphism of $Q(+)$ and, by Theorem 10,

$$A(x_1^n)=\gamma_1x_1+\gamma_2x_2+\ldots+\gamma_nx_n+d$$

for $\gamma_i=\eta\alpha_i$. It is easy to check that $\gamma_i\gamma_j=\gamma_{n-i+1}\gamma_{n-j+1}$ for all $i,j\in N_n$. □

For $n=2$, we obtain the well-known result of Němec and Kepka proved in [5].

Theorem 13. *Every paramedial quasigroup $Q(\cdot)$ can be presented in the form*

$$x\cdot y=\varphi x+\psi y+d,$$

where $Q(+)$ is an abelian group, φ, ψ are automorphisms of $Q(+)$ such that $\varphi^2=\psi^2$ and d is some fixed element of Q.

Finally, we give an example of two paramedial n-quasigroup, one of which is medial.

Example 14. Let $Q = \mathbb{Z}_4$. Them $Q(A)$ and $Q(B)$, where $A(x_1^4) = (x_1 + 3x_2 + 3x_3 + x_4)(\mathrm{mod}\, 4)$ and $B(x_1^4) = (x_1 + 3x_2 + x_3 + 3x_4)(\mathrm{mod}\, 4)$, are parastrophic paramedial 4-quasigroups.The first is medial, the second is not.

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Received February 12, 2026

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