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Reduced Sombor index for some graph transformations

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Abstract. Let G be a simple graph and $d_G(x)$ be the degree of the vertex x in G. The reduced Sombor index of graph G is denoted as

$$So_{red} = \sum_{xy \in E(G)} \sqrt{(d_G(x)-1)^2 + (d_G(y)-1)^2}.$$

The Sombor index for some graph transformation was calculated in [10]. In this paper we calculate So_{red} of that graph transformation.

1. Introduction

Topological and chemical indices have received quite some attention in graph theory already and have various applications in mathematical chemistry. Wiener index is the oldest topological index, was introduced in 1947 by Wiener [13]. As it is the oldest index it has been studied by many different researchers, we refer the reader to papers [5, 6, 7] for more information about it.

One of the most famous of these indices is Sombor and reduced Sombor index. It was introduced in 2021 by Ivan Gutman [8]. It has been studied by many researchers, we refer the reader to papers [2, 3, 4, 11, 12] for more information about it.

Let $G = (V(G), E(G))$ be a graph with n vertices and m edges. Also, in graph theory, the degree of a vertex of a graph is the number of edges that are incident to the vertex. The Sombor and reduced Sombor indices are denoted as SO and SO_{red}, respectively. They are defined for a graph G as:

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$$So = \sum_{xy \in E(G)} \sqrt{d_G(x)^2 + d_G(y)^2},$$
$$So_{red} = \sum_{xy \in E(G)} \sqrt{(d_G(x) - 1)^2 + (d_G(y) - 1)^2}.$$

In [10], was calculated the Sombor index of some graph transformations, we do it for reduced Sombor index.

2. Main results

Subdivision graph.
Let G be a graph with n vertices and m edges, the graph is obtained by inserting a new vertex into each edge of G, is called the subdivision graph and denoted as $S(G)$. So it is clear that $S(G)$ has $m + n$ vertices and $2m$ edges.

Theorem 2.1. *Let G be a graph and $S(G)$ be the subdivision graph of G. Then*

$$SO_{red}(S(G)) = \sum_{u \in V(G)} d_G(u) \sqrt{(d_G(u) - 1)^2 + 1}.$$

Proof. By definition, we have:

$$\begin{aligned} SO_{red}(S(G)) &= \sum_{uv \in E(S(G))} \sqrt{(d_{S(G)}(u) - 1)^2 + (d_{S(G)}(v) - 1)^2} \\ &= \sum_{uv \in E(G)} \sqrt{(d_G(u) - 1)^2 + (2 - 1)^2} \\ &= \sum_{u \in V(G)} d_G(u) \sqrt{(d_G(u) - 1)^2 + 1}. \end{aligned}$$

□

Line graph.
The line graph of an undirected graph G is denoted by $L(G)$, which is constructed in the following way: for each edge in G, make a vertex in $L(G)$; for every two edges in G that have a vertex in common, make an edge between their corresponding vertices in $L(G)$.

Hence, if $e = uv \in E(G)$, then $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.

In the sequel, let G be a graph and A be the set of all paths of length 2 in G.

Theorem 2.2. *Let G be a graph and $L(G)$ be the line graph of G. Then $SO_{red}(L(G))$ is equal to*

$$\sum_{uvw\in A}[(d_G^2(u)+d_G^2(w))-6(d_G(u)+d_G(w)) + 2d_G(v)(d_G(u)+d_G(v)+d_G(w)-6)+18]^{1/2}.$$

Proof. Let $ee' \in E(L(G))$, so we can consider $e = uv$ and $e' = vw$, $(e, e' \in E(G))$. Therefore, uvw is a path of length 2 and so $uvw \in A$. Consequently,

$$\begin{aligned} SO_{red}(L(G)) &= \sum_{ee'\in E(L(G))} \sqrt{(d_{L(G)}(e)-1)^2+(d_{L(G)}(e')-1)^2} \\ &= \sum_{ee'\in E(L(G))} \sqrt{(d_G(u)+d_G(v)-3)^2+(d_G(v)+d_G(w)-3)^2} \\ &= \sum_{uvw\in A}[(d_G^2(u)+d_G^2(w))-6(d_G(u)+d_G(w)) \\ &+ 2d_G(v)(d_G(u)+d_G(v)+d_G(w)-6)+18]^{1/2}. \end{aligned}$$

□

We consider

$$\begin{aligned} E_1 &= \{uv \mid uv \in E(G)\}, \\ E_2 &= \{ef \mid e = uv \in E(G), f = vw \in E(G), e \sim f\}, \\ E_3 &= \{ue \mid e = uv \in E(G)\}. \end{aligned}$$

Total graph.
The total graph $T(G)$ of a graph G is a graph such that the vertex set of it corresponds to the vertices and edges of G and the edge set of $T(G)$ is equal to $E_1 \cup E_2 \cup E_3$.

Hence, if $e = uv \in E(G)$, then $d_{T(G)}(e) = d_G(u)+d_G(v)$ and $d_{T(G)}(u) = 2d_G(u)$.

Semitotal point graph.
The semitotal point graph of G is denoted by $T_1(G)$ and its vertex set is same as the vertex set of the total graph of G and its edge set is the union of E_1 and E_3.

Hence, if $e = uv \in E(G)$, then $d_{T_1(G)}(e) = 2$ and $d_{T_1(G)}(u) = 2d_G(u)$.

Semitotal line graph.
The semitotal line graph of G is denoted by $T_2(G)$ and its vertex set is same

as the vertex set of the total graph of G and its edge set is the union of E_2 and E_3.

Hence, if $e = uv \in E(G)$, then $d_{T_2(G)}(e) = d_G(u)+d_G(v)$ and $d_{T_2(G)}(u) = d_G(u)$.

Theorem 2.3. *Let G be a graph and $T(G), T_1(G)$ and $T_2(G)$, be the total graph, semitotal point graph and semitotal line graph of G, respectively. Then*

$$\begin{aligned}
SO_{red}(T(G)) = &\sum_{uv\in E(G)} \sqrt{4(d_G^2(u)+d_G^2(v)) - 4(d_G(u)+d_G(v)) + 2} \\
&+ \sum_{uvw\in A} [(d_G^2(u)+d_G^2(w)) - 2(d_G(u)+d_G(w)) \\
&\qquad + 2d_G(v)(d_G(u)+d_G(v)+d_G(w)-2)+2]^{1/2} \\
&+ \sum_{u\in V(G)} \sum_{uv\in E(G)} d_G(u)[5d_G^2(u) - 6d_G(u) \\
&\qquad + d_G(v)(d_G(v)+2d_G(u))-2)+2]^{1/2}, \\
SO_{red}(T_1(G)) = &\sum_{uv\in E(G)} \sqrt{4(d_G^2(u)+d_G^2(v)) - 4(d_G(u)+d_G(v)) + 2} \\
&+ \sum_{u\in V(G)} d_G(u)\sqrt{4d_G^2(u) - 4d_G(u)) + 2}. \\
SO_{red}(T_2(G)) = &\sum_{u\in V(G)} \sum_{uv\in E(G)} d_G(u)[2d_G(u)(d_G(u)-2) \\
&\qquad + d_G(v)(d_G(v)+2d_G(u)-2)+2]^{1/2} \\
&+ \sum_{uvw\in A} [(d_G^2(u)+d_G^2(w)) - 2(d_G(u)+d_G(w)) \\
&\qquad + 2d_G(v)(d_G(u)+d_G(v)+d_G(w)-2)+2]^{1/2}.
\end{aligned}$$

Proof. At first, we consider the total graph of G, so we have:

$$\begin{aligned}
SO_{red}(T(G)) &= \sum_{ab\in E(T(G))} \sqrt{(d_{T(G)}(a)-1)^2 + (d_{T(G)}(b)-1)^2} \\
&= \sum_{ab\in E_1} \sqrt{(d_{T(G)}(a)-1)^2 + (d_{T(G)}(b)-1)^2} \\
&+ \sum_{ab\in E_2} \sqrt{(d_{T(G)}(a)-1)^2 + (d_{T(G)}(b)-1)^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{ab\in E_3} \sqrt{(d_{T(G)}(a)-1)^2+(d_{T(G)}(b)-1)^2} \\
&= \sum_{uv\in E_1} \sqrt{(d_{T(G)}(u)-1)^2+(d_{T(G)}(v)-1)^2} \\
&+ \sum_{ef\in E_2} \sqrt{(d_{T(G)}(e)-1)^2+(d_{T(G)}(f)-1)^2} \\
&+ \sum_{ue\in E_3} \sqrt{(d_{T(G)}(u)-1)^2+(d_{T(G)}(e)-1)^2} \\
&= \sum_{uv\in E(G)} \sqrt{(2d_G(u)-1)^2+(2d_G(v)-1)^2} \\
&+ \sum_{uvw\in A} \sqrt{(d_G(u)+d_G(v)-1)^2+(d_G(v)+d_G(w)-1)^2} \\
&+ \sum_{uv\in E(G)} [\sqrt{(2d_G(u)-1)^2+(d_G(u)+d_G(v)-1)^2} \\
&\qquad + \sqrt{(2d_G(v)-1)^2+(d_G(u)+d_G(v)-1)^2}] \\
&= \sum_{uv\in E(G)} \sqrt{4(d_G^2(u)+d_G^2(v))-4(d_G(u)+d_G(v))+2} \\
&+ \sum_{uvw\in A} [(d_G^2(u)+d_G^2(w))-2(d_G(u)+d_G(w)) \\
&\qquad + 2d_G(v)(d_G(u)+d_G(v)+d_G(w)-2)+2]^{1/2} \\
&+ \sum_{u\in V(G)} \sum_{uv\in E(G)} d_G(u)[5d_G^2(u)-6d_G(u) \\
&\qquad + d_G(v)(d_G(v)+2d_G(u))-2)+2]^{1/2},
\end{aligned}
$$

.

Now, we consider the semitotal point graph of G, so we have:

$$
\begin{aligned}
SO_{red}(T_1(G)) &= \sum_{ab\in E(T_1(G))} \sqrt{(d_{T_1(G)}(a)-1)^2+(d_{T_1(G)}(b)-1)^2} \\
&= \sum_{ab\in E_1} \sqrt{(d_{T_1(G)}(a)-1)^2+(d_{T_1(G)}(b)-1)^2} \\
&+ \sum_{ab\in E_3} \sqrt{(d_{T_1(G)}(a)-1)^2+(d_{T_1(G)}(b)-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{uv\in E_1} \sqrt{(d_{T_1(G)}(u)-1)^2 + (d_{T_1(G)}(v)-1)^2} \\
&+ \sum_{ue\in E_3} \sqrt{(d_{T_1(G)}(u)-1)^2 + (d_{T_1(G)}(e)-1)^2} \\
&= \sum_{uv\in E(G)} \sqrt{(2d_G(u)-1)^2 + (2d_G(v)-1)^2} \\
&+ \sum_{uv\in E(G)} \sqrt{(2d_G(u)-1)^2+1} + \sqrt{(2d_G(v)-1)^2+1} \\
&= \sum_{uv\in E(G)} \sqrt{4(d_G^2(u)+d_G^2(v)) - 4(d_G(u)+d_G(v)) + 2} \\
&+ \sum_{u\in V(G)} d_G(u)\sqrt{4d_G^2(u) - 4d_G(u) + 2}.
\end{aligned}
$$

Finally, we consider the semitotal line graph of G, therefore:

$$
\begin{aligned}
SO_{red}(T_2(G)) &= \sum_{ab\in E(T_2(G))} \sqrt{(d_{T_2(G)}(a)-1)^2 + (d_{T_2(G)}(b)-1)^2} \\
&= \sum_{ab\in E_2} \sqrt{(d_{T_2(G)}(a)-1)^2 + (d_{T_2(G)}(b)-1)^2} \\
&+ \sum_{ab\in E_3} \sqrt{(d_{T_2(G)}(a)-1)^2 + (d_{T_2(G)}(b)-1)^2} \\
&= \sum_{ef\in E_2} \sqrt{(d_{T_2(G)}(e)-1)^2 + (d_{T_2(G)}(f)-1)^2} \\
&+ \sum_{ue\in E_3} \sqrt{(d_{T_2(G)}(u)-1)^2 + (d_{T_2(G)}(e)-1)^2} \\
&= \sum_{uvw\in A} \sqrt{(d_G(u)+d_G(v)-1)^2 + (d_G(v)+d_G(w)-1)^2} \\
&+ \sum_{uv\in E(G)} [\sqrt{(d_G(u)-1)^2 + (d_G(u)+d_G(v)-1)^2} \\
&\qquad + \sqrt{(d_G(v)-1)^2 + (d_G(u)+d_G(v)-1)^2}] \\
&= \sum_{u\in V(G)} \sum_{uv\in E(G)} d_G(u)[2d_G(u)(d_G(u)-2) \\
&\qquad + d_G(v)(d_G(v)+2d_G(u)-2) + 2]^{1/2}
\end{aligned}
$$

$$+ \sum_{uvw \in A} [(d_G^2(u) + d_G^2(w)) - 2(d_G(u) + d_G(w)) + 2d_G(v)(d_G(u) + d_G(v) + d_G(w) - 2) + 2]^{1/2}.$$

□

We denote by K_n the complete graph of order n and consider

$$E_1' = \{uv \mid uv \in E(G)\},$$
$$E_2' = \{uv \mid uv \notin E(G)\},$$
$$E_3' = \{uu \mid u \in V(G)\}.$$

Double graph.
Let G be a graph with n vertices and m edges. The double graph of G is denoted by $D(G)$, which is constructed by making two copies of G (including the initial edge set of each) and adding edges u_1v_2 and v_1u_2 for every edge $uv \in E(G)$. The k^{th} iterated double graph of G for $k = 0, 1 \ldots$ is defined as $D^k(G) = D(D^{k-1}(G))$, where $D^0(G) \equiv G$ and $D^1(G) \equiv D(G)$ [10].
Hence, $E(D^k(G)) = E_1' \cup E_2'$ and it is clear that $|E_1'| = 2^k m$ and $|E_2'| = 2^k m(2^k - 1)$. Also, if $u \in V(G)$, then $d_{D^k(G)}(u) = 2^k d_G(u)$.
Strong double graph.[10]
Let G be a graph with n vertices and m edges. The strong double graph $SD(G)$ is obtained by taking two copies of the graph G and joining each vertex v in one copy with the closed neighborhood of the corresponding vertex in another copy. The k^{th} iterated double graph of G for $k = 0, 1 \ldots$ is defined as $SD^k(G) = SD(SD^{k-1}(G))$, where $SD^0(G) \equiv G$ and $SD^1(G) \equiv SD(G)$.

Hence, $E(D^k(G)) = E_1' \cup E_2' \cup E_3'$. Also, if $u \in V(G)$, then $d_{SD^k(G)}(u) = 2^k(d_G(u) + 1) - 1$.

Theorem 2.4. *Let G be a graph with n vertices and m edges and $D^k(G)$, $SD(G)$ and $SD^k(G)$ be the iterated double graph, strong double graph and iterated strong double graph, respectively. Then*

$$SO_{red}(D^k(G)) = \sum_{uv \in E(K_n)} \sqrt{2^{2k}(d_G^2(u) + d_G^2(v)) - 2^{k+1}(d_G(u) + d_G(v)) + 2}.$$

$$SO_{red}(SD(G)) = 2SO(G) + 2 \sum_{uv \in E_2'} \sqrt{(d_G(u))^2 + (d_G(v))^2} + 4\sqrt{2}m.$$

$SO_{red}(SD^k(G)) =$

$$2^k \sum_{uv\in E(G)} [2^{2k}(d_G^2(u)+d_G^2(v))+ \\ 2^{k+2}(2^{k-1}-1)(d_G(u)+d_G(v))+2^3(2^{k-1}-1)^2]^{1/2} \\ + \sum_{uv\in E_2'} [2^{2k}(d_G^2(u)+d_G^2(v))+ \\ 2^{k+2}(2^{k-1}-1)(d_G(u)+d_G(v))+2^3(2^{k-1}-1)^2]^{1/2} \\ + \sqrt{2}(2^k(2m)+(2^k-2)n).$$

Proof. At first, we consider the iterated double graph of G.

$$SO_{red}(D^k(G)) = \sum_{uv\in E(D^k(G))} \sqrt{(d_{D^k(G)}(u)-1)^2+(d_{D^k(G)}(v)-1)^2} \\ = \sum_{uv\in E_1'} \sqrt{(d_{D^k(G)}(u)-1)^2+(d_{D^k(G)}(v)-1)^2} \\ + \sum_{uv\in E_2'} \sqrt{(d_{D^k(G)}(u)-1)^2+(d_{D^k(G)}(v)-1)^2} \\ = \sum_{uv\in E_1'} \sqrt{(2^k d_G(u)-1)^2+(2^k d_G(v)-1)^2} \\ + \sum_{uv\in E_2'} \sqrt{(2^k d_G(u)-1)^2+(2^k d_G(v)-1)^2} \\ = \sum_{uv\in E_1'} \sqrt{(2^{2k}d_G^2(u)-2^{k+1}d_G(u)+1)+(2^{2k}d_G^2(v)-2^{k+1}d_G(v)+1)} \\ + \sum_{uv\in E_2'} \sqrt{(2^{2k}d_G^2(u)-2^{k+1}d_G(u)+1)+(2^{2k}d_G^2(v)-2^{k+1}d_G(v)+1)} \\ = \sum_{uv\in E(K_n)} \sqrt{2^{2k}(d_G^2(u)+d_G^2(v))-2^{k+1}(d_G(u)+d_G(v))+2)}.$$

Now, we consider the strong double graph of G, it is clear that $|E_1'| = |E_2'| = 2m$ and $|E_3'| = n$. We have:

$$SO_{red}(SD(G)) = \sum_{ab\in E(SD(G))} \sqrt{(d_{SD(G)}(a)-1)^2+(d_{SD(G)}(b)-1)^2} \\ = \sum_{uv\in E_1'} \sqrt{(d_{SD(G)}(u)-1)^2+(d_{SD(G)}(v)-1)^2}$$

$$
\begin{aligned}
&+ \sum_{uv \in E_2'} \sqrt{(d_{SD(G)}(u)-1)^2 + (d_{SD(G)}(v)-1)^2} \\
&+ \sum_{uu \in E_3'} \sqrt{(d_{SD(G)}(u)-1)^2 + (d_{SD(G)}(u)-1)^2} \\
&= \sum_{uv \in E(G)} \sqrt{(2d_G(u))^2 + (2d_G(v))^2} \\
&+ \sum_{uv \in E_2'} \sqrt{(2d_G(u))^2 + (2d_G(v))^2} \\
&+ \sum_{u \in V(G)} \sqrt{2(2d_G(u))^2} \\
&= 2SO(G) + 2 \sum_{uv \in E_2'} \sqrt{(d_G(u))^2 + (d_G(v))^2} + 4\sqrt{2}m.
\end{aligned}
$$

Finally, we consider the iterated strong double graph G, we have $|E_1'| = 2^k m$, $|E_2'| = 2^k m(2^k - 1) + 2^{k-1} n(2^k - k - 2)$ and $|E_3'| = n2^{k-1}(k+1)$ therefore:

$$
\begin{aligned}
SO_{red}(SD^k(G)) &= \sum_{ab \in E(SD^k(G))} \sqrt{(d_{SD^k(G)}(a)-1)^2 + (d_{SD^k(G)}(b)-1)^2} \\
&= \sum_{uv \in E_1'} \sqrt{(d_{SD^k(G)}(u)-1)^2 + (d_{SD^k(G)}(v)-1)^2} \\
&+ \sum_{uv \in E_2'} \sqrt{(d_{SD^k(G)}(u)-1)^2 + (d_{SD^k(G)}(v)-1)^2} \\
&+ \sum_{uu \in E_3'} \sqrt{(d_{SD^k(G)}(u)-1)^2 + (d_{SD^k(G)}(u)-1)^2} \\
&= \sum_{uv \in E(G)} [2^{2k}(d_G^2(u) + d_G^2(v)) + \\
&\qquad 2^{k+2}(2^{k-1}-1)(d_G(u) + d_G(v)) + 2^3(2^{k-1}-1)^2]^{1/2} \\
&+ \sum_{uv \in E_2'} [2^{2k}(d_G^2(u) + d_G^2(v)) + \\
&\qquad 2^{k+2}(2^{k-1}-1)(d_G(u) + d_G(v)) + 2^3(2^{k-1}-1)^2]^{1/2} \\
&+ \sum_{u \in V(G)} \sqrt{2}(2^k(d_G(u)+1) - 2)
\end{aligned}
$$

$$= 2^k \sum_{uv \in E(G)} [2^{2k}(d_G^2(u) + d_G^2(v)) + 2^{k+2}(2^{k-1} - 1)(d_G(u) + d_G(v)) + 2^3(2^{k-1} - 1)^2]^{1/2}$$
$$+ \sum_{uv \in E_2'} [2^{2k}(d_G^2(u) + d_G^2(v)) + 2^{k+2}(2^{k-1} - 1)(d_G(u) + d_G(v)) + 2^3(2^{k-1} - 1)^2]^{1/2}$$
$$+ \sqrt{2}(2^k(2m) + (2^k - 2)n).$$

□

Let G be a graph with n vertices and m edges. We consider

$$E_1'' = \{uv \mid uv \in E(G)\}, \quad |E_1''| = m$$
$$E_2'' = \{ue \mid u \in V(G), e \in E(G), u \sim e\} \quad |E_2''| = 2m$$
$$E_3'' = \{uv \mid uv \notin E(G)\}, \quad |E_3''| = \binom{n}{2} - m$$
$$E_4'' = \{ue \mid u \in V(G), e \in E(G), u \nsim e\} \quad |E_4''| = m(n-2).$$

Generalized transformation graph.
The generalized transformation graph G^{xy} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:
(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$, whereas α, β are not adjacent in G if $x = -$.
(b) $\alpha \in V(G)$and $\beta \in E(G)$, α, β are adjacent in G if $y = +$, whereas α, β are not adjacent in G if $y = -$ (see [10]).
Hence, $E(G^{++}) = E_1'' \cup E_2''$, $E(G^{+-}) = E_1'' \cup E_4''$, $E(G^{-+}) = E_2'' \cup E_3''$ and $E(G^{--}) = E_3'' \cup E_4''$.

Proposition 2.5. [1] *Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then degrees of point and line vertices in G^{xy} are*
(i) $d_{G^{++}}(u) = 2d_G(u)$ *and* $d_{G^{++}}(e) = 2$.
(ii) $d_{G^{+-}}(u) = m$ *and* $d_{G^{+-}}(e) = n - 2$.
(iii) $d_{G^{-+}}(u) = n - 1$ *and* $d_{G^{-+}}(e) = 2$.
(iv) $d_{G^{--}}(u) = n + m - 1 - 2d_G(u)$ *and* $d_{G^{--}}(e) = n - 2$.

Theorem 2.6. *Let G be a graph with n vertices and m edges. Then*

$$SO_{red}(G^{++}) = \sum_{uv\in E(G)} 2\sqrt{(d_G^2(u)+d_G^2(v)) - (d_G(u)+d_G(v)) + 1/2}$$
$$+ \sum_{u\in V(G)} 2d_G(u)\sqrt{d_G^2(u) - d_G(u) + 1/2}.$$
$$SO_{red}(G^{+-}) = \sqrt{2}m(m-1) + m(n-2)\sqrt{(m^2+n^2) - 2(m+3n) + 10}.$$
$$SO_{red}(G^{-+}) = \sqrt{2}(n-1)(\binom{n}{2} - m) + 2m\sqrt{n^2-2n+2}.$$
$$SO_{red}(G^{--}) = \sum_{uv\in E(G)} [2(n+m-2)^2 + 4(d_G^2(u)+d_G^2(v))$$
$$- 4(n+m-2)(d_G(u)+d_G(v))]^{1/2}$$
$$+ \sum_{u\in V(G)} (m-d_G(u))\sqrt{(n+m-2-2d_G(u))^2 + (n-3)^2}.$$

Proof. • Let $x = y = +$. We have

$$SO_{red}(G^{++}) = \sum_{ab\in E(G^{++})} \sqrt{(d_{G^{++}}(a)-1)^2 + (d_{G^{++}}(b)-1)^2}$$
$$= \sum_{uv\in E_1''} \sqrt{(d_{G^{++}}(u)-1)^2 + (d_{G^{++}}(v)-1)^2}$$
$$+ \sum_{ue\in E_2''} \sqrt{(d_{G^{++}}(u)-1)^2 + (d_{G^{++}}(e)-1)^2}$$
$$= \sum_{uv\in E(G)} \sqrt{(2d_G(u)-1)^2 + (2d_G(v)-1)^2}$$
$$+ \sum_{ue\in E_2''} \sqrt{(2d_G(u)-1)^2 + (1)^2}$$
$$= \sum_{uv\in E(G)} 2\sqrt{(d_G^2(u)+d_G^2(v)) - (d_G(u)+d_G(v)) + 1/2}$$
$$+ \sum_{ue\in E_2''} 2\sqrt{d_G^2(u) - d_G(u) + 1/2}$$
$$= \sum_{uv\in E(G)} 2\sqrt{(d_G^2(u)+d_G^2(v)) - (d_G(u)+d_G(v)) + 1/2}$$

$$+ \sum_{u \in V(G)} 2d_G(u)\sqrt{d_G^2(u) - d_G(u) + 1/2}.$$

- Let $x = +, y = -$. We have

$$\begin{aligned}
SO_{red}(G^{+-}) &= \sum_{ab \in E(G^{+-})} \sqrt{(d_{G^{+-}}(a)-1)^2 + (d_{G^{+-}}(b)-1)^2} \\
&= \sum_{uv \in E_1''} \sqrt{(d_{G^{+-}}(u)-1)^2 + (d_{G^{+-}}(v)-1)^2} \\
&+ \sum_{ue \in E_4''} \sqrt{(d_{G^{+-}}(u)-1)^2 + (d_{G^{+-}}(e)-1)^2} \\
&= \sum_{uv \in E_1''} \sqrt{(m-1)^2 + (m-1)^2} \\
&+ \sum_{ue \in E_4''} \sqrt{(m-1)^2 + (n-3)^2} \\
&= \sum_{uv \in E_1''} \sqrt{2}(m-1) + \sum_{ue \in E_4''} \sqrt{(m^2+n^2) - 2(m+3n) + 10} \\
&= \sqrt{2}m(m-1) + m(n-2)\sqrt{(m^2+n^2) - 2(m+3n) + 10}.
\end{aligned}$$

- Let $x = -, y = +$. We have

$$\begin{aligned}
SO_{red}(G^{-+}) &= \sum_{ab \in E(G^{-+})} \sqrt{(d_{G^{-+}}(a)-1)^2 + (d_{G^{-+}}(b)-1)^2} \\
&= \sum_{uv \in E_3''} \sqrt{(d_{G^{-+}}(u)-1)^2 + (d_{G^{-+}}(v)-1)^2} \\
&+ \sum_{ue \in E_2''} \sqrt{(d_{G^{-+}}(u)-1)^2 + (d_{G^{-+}}(e)-1)^2} \\
&= \sum_{uv \in E_3''} \sqrt{(n-1)^2 + (n-1)^2} + \sum_{ue \in E_2''} \sqrt{(n-1)^2 + (1)^2} \\
&= \sqrt{2}(n-1)(\binom{n}{2} - m) + 2m\sqrt{n^2 - 2n + 2}.
\end{aligned}$$

- Let $x = y = -$. We have

$$SO_{red}(G^{--}) = \sum_{ab \in E(G^{--})} \sqrt{(d_{G^{--}}(a)-1)^2 + (d_{G^{--}}(b)-1)^2}$$

$$
\begin{aligned}
&= \sum_{uv \in E_3''} \sqrt{(d_{G^{--}}(u)-1)^2 + (d_{G^{--}}(v)-1)^2} \\
&+ \sum_{ue \in E_4''} \sqrt{(d_{G^{--}}(u)-1)^2 + (d_{G^{--}}(e)-1)^2} \\
&= \sum_{uv \in E_3''} \sqrt{(n+m-2-2d_G(u))^2 + (n+m-2-2d_G(v))^2} \\
&+ \sum_{ue \in E_4''} \sqrt{(n+m-2-2d_G(u))^2 + (n-3)^2} \\
&= \sum_{uv \in E(G)} [2(n+m-2)^2 + 4(d_G^2(u) + d_G^2(v)) \\
&- 4(n+m-2)(d_G(u)+d_G(v))]^{1/2} \\
&+ \sum_{u \in V(G)} (m - d_G(u)) \sqrt{(n+m-2-2d_G(u))^2 + (n-3)^2}.
\end{aligned}
$$

□

Let G be a graph with n vertices and m edges. We consider

$$
\begin{aligned}
\hat{E}_1 &= \{uv \mid uv \in E(G)\}, \quad |\hat{E}_1| = m \\
\hat{E}_2 &= \{uv \mid uv \notin E(G)\}, \quad |\hat{E}_2| = \binom{n}{2} - m \\
\hat{E}_3 &= \{ue \mid u \in V(G), e \in E(G), u \nsim e\}, \quad |\hat{E}_3| = m(n-2) \\
\hat{E}_4 &= \{ue \mid u \in V(G), e \in E(G), u \sim e\}, \quad |\hat{E}_4| = 2m \\
\hat{E}_5 &= \{ef \mid e, f \in E(G), e \sim f\} \quad |\hat{E}_1| = \binom{m}{2}.
\end{aligned}
$$

Complement of generalized transformation graph.
Let G denote $\overline{G}$ as complement of G.
Hence, $E(\overline{G^{-+}}) = \hat{E}_1 \cup \hat{E}_3 \cup \hat{E}_5$, $E(\overline{G^{++}}) = \hat{E}_2 \cup \hat{E}_3 \cup \hat{E}_5$, $E(\overline{G^{+-}}) = \hat{E}_2 \cup \hat{E}_4 \cup \hat{E}_5$ and $E(\overline{G^{--}}) = \hat{E}_1 \cup \hat{E}_4 \cup \hat{E}_5$.

Proposition 2.7. [9] *Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then degrees of point and line vertices in $\overline{G^{xy}}$ are*
(i) $d_{\overline{G^{++}}}(u) = n + m - 1 - 2d_G(u)$ *and* $d_{\overline{G^{++}}}(e) = n + m - 3$.
(ii) $d_{\overline{G^{+-}}}(u) = n - 1$ *and* $d_{\overline{G^{+-}}}(e) = m + 1$.
(iii) $d_{\overline{G^{-+}}}(u) = m$ *and* $d_{\overline{G^{-+}}}(e) = n + m - 3$.
(iv) $d_{\overline{G^{--}}}(u) = 2d_G(u)$ *and* $d_{\overline{G^{--}}}(e) = m + 1$.

Theorem 2.8. *Let G be a graph with n vertices and m edges. Then*

$$SO_{red}(\overline{G^{-+}}) = \sqrt{2}m(m-1) + m(n-2)\sqrt{(m-1)^2+(n+m-4)^2} + \binom{m}{2}\sqrt{2}(n+m-4).$$

$$SO_{red}(\overline{G^{++}}) = \sum_{uv\in\hat{E}_2}\sqrt{(n+m-2-2d_G(u))^2+(n+m-2-2d_G(v))^2} + \sum_{u\in V(G)}(m-d_G(u))\sqrt{(n+m-2-2d_G(u))^2+(n+m-4)^2} + \binom{m}{2}\sqrt{2}(n+m-4).$$

$$SO_{red}(\overline{G^{+-}}) = \sqrt{2}(\binom{n}{2}-m)(n-2) + 2m\sqrt{(n-2)^2+m^2} + \sqrt{2}\binom{m}{2}m.$$

$$SO_{red}(\overline{G^{--}}) = \sum_{uv\in\hat{E}_1}\sqrt{(2d_G(u)-1)^2+(2d_G(v)-1)^2} + \sum_{u\in V(G)} d_G(u)\sqrt{(2d_G(u)-1)^2+m^2} + \sqrt{2}m\binom{m}{2}.$$

Proof.

- Let $x=-, y=+$. We have

$$\begin{aligned} SO_{red}(\overline{G^{-+}}) &= \sum_{ab\in E(\overline{G^{-+}})}\sqrt{(d_{\overline{G^{-+}}}(a)-1)^2+(d_{\overline{G^{-+}}}(b)-1)^2} \\ &= \sum_{uv\in\hat{E}_1}\sqrt{(d_{\overline{G^{-+}}}(u)-1)^2+(d_{\overline{G^{-+}}}(v)-1)^2} \\ &+ \sum_{ue\in\hat{E}_3}\sqrt{(d_{\overline{G^{-+}}}(u)-1)^2+(d_{\overline{G^{-+}}}(e)-1)^2} \\ &+ \sum_{ef\in\hat{E}_5}\sqrt{(d_{\overline{G^{-+}}}(e)-1)^2+(d_{\overline{G^{-+}}}(f)-1)^2} \\ &= \sum_{uv\in\hat{E}_1}\sqrt{(m-1)^2+(m-1)^2} \\ &+ \sum_{ue\in\hat{E}_3}\sqrt{(m-1)^2+(n+m-4)^2} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{ef \in \hat{E}_5} \sqrt{(n+m-4)^2 + (n+m-4)^2} \\
&= \sum_{uv \in \hat{E}_1} \sqrt{2}(m-1) + \sum_{ue \in \hat{E}_3} \sqrt{(m-1)^2 + (n+m-4)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{2}(n+m-4) \\
&= \sqrt{2}m(m-1) + m(n-2)\sqrt{(m-1)^2 + (n+m-4)^2} \\
&+ \binom{m}{2}\sqrt{2}(n+m-4).
\end{aligned}
$$

- Let $x = y = +$. We have

$$
\begin{aligned}
SO_{red}(\overline{G^{++}}) &= \sum_{ab \in E(\overline{G^{++}})} \sqrt{(d_{\overline{G^{++}}}(a)-1)^2 + (d_{\overline{G^{-+}}}(b)-1)^2} \\
&= \sum_{uv \in \hat{E}_2} \sqrt{(d_{\overline{G^{++}}}(u)-1)^2 + (d_{\overline{G^{++}}}(v)-1)^2} \\
&+ \sum_{ue \in \hat{E}_3} \sqrt{(d_{\overline{G^{++}}}(u)-1)^2 + (d_{\overline{G^{++}}}(e)-1)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{(d_{\overline{G^{++}}}(e)-1)^2 + (d_{\overline{G^{++}}}(f)-1)^2} \\
&= \sum_{uv \in \hat{E}_2} \sqrt{(n+m-2-2d_G(u))^2 + (n+m-2-2d_G(v))^2} \\
&+ \sum_{ue \in \hat{E}_3} \sqrt{(n+m-2-2d_G(u))^2 + (n+m-4)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{(n+m-4)^2 + (n+m-4)^2} \\
&= \sum_{uv \in \hat{E}_2} \sqrt{(n+m-2-2d_G(u))^2 + (n+m-2-2d_G(v))^2} \\
&+ \sum_{u \in V(G)} (m-d_G(u))\sqrt{(n+m-2-2d_G(u))^2 + (n+m-4)^2} \\
&+ \binom{m}{2}\sqrt{2}(n+m-4).
\end{aligned}
$$

- Let $x = +, y = -$. We have

$$\begin{aligned}
SO_{red}(\overline{G^{+-}}) &= \sum_{ab \in E(\overline{G^{+-}})} \sqrt{(d_{\overline{G^{+-}}}(a)-1)^2 + (d_{\overline{G^{+-}}}(b)-1)^2} \\
&= \sum_{uv \in \hat{E}_2} \sqrt{(d_{\overline{G^{+-}}}(u)-1)^2 + (d_{\overline{G^{+-}}}(v)-1)^2} \\
&+ \sum_{ue \in \hat{E}_4} \sqrt{(d_{\overline{G^{+-}}}(u)-1)^2 + (d_{\overline{G^{+-}}}(e)-1)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{(d_{\overline{G^{+-}}}(e)-1)^2 + (d_{\overline{G^{+-}}}(f)-1)^2} \\
&= \sum_{uv \in \hat{E}_2} \sqrt{(n-2)^2 + (n-2)^2} \\
&+ \sum_{ue \in \hat{E}_4} \sqrt{(n-2)^2 + (m)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{m^2 + m^2} \\
&= \sqrt{2}(\binom{n}{2} - m)(n-2) + 2m\sqrt{(n-2)^2 + m^2} + \sqrt{2}\binom{m}{2}m.
\end{aligned}$$

- Let $x = y = -$. We have

$$\begin{aligned}
SO_{red}(\overline{G^{--}}) &= \sum_{ab \in E(\overline{G^{--}})} \sqrt{(d_{\overline{G^{--}}}(a)-1)^2 + (d_{\overline{G^{--}}}(b)-1)^2} \\
&= \sum_{uv \in \hat{E}_1} \sqrt{(d_{\overline{G^{--}}}(u)-1)^2 + (d_{\overline{G^{--}}}(v)-1)^2} \\
&+ \sum_{ue \in \hat{E}_4} \sqrt{(d_{\overline{G^{--}}}(u)-1)^2 + (d_{\overline{G^{--}}}(e)-1)^2} \\
&+ \sum_{ef \in \hat{E}_5} \sqrt{(d_{\overline{G^{--}}}(e)-1)^2 + (d_{\overline{G^{--}}}(f)-1)^2} \\
&= \sum_{uv \in \hat{E}_1} \sqrt{(2d_G(u)-1)^2 + (2d_G(v)-1)^2} \\
&+ \sum_{ue \in \hat{E}_4} \sqrt{(2d_G(u)-1)^2 + (m)^2} + \sum_{ef \in \hat{E}_5} \sqrt{(m)^2 + (m)^2}
\end{aligned}$$

$$= \sum_{uv \in \hat{E}_1} \sqrt{(2d_G(u)-1)^2 + (2d_G(v)-1)^2}$$
$$+ \sum_{u \in V(G)} d_G(u)\sqrt{(2d_G(u)-1)^2 + m^2} + \sqrt{2}m\binom{m}{2}. \qquad \square$$

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