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On intersection hypergraph of $\mathbb{Z}_n$

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Abstract. The aim of this paper is to study the intersection hypergraph on the subgroups of $\mathbb{Z}_n$. We provide characterizations for hypertrees and star structures of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ and prove that the intersection and co-maximal hypergraph of a finite abelian group are isomorphic. Also, we investigate the planarity and non-planarity of the hypergraph.

1. Introduction

Many interesting studies have been carried out on graphs connecting various algebraic structures, providing deeper insights into these structures. In [7], P. Devi and R. Rajkumar studied the complement of the intersection graph of subgroups of a group. They classified all finite groups for which this graph is totally disconnected, bipartite, complete bipartite, a tree, a star graph, or C_3-free. They also characterized all finite groups whose complement of the intersection graph is planar. In [9], M. Saha *et al.* studied various structural properties of co-maximal subgroup graph of $\mathbb{Z}_n$. In particular, they characterized the value of n for which the graph is Hamiltonian, Eulerian, perfect etc. The study of hypergraphs, a generalization of graph, associated with various algebraic structures has also emerged as an interesting area. In [6], P.J. Cameron *et al.* studied various hypergraphs related to structural properties of elements of group *viz* power hypergraphs, enhanced power hypergraphs, etc. In [8], A. Lucchini introduced hypergraphs on elements of a finite group where hyperedges are minimal generating sets, generalizing the classical generating graph to obtain generating properties for groups requiring more than two generators.

Motivated from these studies we define more hypergraphs on groups, particularly focusing on the subgroups of a group, since subgroup is one of

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the prominent tool to analyse the group. Graphs help us study pairwise relationships between vertices. Clique hypergraph of a group reflects group relationships, not just pairwise. The clique hypergraph of a graph $\mathcal{G}$, denoted by $Cl_{\mathcal{H}}(\mathcal{G})$, is the hypergraph with vertex set $V(\mathcal{G})$ whose edge set is the family of vertex sets of maximal cliques in graph $\mathcal{G}$ [11]. As many results *viz.* star structures, tree structures, girth, chromatic number, etc. of the clique hypergraph does not follow from its original graph, the study of clique hypergraphs are important. In [2] and [1], we have obtained some results about the clique hypergraphs of the complement of intersection graph and co-maximal graph on D_n.

Here are some preliminaries required for our study. A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of vertices and $E(\mathcal{H})$ is a set of hyperedges, where each hyperedge is a subset of $V(\mathcal{H})$. A hypergraph $\mathcal{H}' = (V'(\mathcal{H}'), E'(\mathcal{H}'))$ is called a *subhypergraph* of $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ if $V'(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E'(\mathcal{H}') \subseteq E(\mathcal{H})$. The *girth* of a hypergraph is the length of a shortest cycle it contains. A hypergraph is called a *star* if there is a vertex which belongs to all hyperedges. The *incidence graph (or bipartite representation)* $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$ is a bipartite graph with vertex set $V(\mathcal{H}) \cup E(\mathcal{H})$ and a vertex $v \in V(\mathcal{H})$ is adjacent to a vertex $e \in E(\mathcal{H})$ if and only if $v \in e$ in $\mathcal{H}$. A *proper vertex-coloring* (often simply called a proper coloring) of a hypergraph $\mathcal{H}$ is an assignment of colors to the vertices of $\mathcal{H}$ such that no hyperedge contains all vertices of the same color. The *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the minimum number of colors needed for a proper vertex-coloring of $\mathcal{H}$. Two simple hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *isomorphic* if there exists a bijection between their vertex sets such that any subset of vertices forms a hyperedge in $\mathcal{H}_1$ if and only if the corresponding subset of vertices forms a hyperedge in $\mathcal{H}_2$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic, then we denote it by $\mathcal{H}_1 \cong \mathcal{H}_2$. The *empty hypergraph* is the hypergraph with empty vertex set and empty hyperedge set. For more details on graphs and hypergraphs, one may refer [4, 11, 13], etc.

Definition 1.1. [2] Let G be a group and S be the set of all non-trivial proper subgroups of G. *The intersection hypergraph of* G, denoted by $\tilde{\Gamma}_{\mathcal{H}}(G)$, is a hypergraph whose vertex set,
$V = \{H \in S \mid H \cap K = \{e\} \text{ for some } K \in S\}$ and $E \subseteq V$ is a hyperedge if and only if

1. For distinct $H, K \in E$, $H \cap K = \{e\}$.
2. There does not exist $E' \supset E$ which satisfies (1).

Definition 1.2. [7] The *complement of the intersection graph of subgroups of a group* G, denoted by $\mathscr{I}^c(G)$, is the graph whose vertex set is the set of all nontrivial proper subgroups of G and its distinct vertices H and K are adjacent if and only if $H \cap K = \{e\}$.

Let $\mathscr{I}^c(G)^\star$ denote the graph obtained from $\mathscr{I}^c(G)$ by removing all the isolated vertices. Hence, $\tilde{\Gamma}_{\mathcal{H}}(G)$ is the clique hypergraph of $\mathscr{I}^c(G)^\star$.

In this paper, our aim is to study the intersection hypergraph on the subgroups of $\mathbb{Z}_n$. In Section 2, we study various structural properties of intersection hypergraph of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$, provide characterizations for the hypertree and star structures of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ and prove that intersection and co-maximal hypergraph of a finite abelian group are isomorphic. In Section 3, we discuss the possibilities of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ which can be embedded on the plane, torus and projective plane.

2. Structural properties of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$

In this section, we discuss girth, chromatic number and characterization for star and hypertree structures of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. Also, we prove that intersection and co-maximal hypergraph of a finite abelian group are isomorphic.

The following lemma follows from Lemma 2.1 in [9]. We will use the following simple result at various places.

Lemma 2.1. *Let* $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, *where* $p_1, p_2, \dots, p_k$ *are distinct primes and* $\alpha_1, \alpha_2, \dots, \alpha_k$ *are non-negative integers. If* $H_1 = \langle p_1^{r_1}p_2^{r_2}\dots p_k^{r_k}\rangle$ *and* $H_2 = \langle p_1^{s_1}p_2^{s_2}\dots p_k^{s_k}\rangle$, *where* $0 \leqslant r_i, s_i \leqslant \alpha_i$ *for* $i = 1, 2, \dots, k$, *are two subgroups of* $\mathbb{Z}_n$, *then* $lcm(p_1^{r_1}p_2^{r_2}\dots p_k^{r_k}, p_1^{s_1}p_2^{s_2}\dots p_k^{s_k}) = n$ *if and only if* $H_1 \cap H_2 = \{e\}$.

Lemma 2.2. *For* $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, *where* $p_1, p_2, \dots, p_k$ *are distinct primes and* $\alpha_1, \alpha_2, \dots, \alpha_k$ *are non-negative integers, a proper subgroup* $\langle p_1^{r_1}p_2^{r_2}\dots p_k^{r_k}\rangle$ *of* $\mathbb{Z}_n$, *where* $0 \leqslant r_i \leqslant \alpha_i$ *for* $i = 1, 2, \dots, k$, *is in the vertex set of* $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ *if and only if* $r_i = \alpha_i$ *for some* $i = 1, 2, \dots, k$.

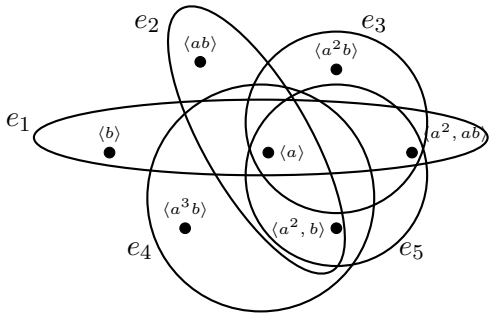
Proof. Follows from Lemma 2.1. □

Theorem 2.3. $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ *is empty if and only if* n *is a power of prime.*

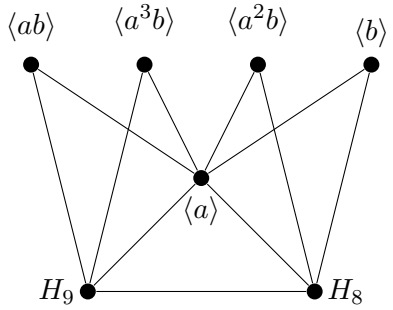
Proof. Suppose that n is a prime power. Then, for any non-trivial proper subgroups H, K of $\mathbb{Z}_n$, either $H \subseteq K$ or $K \subseteq H$. Hence, $H \cap K \neq \{e\}$ and therefore, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is empty.

Conversely, suppose that n is not a power of prime. Then, there exist two distinct prime divisors, say p and q, of n. Hence, by Cauchy's Theorem for Abelian Groups, there exist two distinct subgroups H_p and H_q of order p and q respectively with $H_p \cap H_q = \{e\}$. Thus, $H_p, H_q \in V(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ and therefore, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is nonempty. □

Note. Since $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is empty for $\omega(n) = 1$, we exclude this case and consider $\omega(n) \geqslant 2$ throughout the article, where $\omega(n)$ denotes the number of distinct prime divisors of n. If graph is a star, then its clique hypergraph is also a star hypergraph. But the converse need not be true in general. For example, $Co_{\mathcal{H}}(D_4)$ is the clique hypergraph of $\Gamma^{\star}(D_4)$, i.e., $Co_{\mathcal{H}}(D_4) = Cl_{\mathcal{H}}(\Gamma^{\star}(D_4))$. The Figure 1 depicts that $Co_{\mathcal{H}}(D_4)$ is a star hypergraph, but $\Gamma^{\star}(D_4)$ is not a star graph.



(a) $Co_{\mathcal{H}}(D_4)$



(b) $\Gamma^{\star}(D_4)$

Figure 1

The following theorem characterizes the star structures of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$.

Theorem 2.4. *$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a star hypergraph if and only if $n = p_1^{\alpha} p_2$, where p_1, p_2 are distinct prime numbers and α is a positive integer.*

Proof. Suppose $n = p_1^{\alpha_1} p_2$, where p_1, p_2 are distinct prime numbers and α_1 is a positive integer. Consider the vertex $\langle p_1^{\alpha_1} \rangle$ of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. By Lemma 2.2, a vertex of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ different from $\langle p_1^{\alpha_1} \rangle$ is in the form of $\langle p_1^l p_2 \rangle$, where $0 \leqslant l \leqslant \alpha_1 - 1$. Then, as $lcm(p_1^l p_2, p_1^{\alpha_1}) = p_1^{\alpha_1} p_2$, by Lemma 2.1, we have $\langle p_1^l p_2 \rangle \cap \langle p_1^{\alpha_1} \rangle = \{e\}$. Therefore, $\langle p_1^{\alpha_1} \rangle$ must belong to all the hyperedges of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ by the maximality condition of the hyperedges. Thus, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a star hypergraph.

Conversely, suppose that $n \neq p_1^{\alpha_1} p_2$, where p_1, p_2 are distinct prime numbers and α_1 is a positive integer. Since there exists a hyperedge containing all and only the subgroups of $\mathbb{Z}_n$ with prime order, it is enough to show that for each prime p order subgroup H_p there exists a vertex K in $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ such that $H_p \cap K \neq \{e\}$. To prove this, consider the following cases:

Case 1. Suppose $\omega(n) = 2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1, p_2 are distinct primes and $\alpha_1, \alpha_2 \geqslant 2$. Note that $\langle p_1^{\alpha_1 - 1} p_2^{\alpha_2} \rangle$ and $\langle p_1^{\alpha_1} p_2^{\alpha_2 - 1} \rangle$ are the only subgroups of $\mathbb{Z}_n$ with prime order. Observe that $\langle p_1^{\alpha_1 - 1} p_2^{\alpha_2} \rangle \cap \langle p_2^{\alpha_2} \rangle \neq \{e\}$ and $\langle p_1^{\alpha_1} p_2^{\alpha_2 - 1} \rangle \cap \langle p_1^{\alpha_1} \rangle \neq \{e\}$. Hence, no vertex of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ belongs to all the hyperedges of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ and consequently, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is not a star hypergraph.

Case 2. Suppose $\omega(n) \geqslant 3$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, where $p_1, \dots, p_k$ are distinct primes and $k \geqslant 3$. For each vertex H_{p_i} with order p_i, $H_{p_i} \cap \langle p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_i - 1} p_{i+1}^{\alpha_i + 1} \cdots p_k^{\alpha_k} \rangle \neq \{e\}$. Hence, no vertex of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ belong to all the hyperedges of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ and consequently, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is not a star hypergraph. □

To ensure the connectedness of a graph without cycles, tree is an important tool. If graph is a tree, then its clique hypergraph is also a hypertree. But the converse need not be true in general (Figure 1). While a tree of a graph only tells us whether the graph is connected, a hypertree structure of its clique hypergraph shows whether the maximal cliques satisfy the Helly property. The following theorem characterizes when $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a hypertree.

Definition 2.3. [11] A *host graph* G for a hypergraph is a connected graph on the same vertex set such that every hyperedge induces a connected subgraph of G. A hypergraph $\mathcal{H} = (X, \mathcal{D})$ is called a *hypertree* if there exists a host tree $T = (X, E)$ such that each edge $D \in \mathcal{D}$ induces a subtree in T.

Theorem 2.5. $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a hypertree if and only if $n = p_1^{\alpha} p_2$ or $n = p_1 p_2 p_3$, where p_1, p_2, p_3 are distinct primes and $\alpha \geqslant 1$.

Proof. Consider the following cases:

Case 1. Suppose that $n = p_1^{\alpha_1} p_2$, where p_1, p_2 are distinct primes and $\alpha_1 \geqslant 1$. Then, by Theorem 2.4, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a star hypergraph and therefore, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a hypertree.

Case 2. Now, suppose that $n = p_1 p_2 p_3$, where p_1, p_2, p_3 are distinct primes. The Figure 2(a) depicts a host tree for $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. Hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a hypertree.

Conversely, suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers and $k \geqslant 2$ with $n \neq p_1^{\alpha_1} p_2$ and $n \neq p_1 p_2 p_3$. We will prove that there is no host tree for $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. Let $G_{\mathcal{H}}$ be a host graph for $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. Note that any two vertices in the same hyperedge of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ have a path in $G_{\mathcal{H}}$. Consider the vertices $H_1 = \langle p_1^{\alpha_1} \rangle, H_2 = \langle p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle, H_3 = \langle p_1^{\alpha_1} p_2^{\alpha_2 - 1} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle$ and $H_4 = \langle p_1^{\alpha_1 - 1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle$. Observe that $H_i \cap H_{i+1} = \{e\}, \forall i \in \{1, 2, 3\}$ and $H_1 \cap H_4 = \{e\}$. Thus, there exist a path P_1 from H_1 to H_2, a path P_2 from H_2 to H_3, a path P_3 from H_3 to H_4 and a path P_4 from H_4 to H_1 in $G_{\mathcal{H}}$. Therefore, $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is a cycle in $G_{\mathcal{H}}$ and hence, $G_{\mathcal{H}}$ is not a tree, see Figure 2(b). Since $G_{\mathcal{H}}$ is an arbitrary host graph of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$, we conclude that there does not exist a host tree for $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$. Thus, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is not a hypertree.

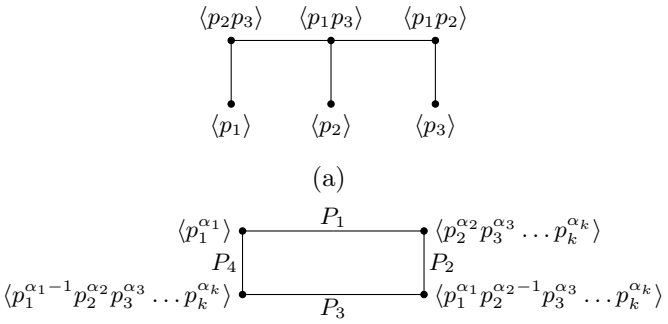


Figure 2

□

Corollary 2.5.1. *$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1 p_2 p_3})$, where p_1, p_2 and p_3 are distinct primes, is a hypertree, but not a star hypergraph.*

Corollary 2.5.2. *If $\omega(n) > 3$, then $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is not a hypertree.*

The existence of a cycle in $\mathcal{G}$ not necessarily imply the existence of a cycle in its clique hypergraph. But the converse is true. For example, $gr(\mathscr{I}^c(\mathbb{Z}_{p_1p_2p_3})^\star) = 3$, but $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3})) = \infty$.

Lemma 2.6. [7]*Let G be a finite group. Then, the following are equivalent.*

1. *G is isomorphic to one of $\mathbb{Z}_{p^\alpha}(\alpha \geqslant 1), Q_{2^\alpha}$ or $\mid G \mid = p^\alpha q^\beta(\alpha, \beta \geqslant 1)$ with G has a unique subgroup of each of distinct prime orders p, q;*
2. *$\mathscr{I}^c(G)$ is bipartite;*
3. *$\mathscr{I}^c(G)$ is C_3- free.*

Remark 2.7. Let $\mathcal{G}$ be a graph and $Cl_{\mathcal{H}}(\mathcal{G})$ denote the clique hypergraph of $\mathcal{G}$. Then, $gr(\mathcal{G}) = gr(Cl_{\mathcal{H}}(\mathcal{G}))$ if and only if $\mathcal{G}$ contains no induced subgraph isomorphic to the graph shown in Figure 3.

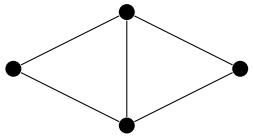


Figure 3

The following result follows from Lemma 2.6 and Remark 2.7.

Theorem 2.8. *Let G be a finite group. Then, $gr(\mathscr{I}^c(G)) = gr(\tilde{\Gamma}_{\mathcal{H}}(G))$ if and only if G is isomorphic to one of $\mathbb{Z}_{p^\alpha}(\alpha \geqslant 1), Q_{2^\alpha}$ or $\mid G \mid = p^\alpha q^\beta(\alpha, \beta \geqslant 1)$ with G has a unique subgroup of each of distinct prime orders p, q.*

Theorem 2.9. *Let p_1, p_2 and p_3 be distinct primes. Then, the girth $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is:*

$$gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = \begin{cases} \infty, \text{ if } n = p_1^{\alpha_1}p_2, \text{where } \alpha_1 \geqslant 1 \text{ or } n = p_1p_2p_3, \\ 4, \text{ if } n = p_1^{\alpha_1}p_2^{\alpha_2}, \text{where } \alpha_1, \alpha_2 \geqslant 2, \\ 2, \text{ otherwise.} \end{cases}$$

Proof. Consider the following cases:

Case 1. If $n = p_1^{\alpha_1}p_2$, where $\alpha_1 \geqslant 1$, then, by Theorem 2.4, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a 2-uniform star hypergraph, i.e., simply a star graph. Hence, there does not exist a cycle and therefore, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = \infty$.

Case 2. If $n = p_1p_2p_3$, then as depicted in the Figure 4(a), $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{pqr})$ does not contain a cycle and therefore, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = \infty$.

Case 3. If $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where $\alpha_1, \alpha_2 \geqslant 2$, then, by Lemma 2.6 and Theorem 2.8, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = gr(\mathscr{I}^c(\mathbb{Z}_n)) \geqslant 4$. The Figure 4 (b) depicts that $\langle p_1p_2^{\alpha_2}\rangle e_1\langle p_1^{\alpha_1}p_2\rangle e_2\langle p_2^{\alpha_2}\rangle e_3\langle p_1^{\alpha_1}\rangle e_4\langle p_1p_2^{\alpha_2}\rangle$ is a cycle in $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ of length 4 and therefore, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 4$.

Case 4. Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$, where α_1, α_2 and α_3 are positive integers and atleast one of α_i is greater than 1. Without loss of generality, assume that $\alpha_1 > 1$. The Figure 4 (c) depicts that $\langle p_1^{\alpha_1}p_2^{\alpha_2}\rangle e_1\langle p_1^{\alpha_1}p_3^{\alpha_3}\rangle e_2\langle p_1^{\alpha_1}p_2^{\alpha_2}\rangle$ is a cycle in $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ of length 2 and consequently, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 2$.

Case 5. If $\omega(n) \geqslant 4$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $k \geqslant 4$, $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers, then Figure 4 (d)

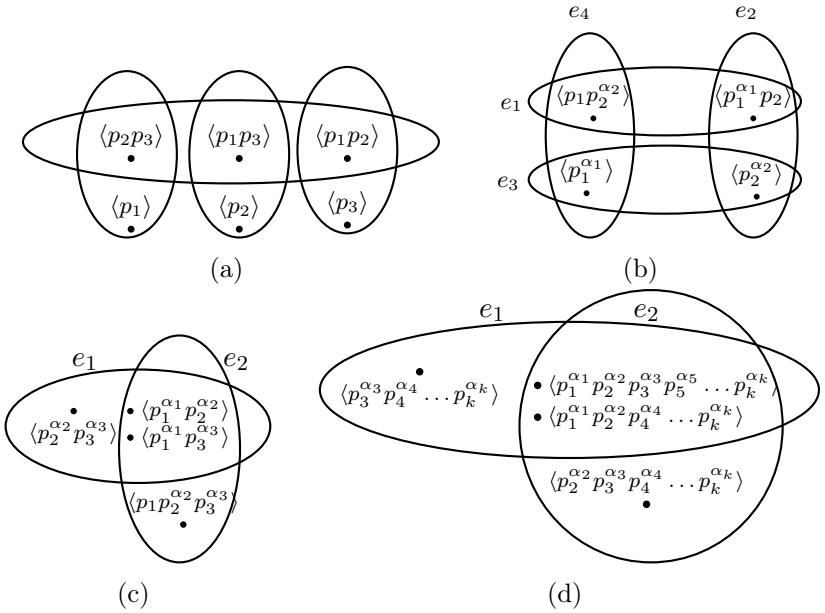


Figure 4

depicts that

$$\langle p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}p_5^{\alpha_5}\dots p_k^{\alpha_k}\rangle e_1\langle p_1^{\alpha_1}p_2^{\alpha_2}p_4^{\alpha_4}\dots p_k^{\alpha_k}\rangle e_2\langle p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}p_5^{\alpha_5}\dots p_k^{\alpha_k}\rangle$$

is a cycle in $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ of length 2 and consequently, $gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 2$. □

Note that the chromatic number of a graph and its clique hypergraph need not be equal in general. But the clique chromatic number of the graph and the chromatic number of its clique hypergraph coincides.

In the following theorem we establish that the chromatic number $\chi(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ of $\mathbb{Z}_n$ is 2 for all n.

Theorem 2.10. *The chromatic number,* $\chi(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 2$ *for all* n.

Proof. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers. Note that $V(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = A \cup B$, where $A\{\langle p_1^{\alpha_1}p_2^{\beta_2}p_3^{\beta_3}\dots p_k^{\beta_k}\rangle \mid 0 \leqslant \beta_i \leqslant \alpha_i \text{ for } i = 2, 3, \dots, k\}$ and $B = \{\langle p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\dots p_k^{\beta_k}\rangle \mid 0 \leqslant \beta_1 \leqslant \alpha_1 - 1 \text{ and } 0 \leqslant \beta_i \leqslant \alpha_i \text{ for } i = 2, 3, \dots, k\}$.

If e_1 is a hyperedge of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ containing the vertices from A only, then, by Lemma 2.1, $\langle p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_k^{\alpha_k}\rangle \cap H = \{e\}$, for all $H \in e_1$. Thus, $\langle p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_k^{\alpha_k}\rangle$ must belong to e_1, otherwise contradicting the maximality condition of e_1. Therefore, e_1 cannot contain vertices from A only. Also, note that no two vertices of the set B are adjacent. Hence, any hyperedge of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ cannot contain only vertices from the set B. Thus, each hyperedge has at least one vertex from A and B both. Assign the color c_1 to the vertices in A and the color c_2 to the vertices in B. This is a proper coloring of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ and consequently, $\chi(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 2$. □

In the next theorem, we establish that the intersection hypergraph and co-maximal hypergraph of a finite abelian group are isomorphic. For this, we need the following.

Definition 2.4. [1] Let G be a group and S be the set of all non-trivial proper subgroups of G. The *co-maximal hypergraph of* G, denoted by $Co_{\mathcal{H}}(G)$, is an undirected hypergraph whose vertex set, $V = \{H \in S \mid HK = G \text{ for some } K \in S\}$ and $E \subseteq V$ is a hyperedge if and only if

1. for distinct $H, K \in E$, $HK = G$,
2. there does not exist $E' \supset E$ which satisfies (1).

A subgroup G has a *dual* if there exists a duality from its subgroup lattice onto the subgroup lattice of a group $\bar{G}$, i.e., there exists a bijective map $\delta : L(G) \to L(\bar{G})$ such that for all $H, K \in L(G), H \leqslant K$ if and only if $\delta(K) \leqslant \delta(H)$. Let σ be an isomorphism from G to $\bar{G}$. Then, the map $\bar{\sigma} : L(G) \to L(\bar{G})$ defined as $\bar{\sigma}(H) = \sigma(H)$ for all subsets H of G is a projectivity from G to $\bar{G}$ and we call $\bar{\sigma}$ the *projectivity induced by* σ [10].

Lemma 2.11. [10]*Let G be a finite abelian group, written multiplicatively, and let $G^\star = Hom(G, \mathbb{C}^\star)$ be the set of all homomorphisms from G into the multiplicative group $\mathbb{C}^\star$ of complex numbers. For $\sigma, \tau \in G^\star$, define $\sigma \circ \tau \in G^\star$ by $x^{\sigma\circ\tau} = x^\sigma x^\tau$.*

1. *Then $(G^\star, \circ)$ is a group isomorphic to G; it is called the character group of G.*
2. *If $1 \neq x \in G$, there exists $\sigma \in G^\star$ such that $x^\sigma \neq 1$.*

Theorem 2.12. [10] *A finite abelian group G is self-dual, that is, there exists a duality from G onto G.*

In the proof of Theorem 2.12, for a subset X of a finite abelian group G, $X^\perp$ is defined as,

$$X^\perp = \{\sigma \in G^\star \mid x^\sigma = 1 \text{ for all } x \in X\}.$$

It is also denoted by $\perp\ (X)$. Observe that $\perp$ is a duality from G onto $G^\star$. Let ρ be the projectivity induced by an isomorphism from $G^\star$ onto G. Hence, $\delta = \rho \perp$, the composition of ρ and $\perp$, is a duality from G onto G.

Theorem 2.13. *For a finite abelian group G, $\tilde{\Gamma}_{\mathcal{H}}(G) \cong Co_{\mathcal{H}}(G)$.*

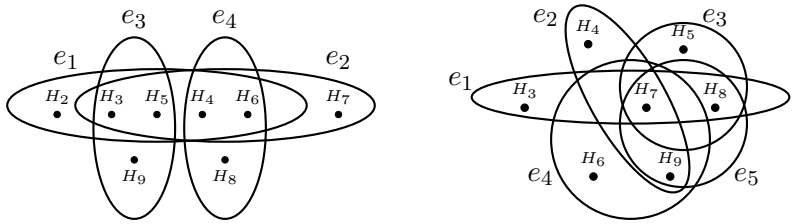
Proof. Let G be a finite abelian group and $H_1 \in V(\tilde{\Gamma}_{\mathcal{H}}(G))$. Then, by the definition, there exists $H_2 \in V(\tilde{\Gamma}_{\mathcal{H}}(G))$ such that $H_1 \cap H_2 = \{e\}$. Note that for subgroups H_1, H_2 of G, $H_1 \vee H_2 = \langle H_1 \cup H_2\rangle$ and $H_1 \wedge H_2 = H_1 \cap H_2$ in $L(G)$, the lattice of subgroups of G. As G is abelian, $\delta(H_1)$ and $\delta(H_2)$ are normal. Therefore, $\delta(H_1)\delta(H_2) = \delta(H_1) \vee \delta(H_2) = \rho \perp (H_1) \vee \rho \perp (H_2) = \rho(\perp\ (H_1) \vee \perp\ (H_2)) = \rho(\perp\ (H_1 \wedge H_2)) = \rho((H_1 \wedge H_2)^\perp) = \rho(\{e\}^\perp) = \rho(G^\star) = G$ Thus, if $H_1 \in V(\tilde{\Gamma}_{\mathcal{H}}(G))$, then $\delta(H_1) \in V(Co_{\mathcal{H}}(G))$.

Define a map $f : V(\tilde{\Gamma}_{\mathcal{H}}(G)) \longrightarrow V(Co_{\mathcal{H}}(G))$ such that $f(H) = \delta(H)$ for each $H \in V(\tilde{\Gamma}_{\mathcal{H}}(G))$. Since δ is a well-defined bijective map, f is also a well-defined bijective map.

Observe that, $H_1 \sim H_2$ if and only if $\rho((H_1 \wedge H_2)^\perp) = \rho(G^\star)$ if and only if $f(H_1)f(H_2) = G$ if and only if $f(H_1) \sim f(H_2)$ in $Co_{\mathcal{H}}(G)$, i.e., $H_1 \sim H_2$ in $\tilde{\Gamma}_{\mathcal{H}(G)}$ if and only if $f(H_1) \sim f(H_2)$ in $Co_{\mathcal{H}}(G)$. Hence, for a positive integer r, $\{H_1, H_2, \ldots, H_r\}$ is a hyperedge of $\tilde{\Gamma}_{\mathcal{H}}(G)$ if and only if $\{f(H_1), f(H_2), \ldots, f(H_r)\}$ is a hyperedge of $Co_{\mathcal{H}}(G)$.

Thus, f is an isomorphism from $V(\tilde{\Gamma}_{\mathcal{H}}(G))$ to $V(Co_{\mathcal{H}}(G))$. Thus, $\tilde{\Gamma}_{\mathcal{H}}(G) \cong Co_{\mathcal{H}}(G)$. □

Remark 2.14. For non-abelian groups, Theorem 2.13 need not be true. For the dihedral group of order 8, $D_4 = \langle a, b \,|\, a^4 = e = b^2, bab^{-1} = a^{-1} \rangle$. The vertex set of $\tilde{\Gamma}_{\mathcal{H}}(D_4)$ is $V = \{H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9\}$, where $H_2 = \langle a^2 \rangle, H_3 = \langle b \rangle, H_4 = \langle ab \rangle, H_5 = \langle a^2 b \rangle, H_6 = \langle a^3 b \rangle, H_7 = \langle a \rangle, H_8 = \langle a^2, b \rangle$ and $H_9 = \langle a^2, ab \rangle$. Note that $\tilde{\Gamma}_{\mathcal{H}}(D_4)$ and $Co_{\mathcal{H}}(D_4)$ are not isomorphic because $\tilde{\Gamma}_{\mathcal{H}}(D_4)$ contains 4 hyperedges whereas $Co_{\mathcal{H}}(D_4)$ contains 5 hyperedges, see Figure 5.



(a) $\tilde{\Gamma}_{\mathcal{H}}(D_4)$ (b) $Co_{\mathcal{H}}(D_4)$

Figure 5

Remark 2.15. The converse of Theorem 2.13 need not be true, i.e., even if $\tilde{\Gamma}_{\mathcal{H}}(G) \cong Co_{\mathcal{H}}(G)$, G need not be abelian. For example, consider the dihedral group $D_3 = \langle a, b \mid a^3 = e = b^2, bab^{-1} = a^{-1} \rangle$ of order 6. Both $\tilde{\Gamma}_{\mathcal{H}}(D_3)$ and $Co_{\mathcal{H}}(D_3)$ consist of a single hyperedge containing 4 subgroups - $\langle a \rangle, \langle b \rangle, \langle ab \rangle$ and $\langle a^2 b \rangle$. Hence, $\tilde{\Gamma}_{\mathcal{H}}(D_3) \cong Co_{\mathcal{H}}(D_3)$, but D_3 is not abelian.

3. Planarity and non-planarity of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$

An *embedding of a graph* on a surface is a continuous and one to one function from a topological representation of the graph into the surface. We denote by S_n the surface obtained from the sphere S_0 by adding n handles. The number n is called the *genus of the surface* $S_n, n \geqslant 0$. The *orientable genus* of a graph G, denoted by $g(G)$, is the minimum genus of a surface in which G can be embedded. A *cross-cap* is a topological object formed by identifying opposite points on the boundary of a circle (or a disk) and is equivalent to gluing a Möbius strip into a hole in a surface. A surface obtained by adding k crosscaps to S_0 is known as the non-orientable surface and we denote it by N_k. The number k is called the crosscap of N_k. The non-orientable genus of a graph G, denoted by $\tilde{g}(G)$, is the smallest integer k such that G can be embedded on N_k. A graph is said to be *planar* if it can be drawn on the

plane in such a way that no edges intersect, except at a common end vertex. A graph is said to be *toroidal* if it can be embedded on a torus and is called *projective* if it can be embedded on a projective plane. Further, note that if H is a subgraph of a graph G, then $g(H) \leqslant g(G)$ and $\tilde{g}(H) \leqslant \tilde{g}(G)$. A hypergraph is *toroidal* if its incidence graph is toroidal and is *projective* if its incidence graph is projective.

In general, the embeddability of the clique hypergraph on a surface cannot be inferred from the embeddability of the associated graph, since the embedding properties of a hypergraph are obtained from its incidence graph. For example, $\mathscr{I}^c(\mathbb{Z}_{2^3.3.5})$ is planar [7], whereas from the following theorem its clique hypergraph $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{2^3.3.5})$ is not planar. Similarly, $\mathscr{I}^c(D_4)$ is not planar as it contains K_5, while its clique hypergraph $\tilde{\Gamma}_{\mathcal{H}}(D_4)$ is planar [2]. To study the planarity of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$, we essentially need the following.

Theorem 3.1. [12]*A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 3.2. [12] *A hypergraph is planar if and only if its incidence graph is planar.*

Theorem 3.3. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are non-negative integers. Then, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar if and only if n is equal to $p_1^{\alpha_1}p_2$ or $p_1^{\alpha_1}p_2^2$ or $p_1p_2p_3$ or $p_1^2p_2p_3$.*

Proof. Consider the following cases:

Case 1. If $n = p_1^{\alpha_1}p_2$, then by Theorem 2.4, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is a 2-uniform hypergraph, i.e., star graph. So, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar.

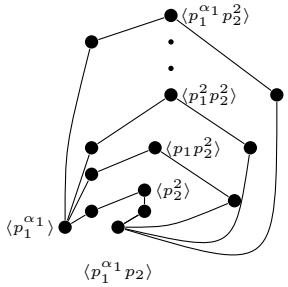
Case 2. If $n = p_1^{\alpha_1}p_2^2$, where $\alpha_1 \geqslant 2$, then Figure 6 (a) depicts an embedding of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ on the plane, where the unlabelled vertices are hyperedges and hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar.

Case 3. If $n = p_1p_2p_3$, then Figure 6 (b) depicts an embedding of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ on the plane and hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar.

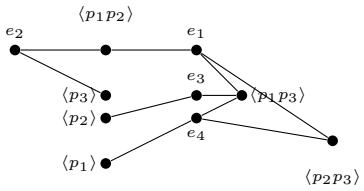
Case 4. If $n = p_1^2p_2p_3$, then Figure 6 (c) depicts an embedding of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ on the plane and hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar.

Conversely, suppose that $n \neq p_1^{\alpha_1}p_2, n \neq p_1^{\alpha_1}p_2^2, n \neq p_1p_2p_3$ and $n \neq p_1^2p_2p_3$. Then, consider the following cases:

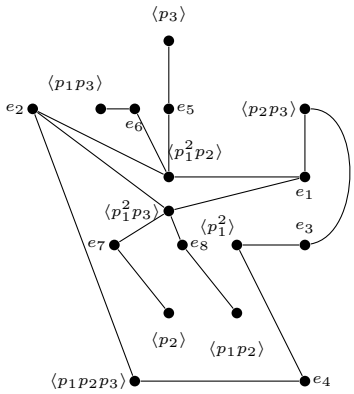
Case 1. If $\omega(n) = 2$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where both α_1 and α_2 are integers greater than 2, then note that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic



(a) Planar embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1} p_2^2}))$



(b) Planar embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1 p_2 p_3}))$

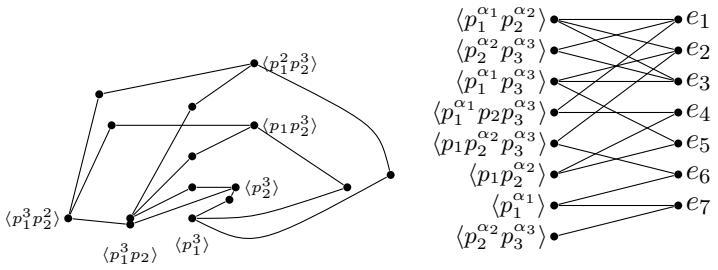


(c) Planar embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2 p_2 p_3}))$

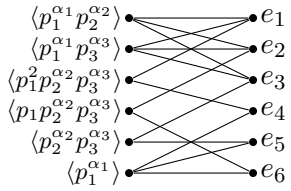
Figure 6

to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^3 p_2^3}))$ and Figure 7(a) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^3 p_2^3}))$ is non-planar. Hence, in this case, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is non-planar.

Case 2. If $\omega(n) = 3$ with $n \neq p_1 p_2 p_3$ and $n \neq p_1^2 p_2 p_3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to either Figure 7(b) or Figure 7(c) where both are subdivisions of $K_{3,3}$. Hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is non-planar.



(a) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^3p_2^3}))$ on a plane (b)



(c)

Figure 7

Case 3. If $\omega(n) \geqslant 4$, then consider the following cases:

Subcase 3.1. Suppose that $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_k^{\alpha_k}$, where $k \geqslant 4$, $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers and $0 \leqslant \alpha_i \leqslant 2$, then it contains a subdivision of $K_{3,3}$ as depicted in Figure 8. Hence, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is non-planar.

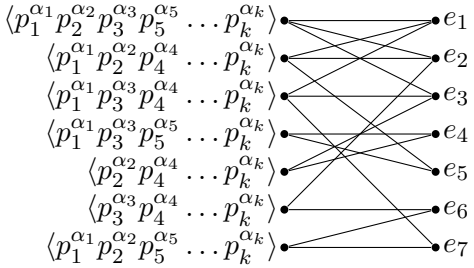


Figure 8

Subcase 3.2. Suppose that $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_k^{\alpha_k}$, where $k \geqslant 4$, $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers and atleast one of α_i is greater than or equal to 3. Without loss of generality,

assume that $\alpha_1 \geqslant 3$. Consider the vertices $H_1 = \langle p_1^{\alpha_1} p_3^{\alpha_3} p_4^{\alpha_4} \dots p_k^{\alpha_k} \rangle$, $H_2 = \langle p_1^{\alpha_1} p_2^{\alpha_2} p_4^{\alpha_4} \dots p_k^{\alpha_k} \rangle$, $H_3 = \langle p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_5^{\alpha_5} \dots p_k^{\alpha_k} \rangle$, $K_1 = \langle p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle$, $K_2 = \langle p_1 p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle$ and $K_3 = \langle p_1^2 p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k} \rangle$. Observe that $H_i \cap H_j = \{e\}$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Also, $K_i \cap K_j \neq \{e\}$ and $H_i \cap K_j = \{e\}$ for all $i, j \in \{1, 2, 3\}$. Hence, there exists three distinct hyperedges e_1, e_2, e_3 such that e_1 contains H_1, H_2, H_3 and K_1, e_2 contains H_1, H_2, H_3 and K_2, and e_3 contains H_1, H_2, H_3 and K_3. Hence, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains $K_{3,3}$ and so $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is non-planar. □

We have the following result about orientable genus of a hypergraph to characterize $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ whose orientable genus is 1.

Theorem 3.4. [12]*For any hypergraph $\mathcal{H}$, $g(\mathcal{H}) = g(\mathcal{I}(\mathcal{H}))$.*

Theorem 3.5. *The orientable genus, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$, of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is equal to 1 if and only if $n = p_1^{\alpha} p_2^3$ with $3 \leqslant \alpha \leqslant 6$ or $n = p_1^4 p_2^4$ or $n = p_1^{\beta} p_2 p_3$ with $3 \leqslant \beta \leqslant 5$, where p_1, p_2 and p_3 are distinct primes.*

Proof. Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $k \geqslant 2$, $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers. Consider the following cases:

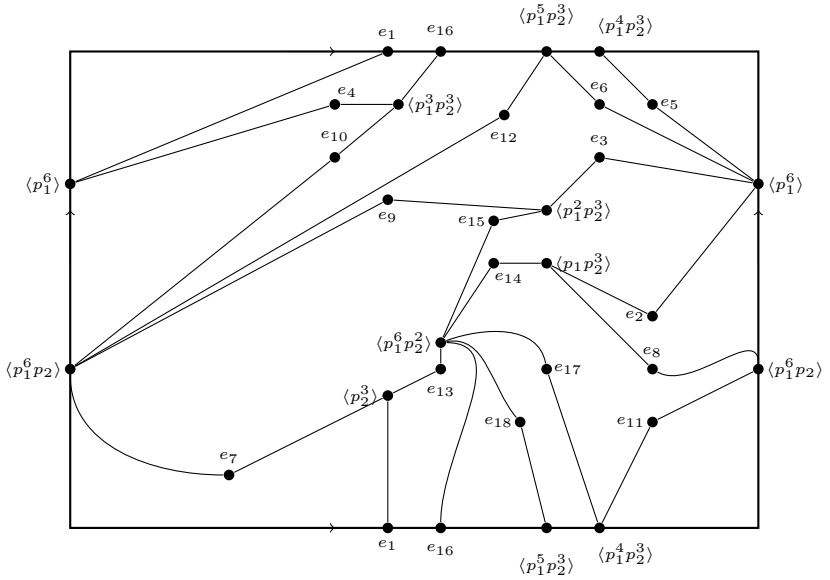
Case 1. Suppose that $\omega(n) = 2$.

Subcase 1.1. If $n = p_1^{\alpha_1} p_2$ or $n = p_1^{\alpha_1} p_2^2$, then by Theorem 3.3 $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar and hence $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

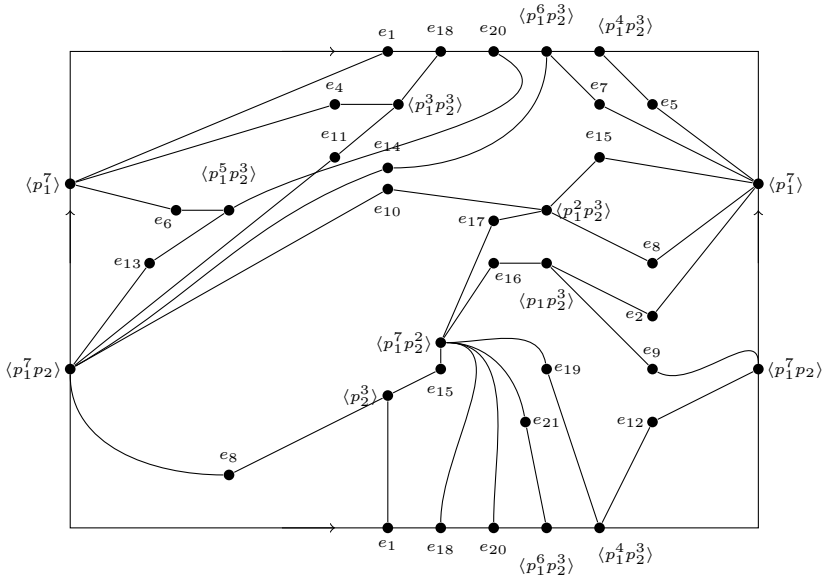
Subcase 1.2. If $n \neq p_1^{\alpha_1} p_2$ and $n \neq p_1^{\alpha_1} p_2^2$, then by Theorem 3.3, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) > 0$. In this subcase, we have only the following subsubcases:

Subsubcase 1.2.1. If n divides $p_1^6 p_2^3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is isomorphic to a subgraph of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6 p_2^3}))$ and Figure 9(a) depicts the embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6 p_2^3}))$ on a torus. Hence, in this case, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is embeddable on torus and therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 1$.

Subsubcase 1.2.2. If n is divisible by $p_1^7 p_2^3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^7 p_2^3}))$ and Figure 9(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^7 p_2^3}))$ is not embeddable on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is not embeddable on a torus. Therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

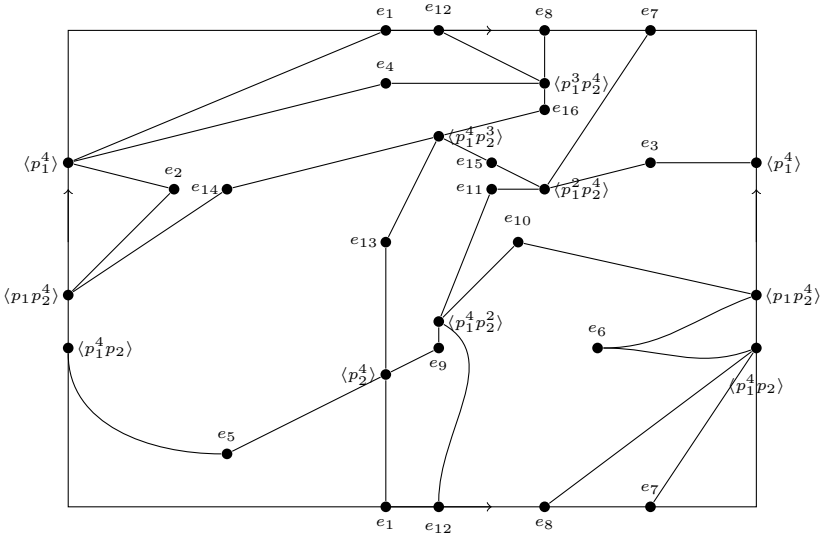


(a) An embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6p_2^3}))$ on a torus

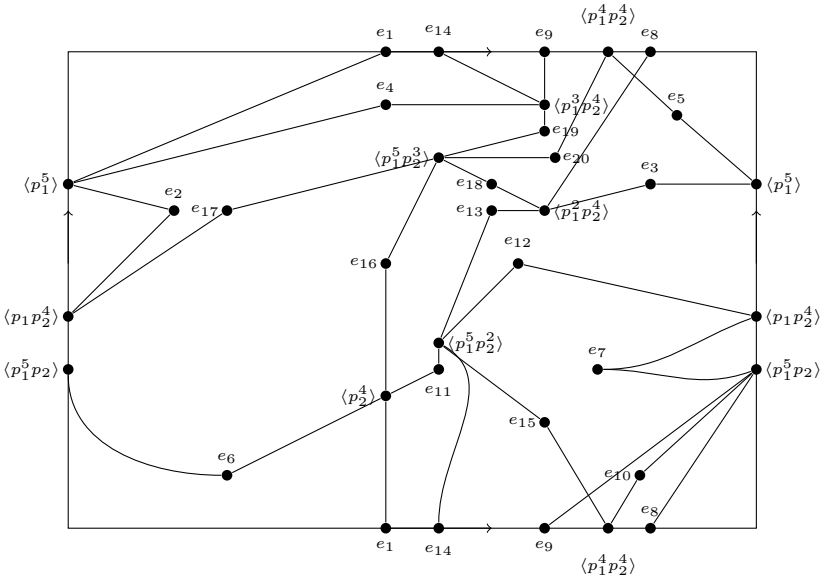


(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^7p_2^3}))$ on a torus

Figure 9



(a) An embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4 p_2^4}))$ on a torus



(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5 p_2^4}))$ on a torus

Figure 10

SUBSUBCASE 1.2.3. If $n = p_1^4 p_2^4$, then Figure 10(a) depicts an embed-

ding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^4}))$ on a torus. Therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^4})) = 1$.

Subsubcase 1.2.4 If n is divisible by $p_1^5p_2^4$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2^4}))$ and Figure 10(b) depicts that in this case $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2^4}))$ is not embeddable on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is also not embeddable on a torus. Therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

Case 2. Suppose $\omega(n) = 3$.

Subcase 2.1. If $n = p_1p_2p_3$ or $n = p_1^2p_2p_3$, then by Theorem 3.3, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar and hence, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

Subcase 2.2. If $n \neq p_1p_2p_3$ or $n \neq p_1^2p_2p_3$, then we have the following cases:

Subsubcase 2.2.1. If n divides $p_1^5p_2p_3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is isomorphic to a subgraph of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2p_3}))$ and Figure 11(a) (on the next page) depicts an embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2p_3}))$ on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is also embeddable on a torus. Therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 1$.

Subsubcase 2.2.2. If n is divisible by $p_1^6p_2p_3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6p_2p_3}))$ and Figure 11(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6p_2p_3}))$ is not embeddable on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is also not embeddable on a torus and therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

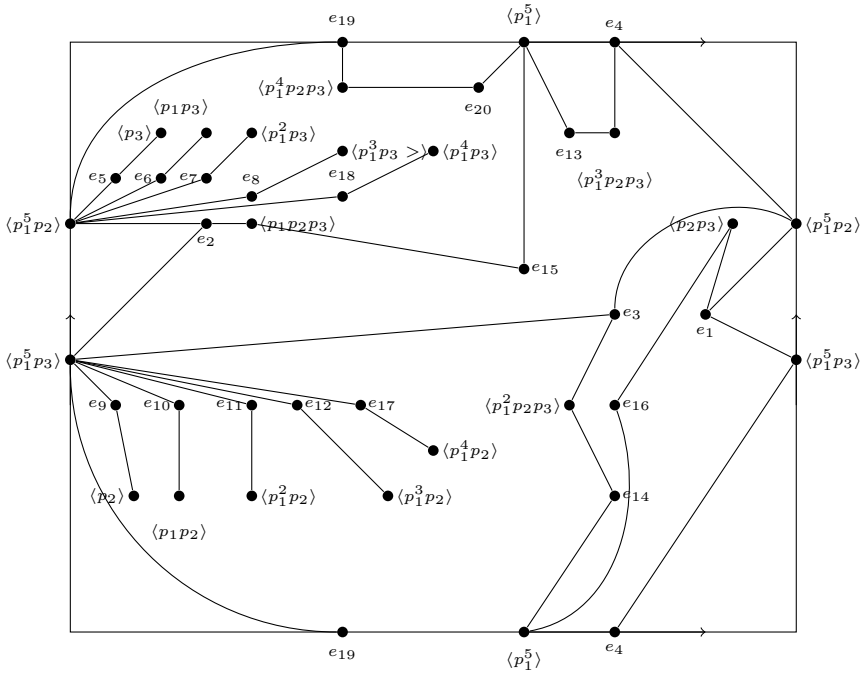
Subsubcase 2.2.3. If n contains at least two prime divisors with power greater than or equal to 2, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ and Figure 12(a) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ is not embeddable on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is also not embeddable on a torus and therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

Case 3. Suppose $\omega(n) > 3$.
Observe that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ and Figure 12(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ is not embeddable on a torus. Hence, in this case, $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is also not embeddable on a torus and therefore, $g(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$. □

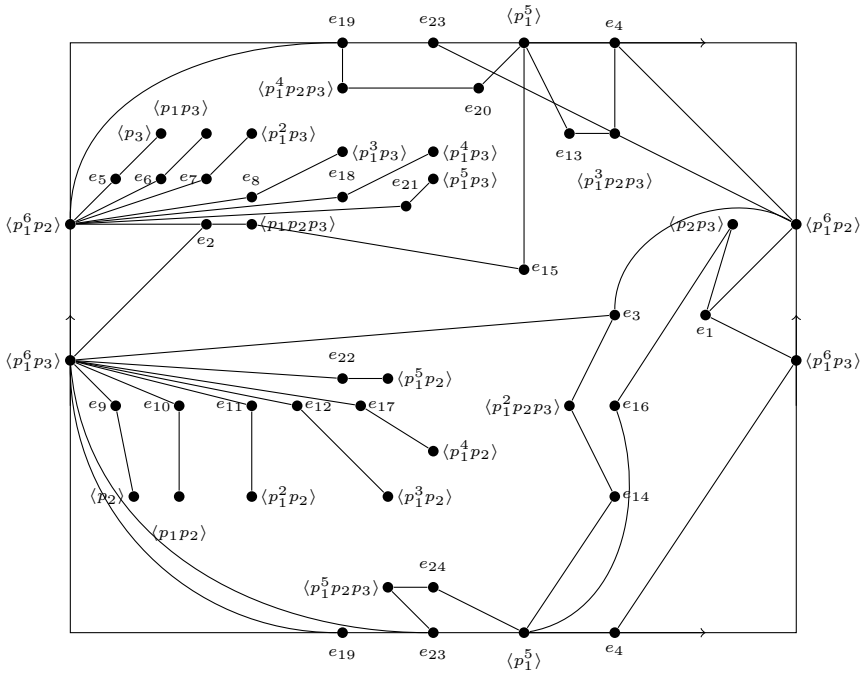
Corollary 3.5.1. *$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is toroidal if and only if $n = p_1^{\alpha_1}p_2^{\alpha}$ with $\alpha = 1, 2$ or $n = p_1^{\beta}p_2^3$ with $3 \leqslant \beta \leqslant 6$, or $n = p_1^4p_2^4$, or $n = p_1^{\gamma}p_2p_3$ with $1 \leqslant \gamma \leqslant 5$, where p_1, p_2, p_3 are distinct primes and α_1 is a positive integer.*

We have the following result about non-orientable genus of a hypergraph to characterize $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ whose non-orientable genus is 1.

Theorem 3.6. [12]*For any hypergraph $\mathcal{H}$, $\tilde{g}(\mathcal{H}) = \tilde{g}(\mathcal{I}(\mathcal{H}))$.*

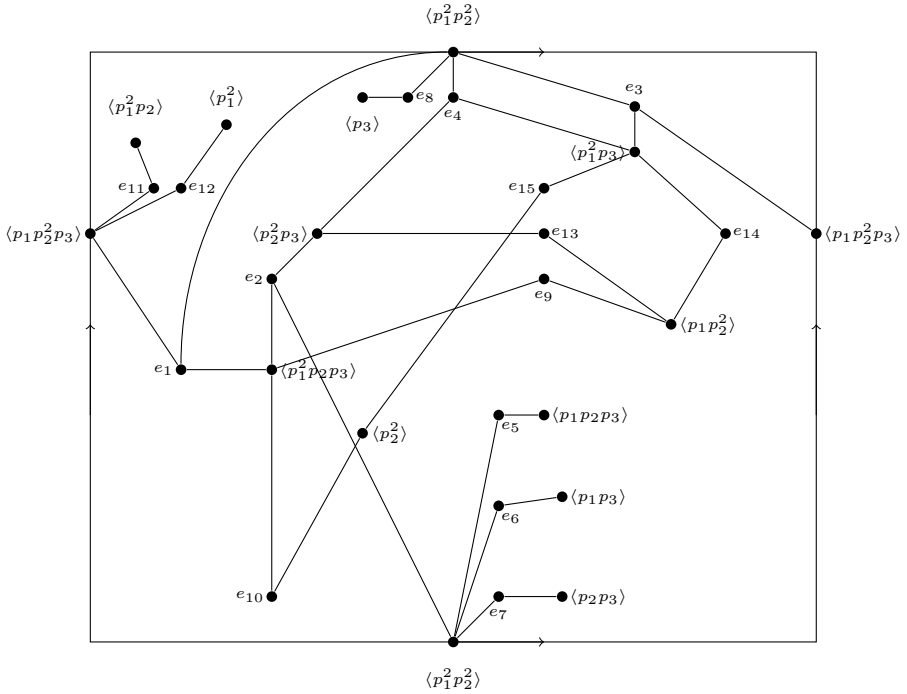


(a) An embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5 p_2 p_3}))$ on a torus

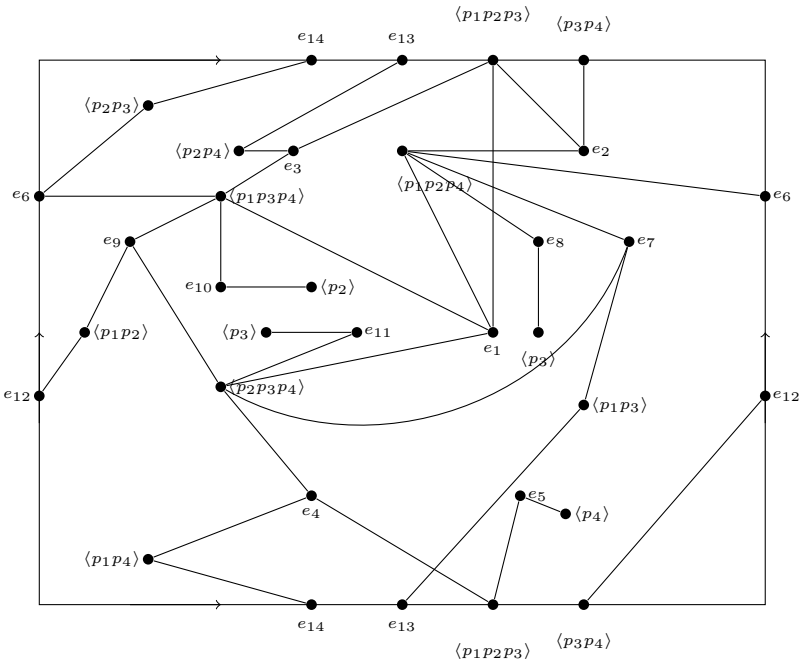


(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^6 p_2 p_3}))$ on a torus

Figure 11



(a) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ on torus



(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ on torus

Figure 12

Theorem 3.7. *The non-orientable genus, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$, of $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is equal to 1 if and only if $n = p_1^{\alpha_1}p_2^3$ with $\alpha_1 = 3, 4$ or $n = p_1^{\alpha_1}p_2p_3$ with $\alpha_1 = 3, 4$, where p_1, p_2 and p_3 are distinct primes.*

Proof. Suppose that $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $k \geqslant 2$, $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers. Consider the following cases:

Case 1. Suppose that $\omega(n) = 2$.

Subcase 1.1. If $n = p_1^{\alpha_1}p_2$ or $n = p_1^{\alpha_1}p_2^2$, then, by Theorem 3.3, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar and hence $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

Subcase 1.2. If $n \neq p_1^{\alpha_1}p_2$ and $n \neq p_1^{\alpha_1}p_2^2$, then, by Theorem 3.3, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) > 0$. In this subcase, we have the only following subsubcases:

Subsubcase 1.2.(a) If n divides $p_1^4p_2^3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is isomorphic to a subgraph of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^3}))$ and Figure 13(a) (next page) depicts an embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^3}))$ on a projective plane. Hence, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 1$.

Subsubcase 1.2.(b) If n is divisible by $p_1^5p_2^3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2^3}))$ and Figure 13(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2^3}))$ is not embeddable on a projective plane. Hence, in this case, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

Subsubcase 1.2.(c) If n is divisible by $p_1^4p_2^4$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^4}))$ and Figure 13(c) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^4}))$ is not embeddable on a projective plane. Hence, in this case, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

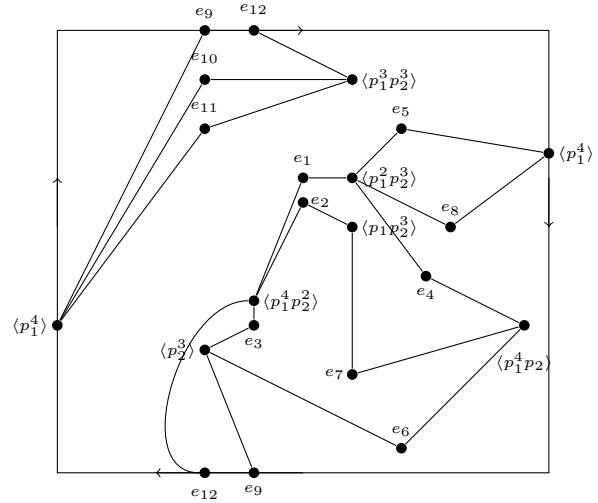
Case 2. Suppose $\omega(n) = 3$.

Subcase 2.1. If $n = p_1p_2p_3$ or $n = p_1^2p_2p_3$, then, by Theorem 3.3, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is planar and hence, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

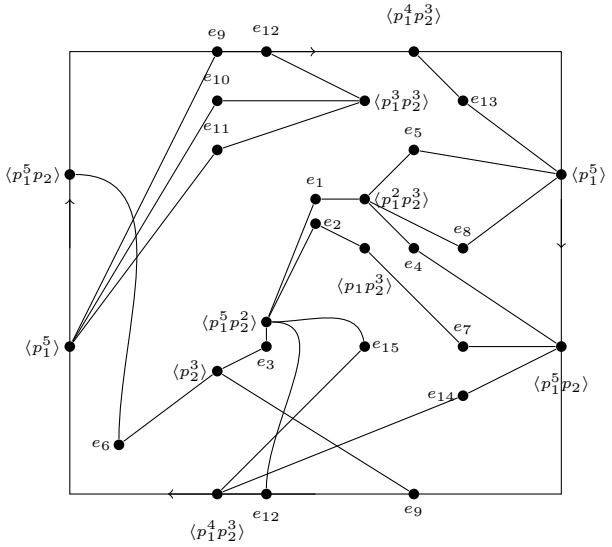
Subcase 2.2. If $n \neq p_1p_2p_3$ or $n \neq p_1^2p_2p_3$, then consider the following cases:

Subsubcase 2.2.(a) If n divides $p_1^4p_2p_3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ is isomorphic to a subgraph of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2p_3}))$ and Figure 14(a) depicts an embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2p_3}))$ on a projective plane. Hence, in this case, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = 1$.

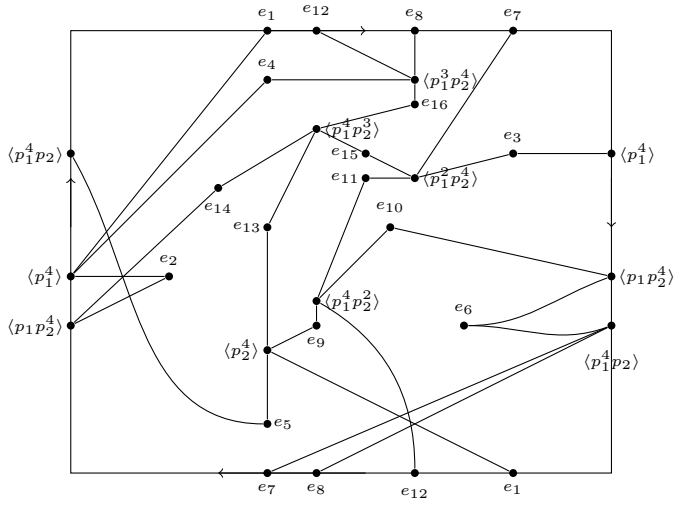
Subsubcase 2.2.(b) If n is divisible by $p_1^5p_2p_3$, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2p_3}))$ and Figure 14(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2p_3}))$ is not embeddable on a projective plane. Hence, in this case, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.



(a) An embedding of $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4 p_2^3}))$ on a projective plane

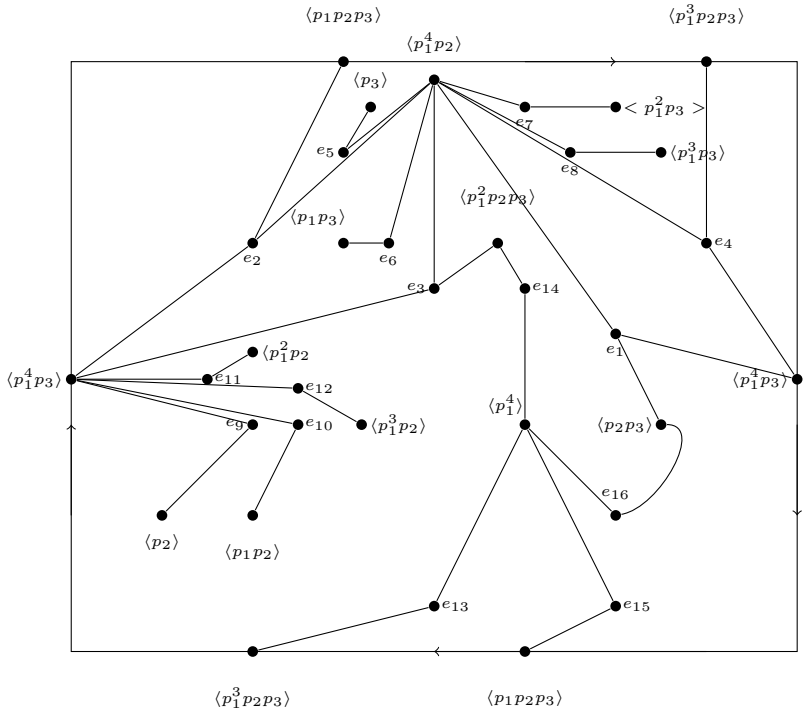


(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5 p_2^3}))$ on projective plane

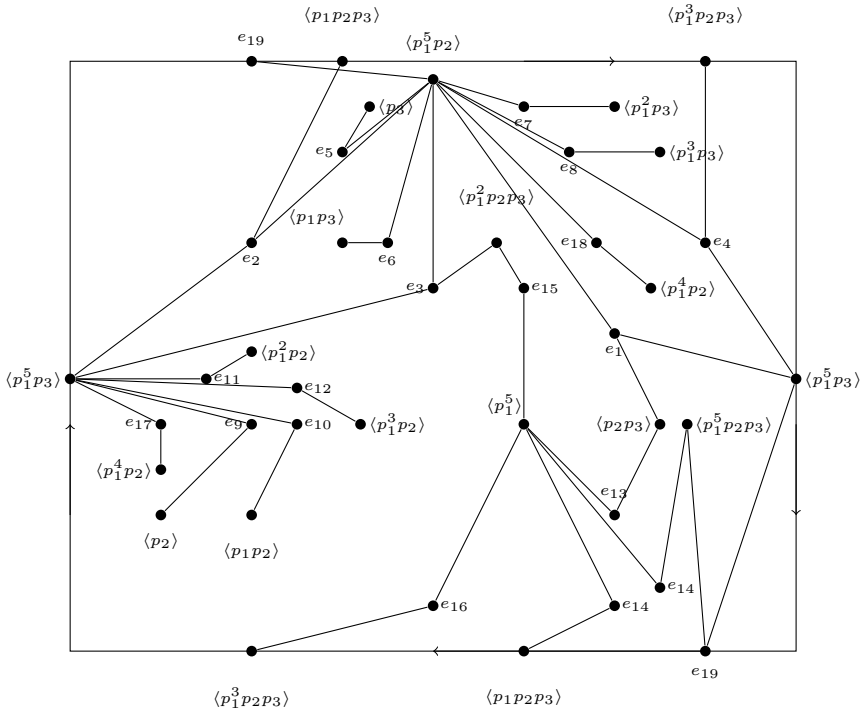


(c) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4 p_2^4}))$ on projective plane

Figure 13

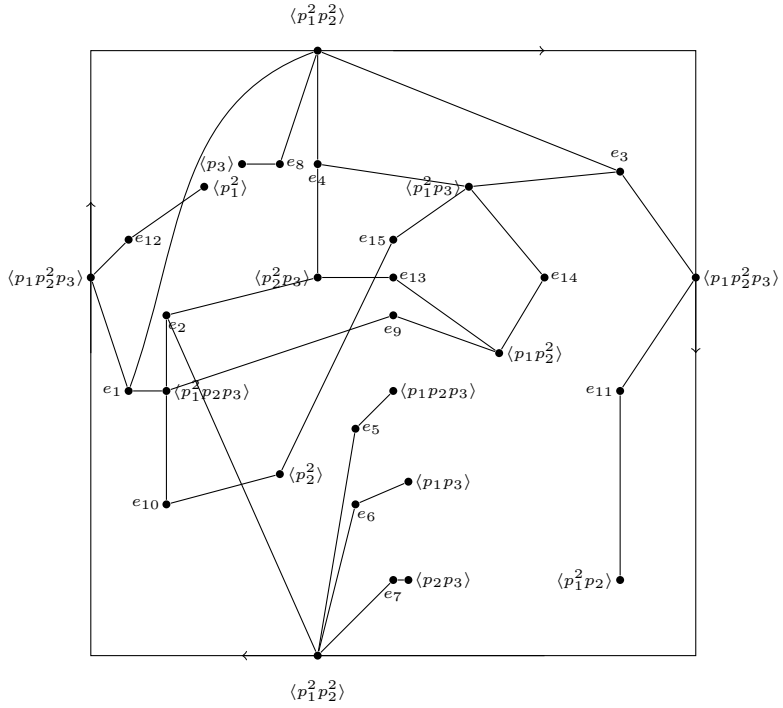


(a) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2p_3}))$ on projective plane

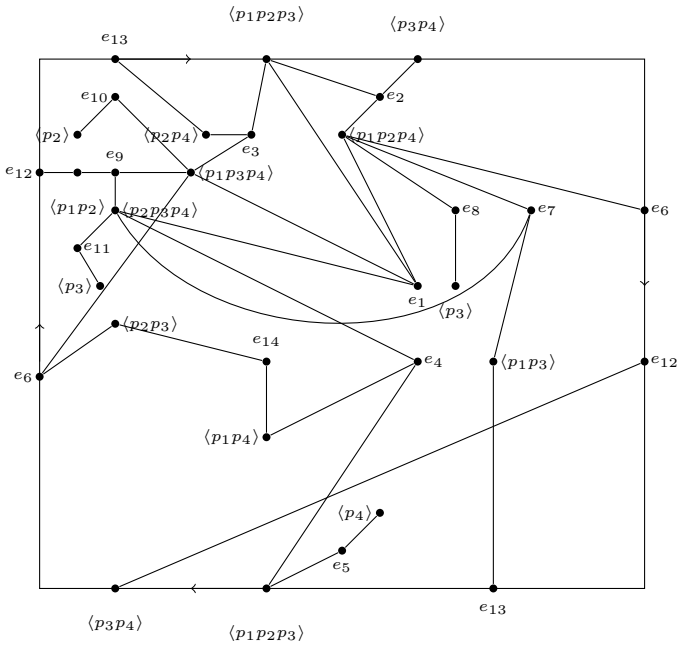


(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^5p_2p_3}))$ on projective plane

Figure 14



(a) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ on a projective plane



(b) $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ on a projective plane

Figure 15

SUBSUBCASE 2.2.(c) If n contains at least two prime divisors with power greater than or equal to 2, then $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ and Figure 15(a) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2^2p_3}))$ is not embeddable on a projective plane. Hence, in this case, $\tilde{g}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$.

CASE 3. Suppose $\omega(n) \geqslant 4$.
Observe that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n))$ contains a subgraph isomorphic to $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ and Figure 15(b) depicts that $\mathcal{I}(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3p_4}))$ is not embeddable on a projective plane and hence, in this case, $\tilde{g}(\Gamma_{\mathcal{H}}(\mathbb{Z}_n)) \neq 1$. □

Corollary 3.7.1. *$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is projective if and only if $n = p_1^{\alpha_1}p_2^{\alpha}$ with $\alpha = 1, 2$ or $n = p_1^{\beta}p_2^3$ with $\beta = 3, 4$ or $n = p_1^{\gamma}p_2p_3$ with $1 \leqslant \gamma \leqslant 4$, where p_1, p_2, p_3 are distinct primes and α_1 is a positive integer.*

In nutshell, the comparison between $\mathscr{I}^c(\mathbb{Z}_n)^\star$ and its clique hypergraph $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$ is given in the following table.

Properties	Graph, $\mathscr{I}^c(\mathbb{Z}_n)^\star$ [7, 9]	Clique hypergraph, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)$
Diameter	$\leqslant 3$	$\leqslant 3$
Girth	$gr(\mathscr{I}^c(\mathbb{Z}_n)^\star) = \begin{cases} \infty, \text{ if } n = p_1^{\alpha_1}p_2, \text{where } \alpha_1 \geqslant 1 \\ 4, \text{ if } n = p_1^{\alpha_1}p_2^{\alpha_2}, \\ 3, \text{ otherwise.} \end{cases}$	$gr(\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_n)) = \begin{cases} \infty, \text{ if } n = p_1^{\alpha_1}p_2, \text{where } \alpha_1 \geqslant 1 \text{ or } n = p_1p_2p_3, \\ 4, \text{ if } n = p_1^{\alpha_1}p_2^{\alpha_2}, \text{where } \alpha_1, \alpha_2 \geqslant 2, \\ 2, \text{ otherwise.} \end{cases}$
Star	$\mathscr{I}^c(\mathbb{Z}_{p_1^{\alpha_1}p_2})^\star$	$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2})^\star$
Tree	$\mathscr{I}^c(\mathbb{Z}_{p_1^{\alpha_1}p_2})^\star$	$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2}), \tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3})$
Chromatic Number	$\begin{cases} 2, \text{ if } \omega(n) = 2, \\ \geqslant 3, \text{ if } \omega(n) = 3, 4 \end{cases}$	2 for all n
Planarity	$\mathscr{I}^c(\mathbb{Z}_{p_1^{\alpha_1}})^\star$, $\mathscr{I}^c(\mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}})^\star(\alpha_1+\alpha_2 \leqslant 5)$, $\mathscr{I}^c(\mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}})^\star(\alpha_1+\alpha_2+\alpha_3 \leqslant 5)$, $\mathscr{I}^c(\mathbb{Z}_{p_1p_2p_3p_4})^\star$	$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2})$, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2^2})$ $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1p_2p_3})$, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^2p_2p_3})$
orientable genus 1	–	$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_2^{\alpha}p_2^3})(3 \leqslant \alpha \leqslant 6)$, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^4p_2^4})$ $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\beta}p_2p_3})(3 \leqslant \beta \leqslant 5)$,
non orientable genus 1	–	$\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2^3})(\alpha_1 = 3, 4)$, $\tilde{\Gamma}_{\mathcal{H}}(\mathbb{Z}_{p_1^{\alpha_1}p_2p_3})(\alpha_1 = 3, 4)$

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