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Uniform elements in N-groups

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Abstract. A zero-symmetric right nearring N is studied. We study relative essential ideal and relative uniform ideal in module over a matrix nearring corresponding to those in N-group (over itself). We explore the properties that show interplay between the ideals of N-group (act on itself) and N^n (over $M_n(N)$). Finally, we show that the finite Goldie dimension ($f.G.d.$) with respect to relative uniform ideal of ${}_NN$ is equal to that of $M_n(N)$-group N^n.

1. Introduction

The concept of uniform dimension in module over a ring generalizes the notion of dimension in finite dimensional vector space. A module in which every non-zero submodule is essential is called uniform. Uniform submodules play a significant role to establish various finite dimension conditions in modules over associative rings [8]. We consider a zero-symmetric right nearring N with 1 and let G stands for an N-group (also called as module over a nearring). In Bhavanari and Kuncham [4], Bhavanari et al. [6], uniform dimension was generalized N-groups and obtained characterization for an N-group to have finite Goldie dimension. Matrix nearrings over arbitrary nearrings were introduced by Meldrum & Van der Walt [9], and the action of G^n ($n \in \mathbb{Z}^+$) over $M_n(N)$ was defined by Van der Walt [18], when G is a locally monogenic N-group. However, N^n becomes an $M_n(N)$-group, by natural action. Bhavanari and Kuncham [5] have established the inter relationship between the ideals of the N-group N and the $M_n(N)$-group

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N^n. Tapatee et al. [16] have obtained a one-to-one correspondence between essential ideals of N-group and those of $M_n(N)$-group N^n. Tapatee et al. [13] have explored the generalized prime ideals of N-groups. The lattice aspects of module analogue, particularly, combinatorial properties of superfluous elements in a lattice were studied by Tapatee et al. [14]. Superfluous submodules are the dualization concept of essential submodules. In Tapatee et al. [15, 17, 11], several properties of matrix nearrings over arbitrary nearrings were explored.

In this paper, we consider N^n, the direct sum of n copies of the group $(N, +)$ and $1 \in N$. We define relative essential ideal and relative uniform ideal in $M_n(N)$-group N^n, corresponding to those in N-group over itself (denoted by $_NN$). We prove some properties which will interplay between the ideals of $_NN$ and the ideals of $M_n(N)$-group N^n. Finally, we obtain that the finite Goldie dimension ($f.G.d.$) with respect to relative uniform ideals of $_NN$ is equal to that of $M_n(N)$-group N^n.

2. Preliminaries

A (right) *nearring* $(N, +, \cdot)$ is an algebraic system [10], where N is additive group (need not be abelian) and multiplicative semigroup, satisfying only one distributive axiom, say right: $(l + m)n = ln + mn$ for all $l, m, n \in N$. Evidently, $0a = 0$, for all $a \in N$, but in general $a0 \neq 0$ for some $a \in N$. Wherever $m0 = 0$, for all $m \in N$ we call N is *zero-symmetric* (denoted by $N = N_0$). An additive group $(G, +)$ with 0_G (additive identity) is called an *N-group* (denoted by $_NG$ or simply by G), if there is a function $N \times G$ to G ($(a, g) := ag$), satisfy: (1) $(l + m)g = lg + mg$, (2) $(lm)g = l(mg)$ for all g in G and l, m in N. We refer to Pilz [10] for necessary definitions and notions of nearring. Throughout, N stands for a (right) nearring with 1 which is zero symmetric, $_NN$ denotes N-group (act on itself). The ideal of an N-group G is denoted by '$\trianglelefteq$'.

From [5], for any $u \in G$, $\langle u \rangle = \bigcup_{i=1}^{\infty} S_{i+1}$, where $S_{i+1} = S_i^* \cup S_i^0 \cup S_i^+$ with $S_0 = \{u\}$, and

$$S_i^* = \{g + y - g : g \in G, y \in S_i\},$$

$$S_i^0 = \{p - q : p, q \in S_i\} \cup \{p + q : p, q \in S_i\},$$

$$S_i^+ = \{n(m + a) - nm : n \in N, m \in G, a \in S_i\}.$$

From ([12]), let $H_1 \trianglelefteq G$. Then H_1 is called *relative essential* (or Δ-*essential*), if there is an ideal $\Delta \neq G$ with $H_1 \nsubseteq \Delta$, for $H_2 \trianglelefteq G$, $H_1 \cap H_2 \subseteq \Delta$ implies $H_2 \subseteq \Delta$. We denote it by $H_1 \leq^e_\Delta G$, we say that H_1 is Δ-essential in G. Further, G is called Δ-*uniform* (or *uniform with respect to* Δ) if for each $K \trianglelefteq G$, $K \nsubseteq \Delta$, then $K \leq^e_\Delta G$.

An ideal H_1 of G is called Δ-*uniform* if whenever $H_2 \trianglelefteq G$, $H_2 \nsubseteq \Delta$ and $H_2 \subseteq H_1$, we have $H_2 \leq^e_\Delta G$.

3. Uniform elements

For $\delta \in G$, we define δ-uniform element, δ-linearly independent element, and provide some properties.

Proposition 3.1. *Let $I \trianglelefteq G$ and $\Delta \neq G$ be an ideal of G. I is uniform with respect to Δ if and only if for any ideals H_1, H_2 contained in I such that $H_1 \cap H_2 \subseteq \Delta$ implies $H_1 \subseteq \Delta$ or $H_2 \subseteq \Delta$.*

Proof. Suppose that I is Δ-uniform. Let $H_1, H_2 \trianglelefteq G$ such that $H_1 \subseteq I$, $H_2 \subseteq I$, and $H_1 \cap H_2 \subseteq \Delta$. Assume that $H_1 \nsubseteq \Delta$. Since I is Δ-uniform, and $H_1 \subseteq I$, we have $H_1 \leq^e_\Delta I$, and since $H_1 \cap H_2 \subseteq \Delta$, we get $H_2 \subseteq \Delta$.

On the other hand, let $J \trianglelefteq G$ with $J \subseteq I$ and $J \nsubseteq \Delta$. To prove $J \leq^e_\Delta I$, let $K \trianglelefteq G$ contained in I such that $J \cap K \subseteq \Delta$. Since $J \nsubseteq \Delta$, by the converse hypothesis, we have $K \subseteq \Delta$. Therefore, $J \leq^e_\Delta I$. □

Example 3.2. Let $N = (\mathbb{Z}, +, \cdot)$ and $G = (\mathbb{Z}_8 \times \mathbb{Z}_3, +)$. Then G is an N-group. Take $\Delta = \mathbb{Z}_8 \times (0)$, an ideal of ${}_NG$. Then the ideal $(2) \times \mathbb{Z}_3$ is Δ-uniform, but not uniform, as the ideals $(4) \times (0)$, $(4) \times \mathbb{Z}_3 \subseteq (2) \times \mathbb{Z}_3$ such that $(4) \times (0) \cap (4) \times \mathbb{Z}_3 = (0) \times (0)$, but $(4) \times (0) \neq (0) \times (0)$ and $(4) \times \mathbb{Z}_3 \neq (0) \times (0)$.

Proposition 3.3. *Let I, J be ideals of G and Δ proper ideal of G. If $I \subseteq J$ and J is uniform with respect to Δ, then so is I.*

Proof. Suppose $H_1, H_2 \trianglelefteq G$ such that $H_1 \subseteq I, H_2 \subseteq I$ and $H_1 \cap H_2 \subseteq \Delta$. Since $I \subseteq J$, we have $H_1 \subseteq J, H_2 \subseteq J$. Now since J is Δ-uniform, it follows that $H_1 \subseteq \Delta$ or $H_2 \subseteq \Delta$. Hence I is Δ-uniform. □

Definition 3.4. Let Δ be a proper ideal of G. We say that $\{I_i : I_i \trianglelefteq G\}_{i \in I}$ is Δ-*direct* if $I_i \cap (\sum_{j \neq i} I_j) \subseteq \Delta$.

Definition 3.5. Let $\delta \in G$. An element $\delta \neq g \in G$ is called a *$n\delta$-uniform element* (briefly, *δ-u-element*) if $\langle u \rangle$ is a $\langle \delta \rangle$-uniform ideal of G.

Definition 3.6. Let $X \subseteq G$ and Δ be a proper ideal in G. Then X is called Δ-*linearly independent* (denoted by, Δ-l.i.) if $\sum_{a \in X} \langle a \rangle$ is Δ-direct. Further, for any $\delta \in G$, the set $\{a_i\}_{i=1}^n$ in G is said to be l.i. with respect to 'δ', if $\sum_{i=1}^n \langle a_i \rangle$ is $\langle \delta \rangle$-direct.

Proposition 3.7. *Let H, Δ (proper) be ideals of G. H is Δ-uniform if and only if for all $0 \neq x, y \in H \setminus \Delta$, $\langle x \rangle \cap \langle y \rangle \nsubseteq \Delta$.*

Proof. Let H be Δ-uniform. Let $0 \neq x, y \in H \setminus \Delta$. Then clearly, $\langle x \rangle$, $\langle y \rangle$ are ideals of G contained in H. On the contrary, suppose $\langle x \rangle \cap \langle y \rangle \subseteq \Delta$. Since H is Δ-uniform, either $\langle x \rangle \subseteq \Delta$ or $\langle y \rangle \subseteq \Delta$. Then $x \in \langle x \rangle \subseteq \Delta$ or $y \in \langle y \rangle \subseteq \Delta$, a contradiction. Hence, $\langle x \rangle \cap \langle y \rangle \nsubseteq \Delta$.

Conversely, suppose $H_1, H_2 \trianglelefteq G$ such that $H_1 \subseteq H, H_2 \subseteq H$ and $H_1 \cap H_2 \subseteq \Delta$. In order to prove that H is Δ-uniform, we need to show that $H_1 \subseteq \Delta$ or $H_2 \subseteq \Delta$. On a contrary, let $H_1 \nsubseteq \Delta, H_2 \nsubseteq \Delta$. Then there exists $0 \neq x \in H_1 \setminus \Delta, 0 \neq y \in H_2 \setminus \Delta$ such that $\langle x \rangle \cap \langle y \rangle \nsubseteq \Delta$, which means, $H_1 \cap H_2 \nsubseteq \Delta$, a contradiction. □

Definition 3.8. An N-group G has *finite Goldie dimension* with respect to a proper ideal Δ of G (we denote as Δ-*f.G.d.*), if G does not contain infinite number of ideals $H_i \nsubseteq \Delta$, such that the sum is Δ-direct.

Proposition 3.9. *Let Δ be a proper ideal of G, and G is Δ-uniform. Then any finite intersection of Δ-essential ideals of G is Δ-essential in G, and converse also holds.*

Proof. Let $\{H_i\}_{i=1}^n$ be a family of Δ-essential ideals of G. Write $H = \bigcap_{i=1}^n H_i$. Clearly $H_i \nsubseteq \Delta$ for each i, and since G is Δ-uniform, $H \nsubseteq \Delta$. To prove H is Δ-essential, we use the induction on the number n of Δ-essential ideals. Suppose that $n = 2$. Let $L \trianglelefteq G$ with $H \nsubseteq \Delta$, $H \cap L \subseteq \Delta$. Then $(H_1 \cap H_2) \cap L \subseteq \Delta$, implies $H_1 \cap (H_2 \cap L) \subseteq \Delta$. Since $H_1 \leq_\Delta^e G$ and $H_2 \cap L \trianglelefteq G$ with $H_2 \cap L \subseteq \Delta$. Again, since $H_2 \leq_\Delta^e G$ and $H_2 \nsubseteq \Delta$, we get $L \subseteq \Delta$. Therefore the statement is true for $n = 2$. We assume the induction hypothesis for $(n-1)$ ideals $\{H_i\}_{i=1}^{n-1}$ of G. Let $L \trianglelefteq G$ such that

$\big(\bigcap_{i=1}^{n} H_i\big) \cap L \subseteq \Delta$ and $H \nsubseteq \Delta$. Then, $\big(\bigcap_{i=1}^{n-1} H_i \cap H_n\big) \cap L \subseteq \Delta$. That is, $\bigcap_{i=1}^{n-1} H_i \cap (H_n \cap L) \subseteq \Delta$. Since $H \subsetneqq \bigcap_{i=1}^{n-1} H_i$ and $H \nsubseteq \Delta$, we have $\bigcap_{i=1}^{n-1} H_i \nsubseteq \Delta$, hence, by induction hypothesis, it follows that $H_n \cap L \subseteq \Delta$. Now since $H_n \leq_\Delta^e G$, $L \trianglelefteq G$, we have $L \subseteq \Delta$, which shows that $H \leq_\Delta^e G$.

Conversely, suppose that $H = \bigcap_{i=1}^{n} H_i \leq_\Delta^e G$. Since $\bigcap_{i=1}^{n} H_i \nsubseteq \Delta$, we get $H_i \nsubseteq \Delta$, for all i. To show $H_i \leq_\Delta^e G$ for every i, $1 \leqslant i \leqslant n$, let $L \trianglelefteq G$ with $H_i \cap L \subseteq \Delta$. Now $H \cap L \subseteq H_i \cap L \subseteq \Delta$ and since $H \leq_\Delta^e G$, it follows that $L \subseteq \Delta$. Since H_i $(1 \leqslant i \leqslant n)$, are arbitrary, we conclude that $H_i \leq_\Delta^e G$ for every i. □

Lemma 3.10. *Let G be an N-group with Δ-$f.G.d.$ Then every ideal K of G, which is not contained in Δ, contains an ideal, uniform with respect to Δ.*

Proof. Suppose that G has Δ-$f.G.d.$ On the contrary, suppose $H \trianglelefteq G$, $K \nsubseteq \Delta$, and it does not contain a strictly Δ-uniform ideal. Then K is not strictly Δ-uniform. So there exist $K_1, K_1' \trianglelefteq G$, $K_1, K_1' \subseteq K$, and $K_1, K_1' \nsubseteq \Delta$ such that $K_1 \cap K_1' \subseteq \Delta$, $K_1 + K_1' \subseteq H$. Then by supposition K_1' is not strictly Δ-uniform, which implies that there exist ideals K_2, K_2' contained in K_1' and $K_2, K_2' \nsubseteq \Delta$ such that $K_2 \cap K_2' \subseteq \Delta, K_2 + K_2' \subseteq K_1'$. If we continue, then we get $\{K_i\}_1^\infty$, $\{K_i'\}_1^\infty$ of two infinite sequences of ideals of G, not contained in Δ such that $K_i \cap K_i' \subseteq \Delta$ and $K_i + K_i' \subseteq K_{i-1}'$, for $i \geq 2$. Thus, the sum $\sum_{i=1}^\infty K_i$ is infinite Δ-direct, a contradiction that G has Δ-$f.G.d.$ □

Lemma 3.11. *Suppose G has $\langle\delta\rangle$-$f.G.d.$ for some $\delta \in G$. If $H \trianglelefteq G$, $H \nsubseteq \langle\delta\rangle$, then H contains a δ-uniform element in G.*

Proof. Let $H \nsubseteq \langle\delta\rangle$ and $H \trianglelefteq G$. Since G has $\langle\delta\rangle$-$f.G.d.$, by Lemma 3.10, there exists an ideal $I \subseteq H$, which is a $\langle\delta\rangle$-uniform. Since I not contained in $\langle\delta\rangle$, there is an element $x \in I$ such that $x \notin \langle\delta\rangle$. Now $\langle x\rangle \subseteq I$ and I is $\langle\delta\rangle$-uniform, by Proposition 3.3, we have $\langle x\rangle$ is $\langle\delta\rangle$-uniform. Therefore, x is δ-uniform element where $x \neq \delta$. □

Note 3.1. Let $\Delta \neq G$ be an ideal of G. If G has Δ-$f.G.d.$, then G contains $\{H_i \subseteq \Delta\}_{i=1}^n$ of Δ-uniform ideals such that their sum is direct and essential with respect to Δ in G (in this case, we denote as $H_1 \oplus \cdots \oplus H_n \leq^e_\Delta G$). The integer '$n$' is independent of Δ-uniform ideals, called the relative dimension of G with respect to Δ, and we write $dim_\Delta(G) = n$.

Lemma 3.12. *Let $\Delta \trianglelefteq G$ be proper and $I \trianglelefteq G$. If $\Delta \subseteq J$ is the maximal among the ideals of G with $I \cap J \subseteq \Delta$, then $I \oplus J \leq^e_\Delta G$.*

Proof. It is sufficient to show the Δ-essentiality. Suppose $K \trianglelefteq G$ such that $(I + J) \cap K \subseteq \Delta$. To show, $K \subseteq \Delta$, first we show that $I \cap (J + K) \subseteq \Delta$. Let $a \in I \cap (J + K)$. Then $a = b + d$, for some $a \in I$, $b \in J$ and $d \in K$, implies $a - b = d \in K \subseteq J$ and $b \in J$, implies $a = (a - b) + b \in J$, so $I \cap J \subseteq \Delta$. Therefore, $I \cap (J + K) \subseteq \Delta$. Now by maximality of J, we have $J + K = J$, shows that $K \subseteq J \subseteq I + J$. Hence, $K = (I + J) \cap K \subseteq \Delta$, shows that $I \oplus J \leq^e_\Delta G$. □

4. Uniform elements in $M_n(N)$-group N^n

Let N^n denotes the n-copies of $(n, +)$. For $a \in N$, $i_i(a) = (0, \cdots, \underbrace{a}_{i^{th}}, \cdots, 0)$, and $\pi_j(a_1, \cdots, a_n) = a_j$, for any $(a_1, \cdots, a_n) \in N^n$ represent i^{th} injective and j^{th} projective maps respectively. The set of $n \times n$-matrices over N, is $M_n(N)$, which is a subnearring of $M(N^n)$, generated by $\{f^r_{ij} : N^n \to N^n : r \in N, 1 \leqslant i, j \leqslant n\}$ where $f^r_{ij}(u_1, \cdots, u_n) := (s_1, s_2, \cdots, s_n)$ with $s_i = ru_j$ and $s_k = 0$ if $k \neq i$. Clearly, $f^r_{ij} = i_i f^r \pi_j$, where $f^r : N \to N$, $r \in N$ by $f^r(x) = rx$, for all $x \in N$. For any ideal $\mathcal{I}$ of N^n, we denote $\mathcal{I}_{**} = \{a \in N : a = \pi_j A,$ for some $A \in \mathcal{I},\ 1 \leqslant j \leqslant n\}$, where π_j is the j^{th} projection map.

In this section, we denote $l.i.$ for linearly independent element and $u - l.i.$ for uniform linearly independent element.

Lemma 4.1. [5] *Let $\mathcal{I}$ be an ideal of N^n. Then*

$$\mathcal{I}_{**} = \{a \in N : (a, 0, \cdots, 0) \in \mathcal{I}\}.$$

Proposition 4.2. [5] *Suppose L is a subset of N. Then L is an ideal of N if and only if L^n is an ideal of N^n.*

Theorem 4.3. [5] *For any $a \in N$, $\langle a \rangle^n = \langle (a, 0, \cdots, 0) \rangle$.*

Proposition 4.4. *Let I, J and Δ (proper) be ideals of N.*

(*i*) $I \subseteq J$ *in* ${}_NN$ *if and only if* $I^n \subseteq J^n$ *in* $M_N(N)$*-group* N^n.

(*ii*) $I \cap J \subseteq \Delta$ *in* ${}_NN$ *if and only if* $(I \cap J)^n \subseteq \Delta^n$ *in* $M_N(N)$*-group* N^n.

Proof. (*i*). Take $\rho = (x_1, \cdots, x_n) \in I^n$. Then

$$\begin{aligned}\rho = (x_1, \cdots, x_n) &= f_{11}^{x_1}(1, 1, \cdots, 1) + f_{22}^{x_2}(1, 1, \cdots, 1) + \cdots + f_{nn}^{x_n}(1, 1, \cdots, 1)\\ &= (f_{11}^{x_1} + f_{22}^{x_2} + \cdots + f_{nn}^{x_n})(1, 1, \cdots, 1)\\ &= A\rho_1 \in I^n.\end{aligned}$$

This implies $\pi_i(A\rho_1) \in I \subseteq J$, for all i, $1 \leqslant i \leqslant n$. Therefore, $A\rho_1 \in J^n$. Hence, $\rho \in J^n$.

Conversely, let $x_1 \in I$. Then $(x_1, \cdots, 0) \in I^n \subseteq J^n$, implies that $x_1 \in (J^n)_{**} = J$, by Proposition 4.2.

(*ii*) Assume that $I \cap J \subseteq \Delta$. Let $\rho = (x_1, \cdots, x_n) \in (I \cap J)^n$. Then there exist $A \in M_n(N)$ and $\rho_1 \in N^n$ such that $\rho = A\rho_1 \in (I \cap J)^n$. This implies $\pi_i(A\rho_1) \in I \cap J \subseteq \Delta$, for all i. Now, $A\rho_1 = (x_1, \cdots, x_n) = \sum_i (0, \cdots, \underbrace{x_i}_{i^{th}}, \cdots, 0) \in \Delta^n$, shows that $(I \cap J)^n \subseteq \Delta^n$.

Conversely, assume that $(I \cap J)^n \subseteq \Delta^n$. Let $x \in I \cap J$. Then $f_{ij}^x(1, \cdots, 1) = (0, \cdots, \underbrace{x}_{i^{th}}, \cdots, 0) \in (I \cap J)^n$. Take $A = f_{ij}^x$ and $\rho_1 = (1, \cdots, 1)$. Then $A\rho_1 \in (I \cap J)^n \subseteq \Delta^n$. Thus, $x = \pi_i(0, \cdots, \underbrace{x}_{i^{th}}, \cdots, 0) = \pi_i(A\rho_1) \in \Delta$. □

Lemma 4.5. [5] *If $\mathcal{I}$ is an ideal of N^n, then $(\mathcal{I}_{\star\star})^n = \mathcal{I}$.*

Remark 4.6. [5] Suppose K, L are ideals of N. Then $(K \cap L)^n = K^n \cap L^n$.

Proposition 4.7. *Let $\delta, l_1, \cdots, l_k \in {}_NN$. Then*

(*i*) *$g \in {}_NN$ is δ-uniform if and only if $(g, 0, ..., 0) \in {}_{M_n(N)}N^n$ is $(\delta, 0, ..., 0)$-uniform.*

(*ii*) *$l_1, \cdots, l_k$ are δ-l.i. in ${}_NN$ if and only if $(l_1, 0, \cdots, 0), (l_2, 0, \cdots, 0), \cdots, (l_k, 0, \cdots, 0)$ are $(\delta, \cdots, 0)$-l.i. in $M_n(N)$-group N^n.*

(*iii*) *$l_1, \cdots, l_k$ are δ-u-l.i. in ${}_NN$ if and only if $(l_1, 0, \cdots, 0), (l_2, 0, \cdots, 0), \cdots, (l_k, 0, \cdots, 0)$ are $(\delta, 0, \cdots, 0)$-u-l.i. element in $M_n(N)$-group N^n.*

Proof. (i). Suppose that g is δ-uniform. On a contrary, let $(g, 0, ..., 0)$ is not $(\delta, 0, ..., 0)$-uniform in $M_n(N)$-group N^n. That is, $\langle(g, 0, ..., 0)\rangle$ is not a $\langle(\delta, 0, ..., 0)\rangle$-uniform ideal in N^n. Then there exist $\mathcal{I}, \mathcal{J} \trianglelefteq N^n$ such that $\mathcal{I}, \mathcal{J} \nsubseteq \langle(\delta, 0, ..., 0)\rangle$ and $\mathcal{I}, \mathcal{J} \subseteq \langle(g, 0, ..., 0)\rangle$, $\mathcal{I} \cap \mathcal{J} \subseteq \langle(\delta, 0, ..., 0)\rangle$. Now by Lemma 4.5, $\mathcal{I} \cap \mathcal{J} \subseteq \langle(\delta, 0, ..., 0)\rangle$, implies $(\mathcal{I}_{**})^n \cap (\mathcal{J}_{**})^n \subseteq \langle(\delta, 0, ..., 0)\rangle$. By Remark 4.6 and Theorem 4.3, we get $(\mathcal{I}_{**} \cap \mathcal{J}_{**})^n \subseteq \langle(\delta, 0, ..., 0)\rangle = \langle\delta\rangle^n$, implies $\mathcal{I}_{**} \cap \mathcal{J}_{**} \subseteq \langle\delta\rangle$. Since $\mathcal{I}, \mathcal{J} \subseteq \langle(g, 0, ..., 0)\rangle = \langle g\rangle^n$, we get $(\mathcal{I}_{**})^n \subseteq \langle g\rangle^n, (\mathcal{J}_{**})^n \subseteq \langle g\rangle^n$. This implies $\mathcal{I}_{**} \subseteq \langle g\rangle, \mathcal{J}_{**} \subseteq \langle g\rangle$. Also, since $\mathcal{I}, \mathcal{J} \nsubseteq \langle(\delta, 0, ..., 0)\rangle = \langle\delta\rangle^n$, we get $\mathcal{I}_{**} \nsubseteq \langle\delta\rangle, \mathcal{J}_{**} \nsubseteq \langle\delta\rangle$, whereas $\mathcal{I}_{**} \cap \mathcal{J}_{**} \subseteq \langle\delta\rangle$, a contradiction.

(ii) Suppose that $\{l_i\}_{i=1}^k$ is δ-l.i. elements of N. Then $\sum_{i=1}^k \langle l_i\rangle$ is $\langle\delta\rangle$-direct $\Leftrightarrow$

$\langle l_i\rangle \cap \big(\sum_{i\neq j}^k \langle l_i\rangle\big) \subseteq \langle\delta\rangle \Leftrightarrow \big[\langle l_i\rangle \cap \big(\sum_{i\neq j}^k \langle l_i\rangle\big)\big]^n \subseteq \langle\delta\rangle^n \Leftrightarrow \big[\langle l_i\rangle^n \cap \big(\sum_{i\neq j}^k \langle l_i\rangle\big)^n\big] \subseteq$

$\langle\delta\rangle^n$. Therefore, $\sum_{i=1}^k \langle l_i\rangle^n$ is $\langle\delta\rangle^n$-direct. Thus $\sum_{i=1}^k \langle(l_i, 0, \cdots, 0)\rangle$ is $\langle(\delta, 0, \cdots, 0)\rangle$-direct.

(iii) Suppose that $l_1, \cdots, l_k$ are δ-u-l.i elements in ${}_NN$. Then $l_1, \cdots, l_k$ are δ-l.i elements and each l_i's are δ-uniform. By (ii), $(l_1, 0, \cdots, 0), (l_2, 0, \cdots, 0)$, $\cdots, (l_k, 0, \cdots, 0)$ are $(\delta, 0, \cdots, 0)$-l.i. elements, and by (i) these elements are $(\delta, 0, \cdots, 0)$-uniform in $M_n(N)$-group N^n. Thus, $(l_1, 0, \cdots, 0), (l_2, 0, \cdots, 0)$, $\cdots, (l_k, 0, \cdots, 0)$ are $(\delta, 0, \cdots, 0)$-u-l.i. element in $M_n(N)$-group N^n. □

Theorem 4.8. *If A is Δ-essential ideal of an N-group over itself, then A^n is Δ^n-essential in $M_n(N)$-group N^n. Moreover, the Δ-f.G.d. of N over itself is same as Δ^n-f.G.d. of $M_n(N)$-group N^n.*

Proof, Suppose $\mathcal{I} \trianglelefteq G$ such that $A^n \cap \mathcal{I} \subseteq \Delta^n$. Now, $\mathcal{I} = (\mathcal{I}_{**})^n$. Then by Remark 4.6, $(A \cap \mathcal{I}_{**})^n \subseteq \Delta^n$, and by Proposition 4.4(i), $A \cap \mathcal{I}_{**} \subseteq \Delta$. Since $A \leq_\Delta^e N$, we get $\mathcal{I}_{**} \subseteq \Delta$. Hence, $\mathcal{I} = (\mathcal{I}_{**})^n \subseteq \Delta^n$, which shows that A^n is Δ^n-essential in $M_n(N)$-group N^n. Suppose ${}_NN$ has Δ-$f.G.d.$ and Δ-dim ${}_NN = n$. Then there exist $\mathcal{I}_i$ $(1 \leqslant i \leqslant n) \nsubseteq \Delta$ such that $\oplus\mathcal{I}_i \leq_\Delta^e N$. Since $\mathcal{I}_i \nsubseteq \Delta$, by Proposition 4.4, we have $\mathcal{I}_i^n \nsubseteq \Delta^n$, for all i $(1 \leqslant i \leqslant n)$ and since $\oplus\mathcal{I}_i \leq_\Delta^e N$, it follows that $\oplus\mathcal{I}_i^n$ is Δ^n-essential in $M_n(N)$-group N^n. Therefore, Δ^n-$f.G.d.$ of $M_n(N)$-group N^n is n. □

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