Quasigroups and Related Systems **33** (2025), 227 – 230
https://doi.org/10.56415/qrs.v33.17

On the simple Suzuki groups $Sz(8)$ by the average orders

Behnam Ebrahimzadeh

Abstract. We prove that simple suzuki group $Sz(8)$, can be uniquely determined by its average order.

1. Introduction

Throughout this paper G is a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. Next, the sum of element orders in a group G denoted by $\psi(G) = \sum_{g\in G} o(g)$ where $o(g)$ denotes the order of $g \in G$ and also we define average order of G as $o(G) = \frac{\psi(G)}{|G|}$. In the way, the function $\psi(G)$ was introduced by Amiri, Jafarian and Isaacs [2]. They proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(Z_n)$. In fact Z_n is characterized by $\psi(Z_n)$ and $|Z_n|$. In this paper we goal discuss about average order of group. We show that the group G is characterized by $o(G)$, when ever there exist the group H, so that if $o(G) = o(H)$, then $G \cong H$. In the way, for example the authors in [1, 3, 5, 6, 8, 12, 15] proved that $PSL(2;5)$ and $PSL(2;7)$ are uniquely determined by their orders and the sum of the element orders. But only some of groups is characterized by average order. For example, in [17] and [21] is proved the alternating group A_5 and the symmetric group S_4 can be determined by average order.

In this paper, we prove that the simple Suzuki group $Sz(8)$, can be uniquely determined by its average order. Namely, we prove

Main Theorem. Let $SZ(8)$ be the simple Suzuki group and G be a droup such that $o(G) = o(Sz(8))$. Then $G \cong Sz(8)$.

2010 Mathematics Subject Classification: 20D06, 20D60.
Keywords: element order, sum of elements order, average order.

Notation and preliminaries

Let $m_i(G)$ be the number of elements of order i and $\pi_e(G)$ denotes the set of element orders of G. We define $\psi(G) = \sum_{i\in\pi_e(G)} im_i(G)$ and $|G| = \sum_{i\in\pi_e(G)} m_i(G)$ and $o(G) = \frac{\psi(G)}{|G|}$

Lemma 2.1. [9] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3. [4] *Let G be a 2-Frobenius group of even order. Then*

!. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*

2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. In particular, every 2-Frobenius group is soluble group.*

Lemma 2.4. [21] *Let G be a group and H a non-trivial normal subgroup of G.*

1. *For each $x \in G$ and $h \in H$, $o(xH) \mid o(xh)$,*
2. *$\frac{\psi(H)-|H|}{|G|}$. In particular $o(G/H) < o(G)$.*

Lemma 2.5. [20] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$*

Lemma 2.6. [2] *Let $P \in Syl_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 2.7. [11] *Let G_1, G_2 be finite groups, p be a prime number and let a, b be positive integers. Then the following hold:*

1. *$\psi(G_1 \times G_2) = \psi(G_1) \times \psi(G_2)$ if and only if $(|G_1|, |G_2|) = 1$ i.e. ψ is multiplicative;*
2. *$\psi(p^a) = \frac{p^{2a+1}+1}{p+1}$,*
3. *$\psi(p^a) \mid \psi(p^b)$ if and only if $2a + 1 \mid 2b + 1$,*
4. *$(\psi(p), \psi(p^2)) = (\psi(p), \psi(p^3)) = (\psi(p^2), \psi(p^3)) = 1$.*

Theorem 2.8. [17] *Let G be a finite group. If $o(G) < \frac{211}{60}$ then G is solvable.*

Proof of the Main Theorem

First, assume $|Sz(8)| = 2^6.5.7.13$ and $o(Sz(8)) = \frac{219311}{29120} = o(G) = 7.53$. Now, by Theorem 2.8 G is not soluble group. Since $o(G) > \frac{211}{60}$. Therefore G is not 2-Frobenius group. Now, we prove G is not a Frobenius group. On opposite, assume G be a Frobenius group. so $t(G) = 2$, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since 13 is an isolated vertex in $\Gamma(G)$. So case(i) $|H| = 13$ and $|K| = \frac{|G|}{13}$ and case(ii) $|K| = 13$ and $|H| = \frac{|G|}{13}$. We can see easily case(ii) not be accurs. Hence, we consider case(i). Since that $|H|$ divided $|K|1so13 \mid \frac{29120}{13} - 1$, where this is impossible. Hence G is not a Frobenius group. Now, by Lemma 2.5 and Lemma 2.4 then $o(G/H) < o(G)$. On the other hand, we know, suzuki group $Sz(q)$ is only group where 3 not divided order of it. So $G/H \cong Sz(q')$, where $q' = 2^{2m+1}$ and $o(G/H) < 7.53$ in special case $q' = 8$. In other words, $G/H \cong Sz(8)$, on the other hand G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ as $H = 1$, so $G \cong Sz(8)$, the proof be completed.

References

[1] **H. Amiri and S.M.J. Amiri**, *Sum of element orders of maximal subgroups of the symmetric group*, Comm. Algebra, **40** (2012), 770 – 778.

[2] **H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs**, *Sums of element orders in finite groups*, Comm. Algebra, **37** (2009), 2978 – 2980.

[3] **M. Baniasad Azad, B. Khosravi**, *On the sum of element orders of $PSL(2;p)$ for some p*, Italian J. Pure Appl. Math., **42** (2019), 18 – 24.

[4] **G.Y. Chen**, *On the structure of Frobenius groups and 2-Frobenius groups*, Southwest China Normal Univ., **20** (1995), 485 – 487.

[5] **S.M. Jafarian Amiri**, *Characterization of A_5 and $PSL(2;7)$ by sum of element orders*, Int. J. Group Theory, **2** (2013), 35 – 39.

[6] **M. Jahania, Y. Marefat, H. Refaghat and B.V. Fasaghandisi**, *A nilpotency criterion for finite groups by the sum of element orders*, Int. J. Nonlinear Anal. Appl, **14** (2023), 2931 – 2937.

[7] **G. Frobenius**, *Verallgemeinerung des sylowschen satze*, Berliner sitz, (1895), 981 – 983.

[8] **M. Gholami, Y. Marefat, H. Doostie and H. Refaghat**, *Remarks on the sum of element orders of non-group semigroups*, J. Algebra and Related Topics, **10** (2022), no.2, 113 – 129.

[9] **D. Gorenstein**, *Finite groups*, Harper and Row, New York, (1980).

[10] **J.B. Li,** *Finite groups with special conjugacy class sizes or generalized permutable subgroups*, (Chongqing: Southwest Univ., (2012).

[11] **M.S. Lazorec**, *A graph related to the sum of element orders of a finite group*, Math.GR, (2022), 1 – 19.

[12] **Y. Marefat, A. Iranmanesh and A. Tehranian**,*On the sum of element orders of finite simple groups*, J. Algebra Appl., **12** (2013), no.7, Article ID 1350026.

[13] **V. D. Mazurov, E.I. Khukhro**, *Unsolved problems in group theory*, The Kourovka Notebook, 16 ed. Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk 2006.

[14] **J. Moori, B.G. Rodrigues, A. Saedi and S. Zandi**, *Designs from maximal subgroups and conjugacy classes of ree groups*, Adv. Math. Commun., **14** (2020), 603 – 611.

[15] **H. Rashidi, M. Etezadi**, *Characterization of* $Alt(5) \times Z_p$*, where* $p \in \{17, 23\}$*, by their product element orders*, Algebraic Structures Appl., **10** (2023), no.2, 99 – 105.

[16] **G.R. Rezaeezadeh, M.R. Darafsheh, M. Bibak and M. Sajjadi**, *OD-characterization of almost simple groups related to* $D_4(4)$, Iran. J. Math. Sci. Inform, **10** (2015), no.1, 23 – 43.

[17] **M. Tarnauceanu**, *A characterzation of* A_5 *by its average order*, Int. J. Group Theory, **14** (2025), no.3, 117 – 123.

[18] **G.E. Wall**, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc., **3** (1963), 1 – 62.

[19] **J.S. Williams**, *Prime graph components of finite groups*, J. Algebra, **69**(1961), no.2,487 – 513.

[20] **H. Xu, G. Chen and Y. Yan,** *A new characterization of simple* K_3*-groups by their orders and large degrees of their irreducible characters*, Comm. Algebra, **42** (2014), 5374 – 5380.

[21] **A. Zarezadeh, B.Khosravi, Z. Akhlaghi**, *Characterizing some finite groups by the average order*, Math. GR, (2023), 1 – 9.

Received February 25, 2025

Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran
Email: behnam.ebrahimzadeh@gmail.com