Quasigroups and Related Systems **33** (2025), 201 − 208

https://doi.org/10.56415/qrs.v33.15

Self-orthogonal n-ary T-quasigroups

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Abstract. We characterize self-orthogonal and strongly self-orthogonal n-ary T-quasigroups in terms of automorphisms of their binary retracts.

1. Two binary quasigroups $Q(A)$ and $Q(B)$ are orthogonal if all $a, b \in Q$ the system of equations

$$\begin{cases} A(x,y) = a \\ B(x,y) = b \end{cases}$$

is uniquely solvable. If this system has a unique solution for $B(x,y) = A(y,x)$, then we say that a quasigroup $Q(A)$ is self-orthogonal. This concept has many generalizations to the n-ary case and is studied by many authors in various directions. Belyavskaya and Mullen investigated in [2] and [3] the properties of orthogonal hypercubes and their connections to the orthogonality of n-ary operations. Dudek and Syrbu in [5] and [10] (see also [4]) described self-orthogonal n-groups. The orthogonality of certain types of n-groups and n-quasigroups was also studied in [7], [8] and [9]. Medial ternary quasigroups are studied in [6].

In this paper, we find necessary and sufficient conditions for a linear or medial n-quasigroup to be self-orthogonal. We also provide a characterization of medial 3-quasigroups for which every triplet of distinct parastrophes is orthogonal. Our results are inspired by the results obtained in [6].

2. The notions and symbols used in this article are the same as in [1].

Recall that an n-quasigroup $Q(A)$ is nonempty set Q with the operation $A : Q^n \to Q$ such that in the expression $A(X_1^n) = x_{n+1}$ each n element uniquely determines the remaining one.

The system $\{A_1, A_2, \ldots, A_t\}$, $t \geqslant n$, of n-ary operations defined on Q is orthogonal if each its subsystem $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ is orthogonal, i.e. the

2010 Mathematics Subject Classification: 20N15.

Keywords: self-orthogonal n-quasigroup, orthogonal system, medial n-quasigroup.

system of equations $\{A_{i_j}(x_1^n) = a_i\}_{j=1}^n$ has a unique solution for all $a_1^n \in Q$. If $t = n$ and each subsystem of $\{A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_n\}$, where all E_i are selectors, i.e. $E_i(x_1^n) = x_i$, $i \in N_n = \{1, 2, \ldots, n\}$, containing n operations is orthogonal, then we say that the system $\{A_1, A_2, \ldots, A_n\}$ is *strongly orthogonal.*

An n-quasigroup $Q(A)$ is called *self-orthogonal* if it has n orthogonal principal parastrophes $A^{\sigma_1}, A^{\sigma_2}, \ldots, A^{\sigma_n}$. If $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ forms a cyclic subgroup in the group $\mathbb{S}_n$, the we say that $Q(A)$ is *cyclically self-orthogonal.* An n-quasigroup that is self-orthogonal for all possible collections of n permutations $\sigma_i \in \mathbb{S}_n$ is called *totally self-orthogonal.*

By fixing $j \in N_{n-2}$ variables in $A(x_1^n)$ we obtain a new $(n-j)$-ary operation B called the $(n-k)$-ary retract of A. All $(n-j)$-ary retracts of an n-quasigroup $Q(A)$ are $(n-2)$-quasigroups and can be used to investigate the orthogonality of the initial n-quasigroup $Q(A)$. One such possibility is provided by the following theorem, which is a modified version of Theorem 3 in [3].

Theorem 1. *An orthogonal set of n-quasigroup operations $A_1 A_2, \ldots, A_n$ defined on a finite set Q, is strongly orthogonal if and only if for each $j \in N_n$ all $(n-j)$-ary retracts are orthogonal.*

3. Let A be an n-ary operation defined on Q and γ be a permutation of Q. The operation γA defined by $(\gamma A)(x_1^n) = \gamma(A(x_1^n))$ is called a *torsion* of A.

Proposition 2. *The set of n-ary operations is orthogonal if and only if the set of their torsions is orthogonal.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be permutations of Q. If the system of equations

$$\left\{A_i(x_1^n) = \alpha_i^{-1}(a_i)\right\}_{i=1}^n$$

has a unique solution for any $a_1^n \in Q$, then the system

$$\{(\alpha_i A_i)(x_1^n) = a_i\}_{i=1}^n$$

also has a unique solution. So it is orthogonal. □

4. An n-quasigroup $Q(A)$ is a T-n-quasigroup if its operation has the form

$$A(x_1^n) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n) + c, \tag{1}$$

where $Q(+)$ is a commutative group, $\varphi_1, \ldots, \varphi_n$ are automorphisms of $Q(+)$ and c is a fixed element of Q.

Proposition 3. *Any parastrophe of a T-n-quasigroup is a T-n-quasigroup.*

Proof. Let $Q(A^\sigma)$ be a σ-parastrophe of $Q(A)$, i.e.

$$A^\sigma(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1^n) = x_{n+1}.$$

Then,

$$A^\sigma(x_1^n) = x_{n+1} \Leftrightarrow A(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}) = x_{\sigma^{-1}(n+1)}$$

Thus, if $Q(A)$ is a T-n-quasigroup, so, by (5), we have

$$\varphi_1(x_{\sigma^{-1}(1)}) + \varphi_2(x_{\sigma^{-1}(2)}) + \dots + \varphi_n(x_{\sigma^{-1}(n)}) + c = x_{\sigma^{-1}(n+1)},$$

i.e.

$$\varphi_1(x_{\sigma^{-1}(1)}) + \varphi_2(x_{\sigma^{-1}(2)}) + \dots + \varphi_n(x_{\sigma^{-1}(n)}) + \varphi_{n+1}(x_{\sigma^{-1}(n+1)}) + c = 0,$$

where $\varphi_{n+1} = -\varepsilon$. Therefore

$$\varphi_{\sigma(1)}(x_1) + \varphi_{\sigma(2)}(x_2) + \dots + \varphi_{\sigma(n)}(x_n) + \varphi_{\sigma(n+1)}(x_{n+1}) + c = 0.$$

Hence $Q(A^\sigma)$ is a T-n-quasigroup. □

From the last equation of the above proof we obtain

Corollary 4. *If a T-n-quasigroup $Q(A)$ has the form* (5)*, then for any $\sigma \in \mathbb{S}_{n+1}$*

$$A^\sigma(x_1^n) = \sum_{i=1}^{n} \varphi_{\sigma(n+1)}^{-1} \varphi_{\sigma(i)} J(x_i) + \varphi_{\sigma(n+1)}^{-1} J(c), \tag{2}$$

where $J = -\varepsilon = \varphi_{n+1}$.

Theorem 5. *The parastrophes $A^{\sigma_1}, \dots, A^{\sigma_n}$ of a T-n-quasigroup $Q(A)$ of the form* (5) *are orthogonal if and only if the determinant*

$$|D| = \begin{vmatrix} \varphi_{\sigma_1(1)} & \varphi_{\sigma_1(2)} & \cdots & \varphi_{\sigma_1(n)} \\ \varphi_{\sigma_2(1)} & \varphi_{\sigma_2(2)} & \cdots & \varphi_{\sigma_2(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{\sigma_n(1)} & \varphi_{\sigma_n(2)} & \cdots & \varphi_{\sigma_n(n)} \end{vmatrix} \tag{3}$$

is an automorphism of the group $Q(+)$.

Proof. According to Proposition 2, orthogonality of the parastrophes $A^{\sigma_1}, A^{\sigma_2}, \dots, A^{\sigma_n}$ is equivalent to orthogonality of their torsions

$$L_c^{-1} \varphi_{\sigma_1(n+1)} J(A^{\sigma_1}), \quad L_c^{-1} \varphi_{\sigma_2(n+1)} J(A^{\sigma_1}), \; \dots, \; L_c^{-1} \varphi_{\sigma_n(n+1)} J(A^{\sigma_1}),$$

i.e. to the fact that the system of equations

$$\begin{cases} L_c^{-1}\varphi_{\sigma_1(n+1)}J(A^{\sigma_1})(x_1^n) = \varphi_{\sigma_1(1)}(x_1) + \ldots + \varphi_{\sigma_1(n)}(x_n) = a_1 \\ L_c^{-1}\varphi_{\sigma_2(n+1)}J(A^{\sigma_2})(x_1^n) = \varphi_{\sigma_2(1)}(x_1) + \ldots + \varphi_{\sigma_2(n)}(x_n) = a_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ L_c^{-1}\varphi_{\sigma_n(n+1)}J(A^{\sigma_n})(x_1^n) = \varphi_{\sigma_n(1)}(x_1) + \ldots + \varphi_{\sigma_n(n)}(x_n) = a_n \end{cases}$$

has a unique solution for each $a_1^n \in Q$.

This system has a unique solution if and only if the system

$$\begin{cases} \varphi_{\sigma_1(1)}(x_1) + \varphi_{\sigma_1(2)}(x_2) \ldots + \varphi_{\sigma_1(n)}(x_n) = b_1 \\ \varphi_{\sigma_2(1)}(x_1) + \varphi_{\sigma_2(2)}(x_2) \ldots + \varphi_{\sigma_2(n)}(x_n) = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_{\sigma_n(1)}(x_1) + \varphi_{\sigma_n(2)}(x_2) \ldots + \varphi_{\sigma_n(n)}(x_n) = b_n \end{cases}$$

has a unique solution.

The last system which can be written as $DX = B$, where

$$D = \begin{pmatrix} \varphi_{\sigma_1(1)} & \varphi_{\sigma_1(2)} & \cdots & \varphi_{\sigma_1(n)} \\ \varphi_{\sigma_2(1)} & \varphi_{\sigma_2(2)} & \cdots & \varphi_{\sigma_2(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{\sigma_n(1)} & \varphi_{\sigma_n(2)} & \cdots & \varphi_{\sigma_n(n)} \end{pmatrix},$$

$X = (x_1, \ldots, x_n)^T$, $B = (b_1, \ldots, b_n)^T$, has a unique solution only in the case when D is an invertible matrix, i.e. only when the determinant of D is an automorphism of the group $Q(+)$. □

Corollary 6. *A T-n-quasigroup $Q(A)$ of the form* (5) *is totally self-orthogonal if and only if all determinants of the form* (4) *are automorphisms of the group $Q(+)$.*

Corollary 7. *If a T-n-quasigroup $Q(A)$ of the form* (5) *is totally self-orthogonal then $\varphi_i \neq \varphi_j$ for $i \neq j$.*

Theorem 8. *A T-n-quasigroup $Q(A)$ of the form* (5) *is totally self-orthogonal if and only if $\varphi_1 + \varphi_2 + \ldots + \varphi_n$ and for all $\sigma_1, \ldots, \sigma_n \in \mathbb{S}_n$ the determinants*

$$|D'| = \begin{vmatrix} \varphi_{\sigma_1(1)} & \varphi_{\sigma_1(2)} & \cdots & \varphi_{\sigma_1(n-1)} & \varepsilon \\ \varphi_{\sigma_2(1)} & \varphi_{\sigma_2(2)} & \cdots & \varphi_{\sigma_2(n-1)} & \varepsilon \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{\sigma_n(1)} & \varphi_{\sigma_n(2)} & \cdots & \varphi_{\sigma_n(n-1)} & \varepsilon \end{vmatrix} \tag{4}$$

are automorphisms of the group $Q(+)$.

Proof. Indeed, $|D| = (\varphi_1 + \varphi_2 + \ldots + \varphi_n)|D'|$. □

5. A T-n-quasigroup in which $\varphi_i\varphi_j = \varphi_j\varphi_i$ for all $i, j \in N_n$ is called *medial.* In other words, an n-quasigroup $Q(A)$ is medial if there exist a commutative group $Q(+)$ and its pairwise commuting automorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$ and an element $c \in Q$ such that

$$A(x_1^n) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n) + c. \tag{5}$$

The following theorem is a modified version of Theorem 4 proved in [6].

Theorem 9. *A medial* 3*-quasigroup* $Q(A)$ *of the form*

$$A(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + c \tag{6}$$

is totally self-orthogonal if and only if

$$\begin{gathered} \varphi_1 - \varphi_2, \quad \varphi_1 - \varphi_3, \quad \varphi_2 - \varphi_3, \quad \varphi_1 + \varphi_2 + \varphi_3, \\ \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \varphi_1\varphi_2 - \varphi_1\varphi_3 - \varphi_2\varphi_3 \end{gathered} \tag{7}$$

are automorphisms of the group $Q(+)$.

Proof. Suppose that $Q(A)$ is a medial 3-quasigroup. Then the operation A has the form (6). The automorphisms $\varphi_1, \varphi_2, \varphi_3$ generate a subring K in the ring of all endomorphisms of the group $Q(+)$. According to Theorem 5, orthogonality of parastrophes $A^{\sigma_1}, A^{\sigma_2}, A^{\sigma_3}$ is equivalent to the fact that the determinant of the corresponding system of equations is an automorphism of $Q(+)$. Thus $Q(A)$ is totally self-orthogonal if determinants of all systems of equations induced by all possible principal parastrophes $A^{\sigma_1}, A^{\sigma_2}, A^{\sigma_3}$ are automorphism of $Q(A)$, i.e. all determinants

$$D = \begin{vmatrix} \varphi_{\sigma_1(1)} & \varphi_{\sigma_1(2)} & \varphi_{\sigma_1(3)} \\ \varphi_{\sigma_2(1)} & \varphi_{\sigma_2(2)} & \varphi_{\sigma_2(3)} \\ \varphi_{\sigma_3(1)} & \varphi_{\sigma_3(2)} & \varphi_{\sigma_3(3)} \end{vmatrix}, \tag{8}$$

where $(\varphi_{\sigma_i(1)}, \varphi_{\sigma_i(2)}, \varphi_{\sigma_i(3)})$ corresponds to the parastrophe A^{σ_i}, are invertible over the subring of K generated by $\varphi_1, \varphi_2, \varphi_3$.

Now permute columns and rows in the determinant (8) we obtain the determinant

$$D_1 = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_{\mu(1)} & \varphi_{\mu(2)} & \varphi_{\mu(3)} \\ \varphi_{\eta(1)} & \varphi_{\eta(2)} & \varphi_{\eta(3)} \end{vmatrix}$$

with $1 \leqslant \mu(1) \leqslant \eta(1))$ and $\mu, \eta \in S_3$. The determinants D and D_1 are equivalent ($D \sim D_1$) in the sense that both are simultaneously invertible or non-invertible.

Add all columns to the last one we can see that

$$D \sim (\varphi_1 + \varphi_2 + \varphi_3) \begin{vmatrix} \varphi_1 & \varphi_2 & \varepsilon \\ \varphi_{\mu(1)} & \varphi_{\mu(2)} & \varepsilon \\ \varphi_{\eta(1)} & \varphi_{\eta(2)} & \varepsilon \end{vmatrix}.$$

Therefore,

$$D \sim \begin{vmatrix} \varphi_1 & \varphi_2 & \varepsilon \\ \varphi_{\mu(1)} & \varphi_{\mu(2)} & \varepsilon \\ \varphi_{\eta(1)} & \varphi_{\eta(2)} & \varepsilon \end{vmatrix}$$

under the condition that the polynomial $\varphi_1 + \varphi_2 + \varphi_3$ is invertible.

If the first or second column has three identical elements, $D = 0$. If one column, let's say the first one, has two identical elements then $\mu(1) = 1$ or $\mu(1) = \eta(1)$. In the first case $\mu(2) = 3$ because for $\mu(2) = 2$ we have two identical rows and $D = 0$. Then

$$D \sim \begin{vmatrix} \varphi_1 & \varphi_2 & \varepsilon \\ \varphi_1 & \varphi_3 & \varepsilon \\ \varphi_{\eta(1)} & \varphi_{\eta(2)} & \varepsilon \end{vmatrix} \sim \begin{vmatrix} \varphi_1 & \varphi_2 & \varepsilon \\ 0 & \varphi_3 - \varphi_2 & 0 \\ \varphi_{\eta(1)} & \varphi_{\eta(2)} & \varepsilon \end{vmatrix} = (\varphi_3 - \varphi_2)(\varphi_1 - \varphi_{\eta(1)}).$$

Since $\varphi_1 \neq \varphi_{\eta(1)}$ we have $\varphi_{\eta(1)} = \varphi_2$ or $\varphi_{\eta(1)} = \varphi_3$. Thus D is invertible if and only if $\varphi_3 - \varphi_2$, $\varphi_1 - \varphi_2$ and $\varphi_1 - \varphi_3$ are invertible.

At last, suppose the variables are different in each row and in each column. Then after permutations of rows and columns we get

$$D \sim \begin{vmatrix} \varphi_1 & \varphi_2 & \varepsilon \\ \varphi_3 & \varphi_1 & \varepsilon \\ \varphi_2 & \varphi_3 & \varepsilon \end{vmatrix} = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \varphi_1\varphi_2 - \varphi_1\varphi_3 - \varphi_2\varphi_3.$$

This completes the proof. □

By Theorem 1, the totally self-orthogonality of the 3-quasigroup $Q(A)$ is equivalent to the orthogonality of every set $\{A^{\sigma_1}, A^{\sigma_2}, A^{\sigma_3}\}$ of its principal parastrophes and to the orthogonality of all binary retracts of $Q(A)$. Therefore, the medial 3-quasigroup $Q(A)$ of the form (6) is strongly orthogonal if and only if each determinant

$$\begin{vmatrix} \varphi_{\sigma_1(1)} & \varphi_{\sigma_1(2)} & \varphi_{\sigma_1(3)} \\ \varphi_{\sigma_2(1)} & \varphi_{\sigma_2(2)} & \varphi_{\sigma_2(3)} \\ \varphi_{\sigma_3(1)} & \varphi_{\sigma_3(2)} & \varphi_{\sigma_3(3)} \end{vmatrix}$$

and all its minors of degree two are invertible, i.e. they are automorphisms of the group $Q(+)$. Proceeding similarly to the proof of Theorem 9, we obtain

Theorem 10. *A medial* 3*-quasigroup* $Q(A)$ *of the form* (6) *is strongly self-orthogonal if and only if the mappings* (7) *and*

$$\varphi_1+\varphi_2, \quad \varphi_1+\varphi_3, \quad \varphi_2+\varphi_3,$$
$$\varphi_1\varphi_2-\varphi_3^2, \quad \varphi_1\varphi_3-\varphi_2^2, \quad \varphi_2\varphi_3-\varphi_1^2$$

are automorphisms of the group $Q(+)$.

Corollary 11. *A medial* 3*-quasigroup* $Q(A)$ *of the form* (6) *is cyclically self-orthogonal if and only if the mappings*

$$\varphi_1+\varphi_2+\varphi_3 \quad \text{and} \quad \varphi_1^2+\varphi_2^2+\varphi_3^2-\varphi_1\varphi_2-\varphi_1\varphi_3-\varphi_2\varphi_3$$

are automorphism of $Q(+)$. *It is strongly self-orthogonal if and only if also*

$$\varphi_1\varphi_2-\varphi_3^2, \quad \varphi_1\varphi_3-\varphi_2^2, \quad \varphi_2\varphi_3-\varphi_1^2$$

are automorphism of $Q(+)$.

Corollary 12. *A medial* 4*-quasigroup* $Q(A)$ *of the form* (5) *is cyclically self-orthogonal if and only if the mappings*

$$\varphi_1+\varphi_2+\varphi_3+\varphi_4, \quad \varphi_1-\varphi_2+\varphi_3-\varphi_4, \quad (\varphi_1-\varphi_3)^2+(\varphi_2-\varphi_4)^2$$

are automorphism of $Q(+)$.

Proposition 13. *The smallest medial totally self-orthogonal ternary quasigroup has the form* $\mathbb{Z}_5(A)$ *where* $A(x,y,z)=x+2y+3z (\bmod 5)$. *The smallest medial strongly self-orthogonal ternary quasigroup has the form* $\mathbb{Z}_{11}(A)$ *where* $A(x,y,z)=x+2y+3z (\bmod 11)$.

Proof. By Corollary 7 should be $\varphi_1 \neq \varphi_2 \neq \varphi_3$, So $\mathbb{Z}_m$ must have at least four elements. The group $\mathbb{Z}_4$ has only two automorphisms; automorphism of the Klein's four group are not commutative. By Theorem 9 $\mathbb{Z}_5(A)$ with $A(x,y,z)=x+2y+3z (\bmod 5)$ is a medial totally self-orthogonal 3-quasigroup. It is the smallest 3-quasigroup with this property.

We can check the second statement in a similar way. □

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Received March 21, 2025
Revised July 10, 2025

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