Quasigroups and Related Systems **33** (2025), 161 – 180

https://doi.org/10.56415/qrs.v33.13

# Eventually semi strongly torsion free acts over monoids

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**Abstract.** We present eventually semi strongly torsion freeness of acts over monoids, which is a generalization of strongly torsion freeness. We say that a right $S$-act $A_S$ is eventually semi strongly torsion free if for every $s \in S$, there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that $as^n = a's^n$ for any $a, a' \in A_S$, implies $ar = a'r$ and $rs^n = s^n$, for some $r \in S$. We show that eventually semi strongly torsion freeness implies GPW-flatness. Also we give some general properties of eventually semi strongly torsion freeness and characterizations of monoids for which this property of their acts implies some other properties and vice versa.

## 1. Introduction

Throughout this paper we used $S$ to denote a monoid. We refer [7, 10] for basic results, definitions and terminologies relating to semigroups and acts over monoids, and to [11] for definitions and results on flatness which are used in the paper.

$S$ is called right (left) reversible if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'(su = s'v)$.

An element $s$ of $S$ is called right $e$-cancellable, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leqslant ker\rho_e$, i.e. $ts = t's$, $t, t' \in S$, implies $te = t'e$. $S$ is called left $PP$ if every element $s \in S$ is right $e$-cancellable, for some idempotent $e \in S$. It is easy to see that $S$ is left $PP$ if and only if for every $s \in S$ there exsits $e \in E(S)$, such that $ker\rho_s = ker\rho_e$. This is equivalent to saying that every principal left ideal of $S$ is projective. Similarly a right $PP$ is defined. An element $s \in S$ is called right semi-cancellative if $ts = t's$, $t, t' \in S$, implies there exists $r \in S$ such that $s = rs$ and $tr = t'r$. $S$ is called left $PSF$ if all principal left ideals of $S$ are strongly flat. It is easy to see that $S$ is left $PSF$ if and only if every element $s \in S$ is right semi-cancellable.

An element $s \in S$ is called regular, if $sxs = s$, for some $x \in S$. $S$ is called regular if all its elements are regular. An element $s$ of $S$ is called left almost regular if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$ such that

$$s_1c_1 = sr_1$$
$$s_2c_2 = s_1r_2$$

2020 Mathematics Subject Classification: 20M30

Keywords: $S$-act; eventually semi strongly torsion free; flat; GPW-flat

$$\cdots$$
$$s_m c_m = s_{m-1} r_m$$
$$s = s_m r s$$

If all elements of $S$ are left almost regular, then $S$ is called left almost regular. We can see that every left almost regular monoid is left $PP$ [10, IV, Proposition 1.3].

A non-empty set $A$ is a right $S$-act, usually denoted by $A_S$, on which $S$ acts unitarian from the right, that is, $(as)t = a(st)$ and $a1 = a$, for every $a \in A$ , $s, t \in S$, where 1 is the identity of $S$. A right $S$-act $A_S$ satisfies Condition $(P)$ if for all $a, a' \in A_S, s, s' \in S$, $as = a's'$, implies that there exist $b \in A_S, u, v \in S$ such that $a = bu, a' = bv$ and $us = vs'$. Recall, from [6] that a right $S$-act $A_S$ satisfies Condition $(P')$ if for all $a, a' \in A_S$ ,$s, t, z \in S$, $as = a't$ and $sz = tz$ imply the existence $b \in A$ and $u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vt$. $A_S$ is said to satisfy Condition $(E)$ if whenever $as = as'$ with $a \in A_S, s, s' \in S$, there exist $a' \in A_S, u \in S$ such that $a = a'u$ and $us = us'$. Recall, from [4, 5] that a right $S$-act $A_S$ satisfies Condition $(E')$ if $as = as'$ and $sz = s'z$, for $a \in A_S$ and $s, s', z \in S$, imply the existence $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = us'$. A right $S$-act $A_S$ satisfies Condition $(EP)$ if $as = at$, for $a \in A_S$, $s, t \in S$, implies the existence $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. Also, we say that $A_S$ satisfies Condition $(E'P)$ if $as = at$ and $sz = tz$, for $a \in A_S$ and $s, t, z \in S$, imply the existence $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(P) \Rightarrow (EP) \Rightarrow (E'P)$, $(E) \Rightarrow (E') \Rightarrow (E'P)$, $(P) \Rightarrow (P') \Rightarrow (E'P)$ and $(E) \Rightarrow (EP)$. We recall from [2, 11] that:

The $S$-act $A_S$ is weakly pullback flat (WPF), if the corresponding $\phi$ is bijective for every pullback diagram $P(S, S, f, g, S)$.

The $S$-act $A_S$ is weakly kernel flat (WKF), if the corresponding $\phi$ is bijective for every pullback diagram $P(I, I, f, f, S)$, where $I$ is a left ideal of $S$.

The $S$-act $A_S$ is principally weakly kernel flat (PWKF), if the corresponding $\phi$ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$.

The $S$-act $A_S$ is translation kernel flat (TKF), if the corresponding $\phi$ is bijective for every pullback diagram $P(S, S, f, f, S)$.

The $S$-act $A_S$ is weakly homoflat (WP), if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_S S$, all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'' \in A_S$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A \otimes_S (Ss \cup St)$ and $f(us') = f(vt')$.

The $S$-act $A_S$ is principally weakly homoflat (PWP), if $as = a's$, for $a, a' \in A_S$, $s \in S$, implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

The $S$-act $A_S$ is called torsion free if for any $a, a' \in A_S$ and for any right cancellable element $c \in S$ the equality $ac = a'c$ implies $a = a'$.

Recall from [14] that the right $S$-act $A_S$ is called strongly torsion free, if for any $a, b \in A_S$ and any $s \in S$, $as = bs$ implies $a = b$.

Recall from [9] that the right $S$-act $A_S$ satisfies Condition $(PWP_{ssc})$, if for $a, a' \in A_S$, $s \in S$, $as = a's$ implies that there exist $u \in S$ such that $au = a'u$ and $us = s$.

Recall from [8] that the right $S$-act $A_S$ is called $GP$-flat if $as = a's$ for $a, a' \in A_S$, $s \in S$ implies that there exists $n \in \mathbb{N}$, such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes Ss^n$. A right $S$-act $A_S$ is called $GPW$-flat if for every $s \in S$, there exists $n = n_{(s, A_S)} \in \mathbb{N}$, such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S (Ss^n)$.(see [12])

## 2. General properties

In this section, we introduce eventually semi strongly torsion freeness and give some of its general properties.

**Definition 2.1.** An element $s \in S$ is called *eventually right semi-cancellative* if there exists a natural number $n \in \mathbb{N}$ such that $ts^n = t's^n$ for any $t, t' \in S$, implies $tr = t'r$ and $rs^n = s^n$, for some $r \in S$. If all elements of $S$ are eventually right semi-cancellative then $S$ is called eventually right semi-cancellative. A monoid $S$ is called *eventually left PSF* if for all $s \in S$ there exists $n \in \mathbb{N}$ such that left ideal $Ss^n$ is strongly flat.

It is easy to see that $S$ is eventually right semi-cancellative if and only if $S$ is eventually left PSF. The concepts eventually left semi-cancellative and eventually right PSF can be defined similarly.

It is clear that every left $PSF$ monoid is eventually left $PSF$. The next example shows that the converse is not true in general.

**Example 2.2.** Consider the monoid $S = \{0, 1, a, b, c\}$ with the multiplication table

| | 0 | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | $a$ | $b$ | $c$ |
| $a$ | 0 | $a$ | $b$ | $c$ | $b$ |
| $b$ | 0 | $b$ | $c$ | $b$ | $c$ |
| $c$ | 0 | $c$ | $b$ | $c$ | $b$ |

We have $aa = ca$ and $r = 1$ be the only element of $S$ such that $ra = a$. But $a1 \neq c1$. Thus $S$ is not left $PSF$ monoid. Let $s \in S$.

Case 1: If $s = a$ and $xa^2 = ya^2$, for $x, y \in S$, then for $r = b = a^2$, we have $xr = xa^2 = ya^2 = yr$ and $ra^2 = bb = b = a^2$.

Case 2: If $s = b$ and $xb = yb$, for $x, y \in S$, then for $r = b$, we have $xr = xb = yb = yr$ and $rb = bb = b$.

Case 3: If $s = c$ and $xc^2 = yc^2$, for $x, y \in S$, then for $r = b = c^2$, we have $xr = xc^2 = yc^2 = yr$ and $rc^2 = bb = b = c^2$.

Thus $S$ is eventually left $PSF$ monoid.

**Definition 2.3.** A right $S$-act $A_S$ is called *eventually semi strongly torsion free* if for any $s \in S$ there exists a natural number $n = n_{(s,A_S)} \in \mathbb{N}$ such that $as^n = a's^n$ for any $a, a' \in A_S$, implies $ar = a'r$ and $rs^n = s^n$, for some $r \in S$. We use the abbreviation $ES - STF$ for eventually semi strongly torsion freeness.

The proof of the next Lemma is a direct consequence of Definitions 2.1 and 2.3.

**Lemma 2.4.** *$S_S$ is $ES - STF$ if and only if $S$ is eventually left $PSF$ monoid.*

Now we establish some general properties.

**Theorem 2.5.** *The following statements are true.*

(1) *$\Theta_S$ is $ES - STF$.*

(2) *If an act is $ES - STF$, then all its subacts are $ES - STF$.*

(3) *Any retract of an $ES-STF$ right $S$-act is $ES-STF$.*

(4) *If $A=\coprod_{i\in I} A_i$, where each $A_i$ is a right $S$-act, is $ES-STF$ then $A_i$ is $ES-STF$ for every $i\in I$.*

*Proof.* (1), (2) and (4) are obvious.

(3). Suppose that the right $S$-act $B_S$ is $ES-STF$. Also, assume that $A_S$ is a retract of $B_S$. Then there exist homomorphisms $f: B_S\to A_S$ and $f': A_S\to B_S$, such that $ff'=id_{A_S}$. Let $s\in S$. Since $B_S$ is $ES-STF$, there exists $n\in\mathbb{N}$ such that $bs^n=b's^n$, for $b,b'\in B_S$, implies $br=b'r$ and $rs^n=s^n$, for some $r\in S$. Let $as^n=a's^n$ for $a,a'\in A_S$. Then, $f'(as^n)=f'(a's^n)$ and so, $f'(a)s^n=f'(a')s^n$. Since $f'(a),f'(a')\in B_S$ and $B_S$ is $ES-STF$, there exists $r\in S$ such that $f'(a)r=f'(a')r$ and $rs^n=s^n$. Now we have $f(f'(ar))=f(f'(a'r))$ and so $ar=a'r$, which means that $A_S$ is $ES-STF$. □

**Theorem 2.6.** *The following statmentes are true.*

(1) *If a right $S$-act is $ES-STF$, then it is $GPW$-flat.*

(2) *If $S$ is eventually left $PSF$ monoid then every $GPW$-flat act is $ES-STF$.*

*Proof.* (1). Let $s\in S$. Suppose $n\in\mathbb{N}$ corresponds to $s$ in the definition of eventually semi strongly torsion free. Let $as^n=a's^n$, for $a,a'\in A_S$. Thus there exists $r\in S$ such that $ar=a'r$ and $rs^n=s^n$. We have:

$$a\otimes s^n=a\otimes rs^n=ar\otimes s^n=a'r\otimes s^n=a'\otimes rs^n=a'\otimes s^n$$

in $A\otimes_S Ss^n$. Hence $A_S$ is $GPW$-flat.

(2). Let $S$ be a eventually left $PSF$ monoid and $A_S$ be a $GPW$-flat right $S$-act and $s\in S$. Since $S$ is eventually left $PSF$, there exists $n\in\mathbb{N}$ such that $ts^n=t's^n$, for any $t,t'\in S$, implies $tr=t'r$ and $rs^n=s^n$ for some $r\in S$. Let $as^n=a's^n$ for $a,a'\in A_S$. Hence by [12, Proposition 2.3],

$$\begin{array}{cc} a=a_1s_1 & \\ a_1t_1=a_2s_2 & s_1s^n=t_1s^n \\ a_2t_2=a_3s_3 & s_2s^n=t_2s^n \\ \dots & \dots \\ a_kt_k=a' & s_ks^n=t_ks^n, \end{array}$$

for $k\in\mathbb{N}$, $a_1,...,a_k\in A_S$ and $s_1,t_1,...,s_k,t_k\in S$. Since $S$ is eventually left $PSF$, the equality $s_1s^n=t_1s^n$ implies there exists $r_1\in S$ such that $s_1r_1=t_1r_1$ and $r_1s^n=s^n$. The equality $s_2s^n=t_2s^n$ implies that $s_2r_1s^n=t_2r_1s^n$. Thus there exists $r_2\in S$ such that $s_2r_1r_2=t_2r_1r_2$ and $r_2s^n=s^n$, by assumption. Let $r=r_1r_2$. Thus $rs^n=r_1r_2s^n=s^n$. Also $s_1r=s_1r_1r_2=t_1r_1r_2=t_1r$ and $s_2r=t_2r$. By continuing this process, we can find $l\in S$ such that $ls^n=s^n$ and $s_il=t_il$ for every $i$,$(1\leqslant i\leqslant k)$. Therefore $al=a_1s_1l=a_1t_1l=a_2s_2l=...=a_kt_kl=a'l$, as required. □

The following example shows that the converse of part (1) of Theorem 2.6, is not true in general. This shows also flatness does not imply eventually semi strongly torsion freeness.

**Example 2.7.** Let $K$ be a proper right ideal of $S$. If $x, y$ and $z$ denote elements not belonging to $S$, define $A(K) = (\{x, y\} \times (S \setminus K)) \cup (\{z\} \times K)$, and define a right $S$-action on $A(K)$ by

$$(x,u)s = \begin{cases} (x, us), & if \ \ us \notin K \\ (z, us), & if \ \ us \in K. \end{cases}$$

$$(y,u)s = \begin{cases} (y, us), & if \ \ us \notin K \\ (z, us), & if \ \ us \in K. \end{cases}$$

$$(z,u)s = (z, us).$$

Then clearly $A(K)$ is a right $S$-act. Let $S = \{a^n \mid n \in \mathbb{N}\} \cup \{e, f, 0\}$ where $e^2 = e, f^2 = f, ef = fe = 0$ and $a^n e = ea^n = fa^n = a^n f = 0$ for all $n \in \mathbb{N}$. If $J = \{0, e\}$, then $J$ is a right ideal of $S$. Because $0 \in J0$ and $e \in Je$, $A(J)$, by [10, III, Proposition 12.19], is a flat $S$-act and so is $GPW$-flat, but it does not $ES-STF$. Otherwise $(x, f)a^n = (y, f)a^n$ implies that there exists $r \in S$ such that $(x, f)r = (y, f)r$ and $ra^n = a^n$. But $r = 1$ is the only element of $S$ such that $rs = s$ and $(x, f)1 \neq (y, f)1$, which is contradiction.

**Theorem 2.8.** *For any family $\{A_i\}_{i\in I}$ of right $S$-acts, if $\prod_{i\in I} A_i$ is $ES-STF$, then $A_i$ is $ES-STF$, for every $i \in I$.*

*Proof.* Let $s \in S$ and $i \in I$. By our assumption there exists $n \in \mathbb{N}$ such that $as^n = a's^n$, for $a, a' \in \prod_{i\in I} A_i$, implies $ar = a'r$ and $rs^n = s^n$, for some $r \in S$. Let $a_i s^n = a'_i s^n$ for any $a_i, a'_i \in A_i$, and let $a_j$ be an arbitrary in $(A_j)_S$ for $j \neq i$. If

$$c_k = \begin{cases} a_i & if \ \ k = i \\ a_k & if \ \ k \neq i \end{cases}$$

$$c'_k = \begin{cases} a'_i & if \ \ k = i \\ a_k & if \ \ k \neq i \end{cases}$$

then $(c_k)_I s^n = (c'_k)_I s^n$ and so, by assumption $(c_k)_I r = (c'_k)_I r$ and $rs^n = s^n$, for $r \in S$. Now we have $a_i r = a'_i r$, $rs^n = s^n$ and hence, $A_i$ is $ES-STF$. □

**Lemma 2.9.** *Let $S$ be a commutative monoid, $A_S$ a right $S$-act and $s \in S$. Suppose that there exists $n \in \mathbb{N}$ such that $as^n = a's^n$, $a, a' \in A_S$, implies $ar = a'r$ and $rs^n = s^n$. Let $m \in \mathbb{N}$ and $m > n$. If $as^m = a's^m$, $a, a' \in A_S$, then $ar = a'r$ and $rs^m = s^m$ for some $r \in S$.*

*Proof.* Since $m > n$ there exists $k \in \mathbb{N}$ such that $kn \leqslant m < (k+1)n$. Suppose that $as^m = a's^m$, for $a, a' \in S$. Thus $(as^{(m-n)})s^n = (a's^{(m-n)})s^n$. By assumption there exists $r_1 \in S$ such that $(as^{(m-n)})r_1 = (a's^{(m-n)})r_1$ and $r_1 s^n = s^n$. By the equality $r_1 s^n = s^n$ we gain $r_1 s^m = s^m$. Since $S$ is commutative, we have $ar_1 s^{(m-n)} = a'r_1 s^{(m-n)}$. By the equality $ar_1 s^{(m-n)} = a'r_1 s^{(m-n)}$ we have $(ar_1 s^{(m-2n)})s^n = (a'r_1 s^{(m-2n)})s^n$. By assumption there exists $r_2 \in S$ such that $as^{(m-2n)} r_1 r_2 = a's^{(m-2n)} r_1 r_2$ and $r_2 s^n = s^n$. By the equality $r_2 s^n = s^n$ we gain $r_2 s^m = s^m$. Also $r_1 r_2 s^m = s^m$. Continuing this procedure there exist $r_1, r_2, ..., r_k \in S$ such that $as^{(m-kn)} r_1 r_2 ... r_k = a's^{(m-kn)} r_1 r_2 ... r_k$ and $r_1 r_2 ... r_k s^m = s^m$. We have two cases:

Case 1. If $m = kn$ then by $as^{(m-kn)} r_1 r_2 ... r_k = a's^{(m-kn)} r_1 r_2 ... r_k$ we have $ar_1 r_2 ... r_k = a'r_1 r_2 ... r_k$. Let $r = r_1 r_2 ... r_k$. Then $ar = a'r$ and $rs^m = s^m$ and we are done.

Case 2. If $m \neq kn$ then by multiplying the equality

$$ar_1 r_2 ... r_k s^{(m-kn)} = a'r_1 r_2 ... r_k s^{(m-kn)}$$

by $s^{(k+1)n-m}$ we gain $ar_1r_2...r_ks^n = a'r_1r_2...r_ks^n$ and so there exists $r_{k+1} \in S$ such that $ar_1r_2...r_kr_{k+1} = a'r_1r_2...r_kr_{k+1}$ and $r_{k+1}s^n = s^n$. By the equality $r_{k+1}s^n = s^n$ we gain $r_{k+1}s^m = s^m$. Also $r_1r_2...r_{k+1}s^m = s^m$. Let $r = r_1r_2...r_kr_{k+1}$. Then $ar = a'r$ and $rs^m = s^m$, and so we are done. □

**Theorem 2.10.** *For a commutative monoid $S$, $\prod_{i=1}^{m} A_i$, where $A_i$, $1 \leqslant i \leqslant m$ are right $S$-acts, is $ES-STF$ if and only if $A_i$ is $ES-STF$ for every $1 \leqslant i \leqslant m$.*

*Proof.* Necessity. It is obvious by Theorem 2.8.
Sufficiency. Suppose $A_i$ is $ES-STF$, for every $1 \leqslant i \leqslant m$, and let $s \in S$. Then, there exists $n_i \in \mathbb{N}$ such that $a_is^{n_i} = a'_is^{n_i}$ for $a_i, a'_i \in A_i$ implies that $a_ir = a'_ir$ and $rs^{n_i} = s^{n_i}$ for some $r \in S$. Let $n = max\{n_1, n_2, ..., n_m\}$. If $(a_1, a_2, ..., a_m)s^n = (a'_1, a'_2, ..., a'_m)s^n$ for $a_i, a'_i \in A_i$, $1 \leqslant i \leqslant m$. By Lemma 2.9 the equality $a_1s^n = a'_1s^n$ implies that there exists $r_1 \in S$ such that $a_1r_1 = a'_1r_1$ and $r_1s^n = s^n$. The equality $a_2s^n = a'_2s^n$ implies $a_2r_1s^n = a'_2r_1s^n$ and so by Lemma 2.9, implies $a_2r_1r_2 = a'_2r_1r_2$ and $r_2s^n = s^n$ for some $r_2 \in S$. Therefor $a_1r_1r_2 = a'_1r_1r_2$, $a_2r_1r_2 = a'_2r_1r_2$ and $r_1r_2s^n = s^n$. Continuing this procedure after $m$ steps, there exist $r_1, r_2, ..., r_m \in S$ such that for each $i$, $a_ir_1r_2...r_m = a'_ir_1r_2...r_m$, and $r_1r_2...r_ms^n = s^n$. Let $r = r_1r_2...r_m$. Then for each $i$, $a_ir = a'_ir$ and $rs^n = s^n$. Hence $(a_1, a_2, ..., a_m)r = (a'_1, a'_2, ..., a'_m)r$ and $rs^n = s^n$, as required. □

**Theorem 2.11.** *$S$ is eventually left $PSF$ monoid if and only if the right $S$-act $S_S^m$ is $ES-STF$ for any $m \in \mathbb{N}$.*

*Proof.* Necessity. Suppose that $S$ is eventually left $PSF$ monoid. Let $s \in S$. Then there exist $n \in \mathbb{N}$ such that $ts^n = t's^n$ for any $t, t' \in S$, implies $tr = t'r$ and $rs^n = s^n$, for some $r \in S$. If $(a_1, a_2, ..., a_m)s^n = (a'_1, a'_2, ..., a'_m)s^n$ for $a_i, a'_i \in S$, $1 \leqslant i \leqslant m$, then $a_is^n = a'_is^n$ for any $1 \leqslant i \leqslant m$. By assumption, there exists $r_1 \in S$ such that $a_1r_1 = a'_1r_1$ and $r_1s^n = s^n$. The equalities $a_2s^n = a'_2s^n$ and $r_1s^n = s^n$ imply $a_2r_1s^n = a'_2r_1s^n$. Again by assumption there exists $r_2 \in S$ such that $a_2r_1r_2 = a'_2r_1r_2$ and $r_2s^n = s^n$. Therefore $a_1r_1r_2 = a'_1r_1r_2$, $a_2r_1r_2 = a'_2r_1r_2$ and $r_1r_2s^n = s^n$. Continuing this procedure after $m$ steps, there exist $r_1, r_2, ..., r_m \in S$ such that for each $i$, $a_ir_1r_2...r_m = a'_ir_1r_2...r_m$, and $r_1r_2...r_ms^n = s^n$. Let $r = r_1r_2...r_m$. Then for each $i$, $a_ir = a'_ir$ and $rs^n = s^n$. Hence $(a_1, a_2, ..., a_m)r = (a'_1, a'_2, ..., a'_m)r$ and $rs^n = s^n$, as required.
Sufficiency. If the right $S$-act $S^m$ is $ES-STF$ then $S_S$ is $ES-STF$, by Theorem 2.8. Thus $S$ is eventually left $PSF$ monoid, by Lemma 2.4. □

Recall, from [1] that for $S$, the cartesian product $S \times S$ equipped with the right $S$-action $(s, t)u = (su, tu)$, $s, t, u \in S$, is called the *diagonal act* of $S$ and it is denoted by $D(S)$.

In the following theorem we obtain equivalent condition for $S_S^n$ to be $ES-STF$ .

**Theorem 2.12.** *The following statements are equivalent:*

(1) *$S_S^n$ is $ES-STF$, for any $n \in \mathbb{N}$.*

(2) *There exists $n \in \mathbb{N}$ such that $S_S^n$ is $ES-STF$.*

(3) *$D(S)$ is $ES-STF$.*

(4) *$S$ is eventually left $PSF$.*

*Proof.* Implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (4). Define $\psi : S_S \to S_S^n$ by $\psi(s) = (s, s, ..., s)$. It is obvious that $\psi$ is monomorphism. Thus $S_S \cong Im\psi \leqslant S_S^n$ and so, by part (2) of Theorem 2.5, $Im\psi$ is $ES-STF$. Thus $S_S$ is $ES-STF$ and so, $S$ is eventually left $PSF$ by Lemma 2.4.

(3) $\Rightarrow$ (4). This easily follows from proof (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). It follows from Theorem 2.11. □

We recall from [10], that a right ideal $K$ of a monoid $S$ is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

**Definition 2.13.** A right ideal $K$ of $S$ is called *GPW-left stabilizing* if for every $s \in S$ there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$, implies that $ls^n = ks^n$ for some $k \in K$.

It is clear that every left stabilizing right ideal of $S$ is $GPW$-left stabilizing.

**Theorem 2.14.** *Let $K$ be a proper right ideal of $S$. Then the following statements are equivalent:*

1. $A_S = S\coprod^K S$ *is* $ES-STF$.
2. $K$ *is* $GPW$*-left stabilizing and* $S$ *is eventually left* $PSF$.

*Proof.* (1) $\Rightarrow$ (2). By part (1) of Theorem 2.6, $A_S = S\coprod^K S$ is $GPW$-flat and so, by [12, Theorem 2.10], $K$ is $GPW$-left stabilizing. On the other hand $A_S = S\coprod^K S = B_S \cup C_S$, where $B_S = \{(l,x)|l \in S \setminus K\} \cup K$, $C = \{(t,y)|t \in S \setminus K\} \cup K$, $B_S, C_S \leqslant A_S$ and $B_S \cong S_S \cong C_S$. By part (2) of Theorem 2.5, $B_S$ is $ES-STF$. Therefore by isomorphism $B_S \cong S_S$, $S_S$ is $ES-STF$ and so, $S$ is eventually left $PSF$, by Lemma 2.4.

(2) $\Rightarrow$ (1). Since $K$ is $GPW$-left stabilizing, it follows by [12, Theorem 2.10] that $A_S = S\coprod^K S$ is $GPW$-flat. On the other hand since $S$ is eventually left $PSF$, $GPW$-flatness and eventually semi strongly torsion freeness are equivalent by part (2) of Theorem 2.6. Hence $A_S = S\coprod^K S$ is $ES-STF$. □

The following example shows that if $S$ be eventually left $PSF$, then any right ideal of $S$ is not $GPW$-left stabilizing, in general.

**Example 2.15.** Let $S = (\mathbb{N}, .)$. Thus $S$ is cancellative and commutative. Hence $S$ is left PSF and so eventually left PSF. But $K = \mathbb{N} \setminus \{1\}$ is not $GPW$-left stabilizing ideal.

Golchin in [3] showed that if $S = G\dot{\cup} I$, where $G$ is a group and $I$ is an ideal of $S$, and $A$ is a right $S$-act such that it is ((principaly) weakly) flat, torsion free, satisfies Condition $(P)$ or $(P_E)$ as a right $I^1$-act, then it has these properties as a right $S$-act. Similarly, we can show the following theorem for eventually semi strongly torsion freeness.

**Theorem 2.16.** *Let $S = G \dot{\cup} I$ where $G$ is a group and $I$ is an ideal of $S$ and $A$ be a right $S$-act. $A$ is $ES-STF$ as a right $I^1$-act if and only if it is $ES-STF$ as a right $S$-act.*

*Proof.* Necessity. Suppose $A$ is $ES-STF$ as a right $I^1$-act and $s \in S$. Then there are two cases as follows:

Case 1. $s \in G$. Then $Ss = S$ and so, for every $n \in \mathbb{N}$, $Ss^n = Ss = S$. If $as = a's$ for $a, a' \in A$, then $a = a'$. By putting $r = 1$ the result follows.

Case 2. $s \in I \subseteq I^1$. Since $A$ is $ES-STF$ as a right $I^1$-act, there exists a natural number $n \in \mathbb{N}$ such that for $a, a' \in A$, $as^n = a's^n$ implies $ar = a'r$ and $rs^n = s^n$, for $r \in I^1 \subseteq S$. Hence $A$ is $ES-STF$ as a right $S$-act.

Sufficiency. Suppose $A$ is $ES-STF$ as right $S$-act. Let $i \in I^1 \subseteq S$. By assumption there exists $n \in \mathbb{N}$ such that $ai^n = a'i^n$, for $a, a' \in A$, implies $ar = a'r$ and $ri^n = i^n$ for some $r \in S$. We have two cases:

Case 1. If $r \in I^1$ then we are done.

Case 2. If $r \in G$ then equality $ar = a'r$ implies $a = a'$. Let $r' = 1 \in I^1$. Hence $ar' = a'r'$ and $r'i^n = i^n$ and so we are done. □

**Corollary 2.17.** *Let $S = G \dot{\cup} I$ where $G$ is a group and $I$ is an ideal of $S$. If all right $I^1$-acts are $ES-STF$, then all right $S$-acts are $ES-STF$.*

# 3. Eventually semi strongly torsion free cyclic acts

**Theorem 3.1.** *Let $\rho$ be a right congruence on $S$. Then the right $S$-act $S/\rho$ is $ES-STF$ if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for $x, y \in S$, $(xs^n)\rho(ys^n)$ implies $(xr)\rho(yr)$ and $rs^n = s^n$ for some $r \in S$.*

The proof of Theorem 3.1 is clear.

**Corollary 3.2.** *The principal right ideal $zS$ is $ES-STF$ if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for any $x, y \in S$, $zxs^n = zys^n$ implies $zxr = zyr$ and $rs^n = s^n$, for some $r \in S$.*

*Proof.* Since $zS \cong S/ker\lambda_z$, apply Theorem 3.1 with $\rho = ker\lambda_z$. □

Now we give an equivalence for Rees factor $S$-acts that are $ES-STF$.

**Theorem 3.3.** *Let $K$ be a right ideal of $S$. The right Rees factor $S$-act $S/K$ is $ES-STF$ if and only if for every $s \in S$ there exists a natural number $n \in \mathbb{N}$ such that $K$ fulfills conditions*

$$(I)\ (\forall x, y \in S)\ [(xs^n = ys^n \in S \setminus K) \Rightarrow (\exists r \in S)(rs^n = s^n \wedge xr = yr)],$$

$$(II)\ (\forall x, y \in S)\ \ [(xs^n, ys^n \in K) \Rightarrow (\exists r \in S)(rs^n = s^n \wedge (xr = yr \vee xr, yr \in K))].$$

*Proof.* Necessity. Suppose that the right Rees factor $S$-act $S/K$ is $ES-STF$ and $s \in S$. Then there exists a natural $n \in \mathbb{N}$ such that $(xs^n)\rho_K(ys^n)$, for all $x, y \in S$, implies $(xr)\rho_K(yr)$ and $rs^n = s^n$, for some $r \in S$. Let $xs^n = ys^n \in S \setminus K$, for $x, y \in S$. Then $(xs^n)\rho_K(ys^n)$, which implies the existance of $r \in S$ such that $(xr)\rho_K(yr)$ and $rs^n = s^n$, by Theorem 3.1. If $xr, yr \in K$ then $ys^n = xs^n = xrs^n \in K$ which is contradiction. Hence $xr = yr$ and $rs^n = s^n$ and so, $(I)$ is obtained. Now let $xs^n, ys^n \in K$, for $x, y \in S$. Thus $(xs^n)\rho_K(ys^n)$, and so there exists $r \in S$ such that $(xr)\rho_K(yr)$ and $rs^n = s^n$ by Theorem 3.1. Hence $xr = yr$ or $xr, yr \in K$, as required.

Sufficiency. Note that if $K = S$ then by Theorem 2.5, $S/K \cong \Theta_S$ is $ES-STF$. Assume $K$ be a proper right ideal of $S$ and $s \in S$. By assumption there exists a natural number $n \in \mathbb{N}$ such that conditions $(I)$, $(II)$ are satisfied. Let $(xs^n)\rho_K(ys^n)$, for $x, y \in S$. Then $xs^n, ys^n \in K$ or $xs^n = ys^n$. If $xs^n, ys^n \in K$ then by condition $(II)$, there exists $r \in S$ such that $rs^n = s^n$ and $(xr)\rho_K(yr)$, as required. If $xs^n = ys^n$ then there are two cases as follows:

Case 1: $xs^n = ys^n \in K$. We again have the before case.

Case 2: $xs^n = ys^n \in S \setminus K$. By condition $(I)$, there exists $r \in S$ such that $rs^n = s^n$ and $xr = yr$. Hence $rs^n = s^n$ and $(xr)\rho_K(yr)$. Therefore by Theorem 3.1, $S/K$ is $ES-STF$. □

# 4. Classification

In this section we give a classification of monoids when acts with other properties are $ES-STF$ and vice versa. We also give a classification of monoids when all their acts are $ES-STF$.

Recall, from [12] that an element $s \in S$ is called *eventually regular* if $s^n$ is regular for some $n \in \mathbb{N}$. That is, $s^n = s^nxs^n$ for some $n \in \mathbb{N}$ and $x \in S$. $S$ is called eventually regular if every $s \in S$ is eventually regular. Obviously every regular monoid is eventually regular. But the converse of it is not true in general.

An element $s \in S$ is called eventually left almost regular if

$$\begin{gathered} s_1c_1 = s^nr_1 \\ s_2c_2 = s_1r_2 \\ \dots \\ s_mc_m = s_{m-1}r_m \\ s^n = s_mrs^n, \end{gathered}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, ..., s_m, r, r_1, ..., r_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$. In other words $s \in S$ is called eventually left almost regular if $s^n$ is left almost regular for some $n \in \mathbb{N}$.

If every element of $S$ is eventually left almost regular, then $S$ is called eventually left almost regular.

It is clear that every left almost regular is eventually left almost regular and also every eventually regular is eventually left almost regular.

**Lemma 4.1.** *Every eventually left almost regular is eventually left $PSF$.*

*Proof.* Let $S$ be eventually left almost regular and $s \in S$. By definition,

$$\begin{gathered} s_1c_1 = s^nr_1 \\ s_2c_2 = s_1r_2 \\ \dots \\ s_mc_m = s_{m-1}r_m \\ s^n = s_mrs^n, \end{gathered}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, ..., s_m, r, r_1, ..., r_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$. Suppose that $t_1s^n = t_2s^n$, $t_1, t_2 \in S$. Hence we conclude that

$$t_1s^nr_1 = t_2s^nr_1 \Rightarrow t_1s_1c_1 = t_2s_1c_1 \Rightarrow t_1s_1 = t_2s_1 \Rightarrow t_1s_1r_2 = t_2s_1r_2$$
$$\Rightarrow t_1s_2c_2 = t_2s_2c_2 \Rightarrow t_1s_2 = t_2s_2.$$

Continuing in this manner we finally obtain $t_1s_i = t_2s_i$, for every $1 \leqslant i \leqslant m$, which implies $t_1s_m = t_2s_m$. Thus $t_1s_mr = t_2s_mr$. Since $s^n = s_mrs^n$, so $S$ is eventually left $PSF$. □

**Theorem 4.2.** *The following statments are equivalent:*

(1) *All right $S$-acts are $ES - STF$.*

(2) *All cyclic right $S$-acts are $ES - STF$.*

(3) *All right Rees factor acts of $S$ are $ES - STF$.*

(4) *All divisible right $S$-acts are $ES - STF$.*

(5) *All principally weakly injective right $S$-acts are $ES - STF$.*

(6) *All fg-weakly injective right $S$-acts are $ES - STF$.*

(7) *All weakly injective right $S$-acts are $ES - STF$.*

(8) *All injective right $S$-acts are $ES - STF$.*

(9) *All cofree right $S$-acts are $ES - STF$.*

(10) *$S$ is eventually regular.*

*Proof.* Implications (1) ⇒ (2) ⇒ (3) are obvious.

Since cofree ⇒ injective ⇒ weakly injective ⇒ fg-weakly injective ⇒ principally weakly injective ⇒ divisible, implications (1) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (9) are obtained immediately.

(3) ⇒ (10). By part (1) of Theorem 2.6, all right Rees factor acts of $S$ are $GPW$-flat. It follows by [12, Theorem 4.5] that $S$ is eventually regular.

(9) ⇒ (10). Since every right $S$-act can be embedded in a cofree right $S$-act, by assumption, every right $S$-act is a subact of $ES - STF$ right $S$-act. By part (2) of Theorem 2.5, all right $S$-acts are $ES - STF$. It follows, by Theorem 2.6, that all right $S$-acts are $GPW$-flat. Thus by [12, Theorem 4.5], $S$ is eventually regular.

(10) ⇒ (1). By [12, Theorem 4.5], all right $S$-acts are $GPW$-flat. Since every eventually regular monoid is eventually left almost regular and by Lemma 4.1, every eventually left almost regular monoid is eventually left $PSF$, by part (2) of Theorem 2.6, all right $S$-acts are $ES - STF$. □

**Theorem 4.3.** *Suppose that $(U)$ be a property of $S$-acts which implies Condition $(PWP)$ and $S_S$ satisfies the property $(U)$. Then the following statements are equivalent:*

(1) *All right $S$-acts satisfying property $(U)$, are $ES - STF$.*

(2) *All finitely generated right $S$-acts satisfying property $(U)$, are $ES - STF$.*

(3) *All cyclic right $S$-acts satisfying property $(U)$, are $ES - STF$.*

(4) *$S$ is eventually left $PSF$.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since $S_S$ is a cyclic act satisfying property $(U)$, by assumption $S_S$ is $ES-STF$ and so, by Lemma 2.4, $S$ is eventually left $PSF$.

(4) $\Rightarrow$ (1). Suppose that $A_S$ be a right $S$-act satisfying property $(U)$. Let $s \in S$. Since $S$ is eventually left $PSF$, there exists a natural number $n \in \mathbb{N}$ such that $ts^n = t's^n$, $t, t' \in S$, implies $tr = t'r$ and $rs^n = s^n$ for some $r \in S$. Let $as^n = a's^n$, for $a, a' \in A_S$. Since $A_S$ satisfies Condition $(PWP)$, there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$. Using assumption for $us^n = vs^n$, we get $r \in S$ such that $ur = vr$ and $rs^n = s^n$. Thus, $ar = a''ur = a''vr = a'r$ and so, $A_S$ is $ES-STF$. □

Notice that in the above theorem Property (U) can be replaced by free, projective, projective generator, strongly flat, $WPF$, $WKF$, $PWKF$, $TKF$, $(WP)$, Condition $(P)$, Condition $(P')$ and Condition $(PWP)$.

**Theorem 4.4.** *The following statements are equivalent:*

(1) *All right $S$-acts are $ES-STF$.*

(2) *All generators right $S$-acts are $ES-STF$.*

(3) *$S \times A_S$ is $ES-STF$ for every right $S$-act $A_S$.*

(4) *$S \times A_S$ is $ES-STF$ for every generator right $S$-act $A_S$.*

(5) *The right $S$-act $A_S$ is $ES-STF$ if $Hom(A_S, S_S) \neq \emptyset$.*

(6) *$S$ is eventually regular.*

*Proof.* Implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are obvious.

(1) $\Leftrightarrow$ (6). This follows from Theorem 4.2.

(2) $\Rightarrow$ (3). Suppose that $A_S$ be a right $S$-act. Indeed, the mapping $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$, for $a \in A_S$ and $s \in S$, is an epimorphism in $Act - S$. Then by [10, II, Theorem 3.16], $S \times A_S$ is generator. Thus by assumption $S \times A_S$ is $ES-STF$.

(3) $\Rightarrow$ (1). This statement immediatly follows from Theorem 2.8.

(4) $\Rightarrow$ (3). Suppose that $A_S$ be a right $S$-act. By proof (2) $\Rightarrow$ (3), $S \times A_S$ is a generator right $S$-act and so, by assumption, $S \times (S \times A_S)$ is $ES-STF$. Then Theorem 2.8, shows that, $S \times A_S$ is $ES-STF$.

(5) $\Rightarrow$ (3). Suppose $A_S$ be a right $S$-act. By proof (2) $\Rightarrow$ (3), $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$, for all $a \in A_S$ and $s \in S$, is an epimorphism in $Act - S$. Then $\mathrm{Hom}(S \times A_S, S_S) \neq \emptyset$. Thus $S \times A_S$ is $ES-STF$ by assumption. □

**Theorem 4.5.** *The following statements are equivalent:*

(1) *All torsion free right $S$-acts are $ES-STF$.*

(2) *All torsion free finitely generated right $S$-acts are $ES-STF$.*

(3) *All torsion free cyclic right $S$-acts are $ES-STF$.*

(4) *All torsion free right Rees factor acts of $S$ are $ES-STF$.*

(5) *$S$ is eventually left almost regular.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By (1) of Theorem 2.6, all torsion free right Ress factor acts of $S$ are $GPW$-flat. This follows from [12, Theorem 4.4] that $S$ is eventually left almost regular.

(5) $\Rightarrow$ (1). By [12, Theorem 4.4], all torsion free right $S$-acts are $GPW$-flat. On the other hand, by Lemma 4.1, every eventually left almost regular monoid is eventually left $PSF$. So by part (2) of Theorem 2.6, all torsion free right $S$-acts are $ES-STF$. □

Recall, from [13] that the right $S$-act $A_S$ is called *$\mathcal{R}$-torsion free*, if for any $a, b \in A_S$ and for any right cancellable element $c \in S$, $ac = bc$ and $a\mathcal{R}b$ imply $a = b$, where $\mathcal{R}$ is a Green relation that is $a\mathcal{R}b$ if and only if $aS = bS.(a, b \in A_S)$

**Theorem 4.6.** *The following statments are equivalent:*

(1) *All $\mathcal{R}$-torsion free right $S$-acts are $ES-STF$.*

(2) *All $\mathcal{R}$-torsion free finitely generated right $S$-acts are $ES-STF$.*

(3) *All $\mathcal{R}$-torsion free right $S$-acts generated by at most two elements are $ES-STF$.*

(4) *$S$ is eventually regular.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $s \in S$. Since $S_S$ is $\mathcal{R}$-torsion free right $S$-act, by Lemma 2.4, it is eventually left $PSF$. Then there exists $n \in \mathbb{N}$ such that $ts^n = t's^n$ for $t, t' \in S$, implies $tr = t'r$ and $rs^n = s^n$ for some $r \in S$. If $s^nS = S$, then there exists $x \in S$ such that $s^nx = 1$ and so, $s^nxs^n = s^n$. Thus $s$ is eventually regular element. Now assume that $s^nS \neq S$. Set

$$A_S = S\coprod^{s^nS} S = \{(l,x)|l \in S \setminus s^nS\} \dot{\cup} s^nS \dot{\cup} \{(t,y)|t \in S \setminus s^nS\}.$$

Indeed,

$$B_S = \{(l,x)|l \in S \setminus s^nS\} \dot{\cup} s^nS \cong S_S \cong \{(t,y)|t \in S \setminus s^nS\} \dot{\cup} s^nS = C_S.$$

Since $A_S = B_S \cup C_S$, $A_S$ is generated by different two elements $(1,x)$ and $(1,y)$. By the above isomorphism, $B_S$ and $C_S$ satisfy Condition $(E)$ and so, $A_S$ satisfies Condition $(E)$. By [13, Proposition 1.2], $A_S$ is $\mathcal{R}$-torsion free and so, by assumption, $A_S$ is $ES-STF$. Therefore the equality $(1,x)s^n = (1,y)s^n$ implies that there exists $r \in S$ such that $rs^n = s^n$ and $(1,x)r = (1,y)r$. The last equality implies that $r \in s^nS$ and so, there exists $x \in S$ such that $r = s^nx$. Therefore $s^n = rs^n = s^nxs^n$. Hence $S$ is eventually regular.

(4) $\Rightarrow$ (1). By Theorem 4.2, the result follows. □

**Theorem 4.7.** *The following statements are equivalent:*

(1) *All right $S$-acts are $ES-STF$.*

(2) *All right $S$-acts satisfying Condition $(E'P)$, are $ES-STF$.*

(3) *All right $S$-acts satisfying Condition $(EP)$, are $ES-STF$.*

(4) *All right $S$-acts satisfying Condition $(E')$, are $ES-STF$.*

(5) *All right $S$-acts satisfying Condition $(E)$, are $ES-STF$.*

(6) *$S$ is eventually regular.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious because $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E')$.

(5) $\Rightarrow$ (6). Since $S_S$ satisfies Condition $(E)$, similar to Theorem 4.6, the result is obtained.

(6) $\Rightarrow$ (1). This follows by Theorem 4.2. □

Similar to Theorem 4.6, it can be resulted that Theorem 4.7, is true for finitely generated $S$-acts and $S$-acts generated by at most two elements.

We recall, from [10] that a right $S$-act $A_S$ is (strongly) faithful if for $s, t \in S$, the validity of $as = at$, for (some)all $a \in A$, implies the equality $s = t$.

**Theorem 4.8.** *The following statements are equivalent:*

(1) *All right $S$-acts are $ES - STF$.*

(2) *All faithful right $S$-acts are $ES - STF$.*

(3) *All finitely generated faithful right $S$-acts are $ES - STF$.*

(4) *All faithful right $S$-acts generated by at most two elements are $ES - STF$.*

(5) *$S$ is eventually regular.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Since $S_S$ is faithful right $S$-act, similar to Theorem 4.6, the result is obtained.

(5) $\Rightarrow$ (1). By Theorem 4.2, the result follows. □

A right $S$-act $A_S$ is called decomposable if there exist two subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \varnothing$. In this case $A_S = B_S \cup C_S$ is called a decomposition of $A_S$. Otherwise $A_S$ is called indecomposable. $A_S$ is called locally cyclic if for any $a, a' \in A_S$ there exists $a'' \in A_S$ such that $a, a' \in a''S$.

**Theorem 4.9.** *The following statements are equivalent:*

(1) *All right $S$-acts are $ES - STF$.*

(2) *All indecomposable right $S$-acts are $ES - STF$.*

(3) *All finitely generated indecomposable right $S$-acts are $ES - STF$.*

(4) *All indecomposable right $S$-acts generated by at most two elements are $ES - STF$.*

(5) *All locally cyclic $S$-acts are $ES - STF$.*

(6) *All finitely generated locally cyclic right $S$-acts are $ES - STF$.*

(7) *All locally cyclic right $S$-acts generated by at most two elements are $ES - STF$.*

(8) *$S$ is eventually regular.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(4) $\Rightarrow$ (8). Since $S_S$ is indecomposable right $S$-act also $S\coprod^{I} S$ is indecomposable, for every proper right ideal $I$ of $S$, similar to Theorem 4.6, the result is obtained.

(7) $\Rightarrow$ (8). By assumption all cyclic right $S$-acts are $ES - STF$ and so, $S$ is eventually regular by Theorem 4.2.

(8) $\Rightarrow$ (1). By Theorem 4.2, the result follows.

□

**Theorem 4.10.** *The following statements are equivalent:*

(1) *All $ES-STF$ right $S$-acts are (strongly) faithful.*

(2) *All $ES-STF$ finitely generated right $S$-acts are (strongly) faithful.*

(3) *All $ES-STF$ cyclic right $S$-acts are (strongly) faithful.*

(4) *All $ES-STF$ right Rees factor acts of $S$ are (strongly) faithful.*

(5) $S=\{1\}$.

*Proof.* Implications (1) ⇒ (2) ⇒ (3)⇒ (4) are obvious.

(4) ⇒ (5). It follows by part (1) of Theorem 2.5, that $S/S_S \cong \Theta_S$ is $ES-STF$. Thus by assumption, $\Theta_S$ is (strongly) faithful. Let $s,t \in S$. Then $\theta s = \theta t$ implies $s=t$ and so, $S=\{1\}$.

(5) ⇒ (1). If $S=\{1\}$ then all right $S$-acts are strongly faithful and so, (1) is obtained. □

**Theorem 4.11.** *The following statements are equivalent:*

(1) *All $ES-STF$ right $S$-acts are (projective) generators.*

(2) *All $ES-STF$ finitely generated right $S$-acts are (projective) generators.*

(3) *All $ES-STF$ cyclic right $S$-acts are (projective) generators.*

(4) *All $ES-STF$ right Rees factor acts of $S$ are (projective) generators.*

(5) $S=\{1\}$.

*Proof.* Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). By part (1) of Theorem 2.5, the right Rees factor $S$-act $S/S_S \cong \Theta_S$ is $ES-STF$. Thus by assumption, $\Theta_S$ is (projective) generator. By [10, II, Theorem 3.16] there exists an epimorphism $\phi : \Theta_S \to S_S$. Thus $S=\{1\}$.

(5) ⇒ (1). If $S=\{1\}$ then all right $S$-acts are (projective) generators and so, the result is obtained. □

**Theorem 4.12.** *The following statements are equivalent:*

(1) *All $ES-STF$ right $S$-acts are free.*

(2) *All $ES-STF$ finitely generated right $S$-acts are free.*

(3) *All $ES-STF$ cyclic right $S$-acts are free.*

(4) *All $ES-STF$ right Rees factor acts of $S$ are free.*

(5) $S=\{1\}$.

*Proof.* Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). By assumption, all $ES-STF$ right Rees factor acts of $S$ are generators. This follows from the previous theorem, that $S=\{1\}$.

(5) ⇒ (1). If $S=\{1\}$ then all right $S$-acts are free and so, (1) is obtained. □

**Theorem 4.13.** *The following statements are equivalent:*

(1) *All strongly faithful right $S$-acts are $ES-STF$.*

(2) *All finitely generated strongly faithful right $S$-acts are $ES-STF$.*

(3) *All strongly faithful right $S$-acts generated by at most two elements are $ES-STF$.*

(4) *Either $S$ is not left cancellative or it is eventually regular.*

(5) *Either $S$ is not left cancellative or $S$ is a group.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $S$ be left cancellative and $s \in S$. Then by [9, Lemma 3.7], $S_S$ is strongly faithful. Now by assumption, similar to proof of Theorem 4.6, the result is obtained.

(3) $\Rightarrow$ (5). Let $S$ be left cancellative and $s \in S$. Then by [9, Lemma 3.7], $S_S$ is strongly faithful. Now by assumption, similar to proof of Theorem 4.6, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = s^n x s^n$. Since $S$ is left cancellative, we have $xs^n = 1$ and so $s$ is left invertible. Thus $S$ is a group.

(4) $\Rightarrow$ (1). If $S$ is not left cancellative then by [9, Lemma 3.7], there exists no strongly faithful right $S$-act and so, the result follows. If $S$ is left cancellative, then by assumption $S$ is eventually regular. Thus by Theorem 4.2, all right $S$-acts are $ES - STF$.

(5) $\Rightarrow$ (1). If $S$ is not left cancellative then by [9, Lemma 3.7], there exists no strongly faithful right $S$-act and so, the result follows. If $S$ is left cancellative, then by assumption $S$ is a group and so eventually regular. Thus by Theorem 4.2, all right $S$-acts are $ES - STF$. □

**Theorem 4.14.** *The following statements are equivalent:*

(1) *All strongly faithful cyclic right $S$-acts are $ES - STF$.*

(2) *All strongly faithful monocyclic right $S$-acts are $ES - STF$.*

(3) *Either $S$ is not left cancellative or it is eventually left $PSF$.*

*Proof.* Implication (1) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (3). Suppose that $S$ be left cancellative. Then by [9, Lemma 3.7], $S_S$ is strongly faithful. Now by isomorphisms $S/\rho(1,1) \cong S/\Delta_S \cong S_S$ and assumption, $S_S$ is $ES - STF$ and so, by Lemma 2.4, $S$ is eventually left $PSF$.

(3) $\Rightarrow$ (1). Suppose that $S$ is not left cancellative. Then by [9, Lemma 3.7], there exist no strongly faithful right $S$-act and so, the result follows. Now let $S$ be left cancellative and so is eventually left $PSF$ by assumption. Since S is eventually left $PSF$, by Lemma 2.4, $S_S$ is $ES - STF$. Let $A_S = aS$ be a cyclic strongly faithful right $S$-act, we define $f : aS \to S_S$ by $f(as) = s$. Then $f$ is an isomorphism of right $S$-acts. Now by isomorphism $aS \cong S_S$, the result follows. □

**Theorem 4.15.** *The following statements are equivalent:*

(1) *There exists at least a strongly faithful cyclic right $S$-act such that is $ES - STF$.*

(2) *There exists at least a strongly faithful monocyclic right $S$-act such that is $ES - STF$.*

(3) *$S$ is left cancellative and each strongly faithful cyclic right $S$-act is $ES - STF$.*

(4) *$S$ is left cancellative and each strongly faithful monocyclic right $S$-act is $ES-STF$.*

(5) *$S$ is left cancellative and eventually left $PFS$.*

*Proof.* Implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are obvious.

(1) $\Rightarrow$ (3). By assumption and [9, Lemma 3.7], $S$ is left cancellative. If $S/\rho$ be a strongly faithful cyclic right $S$-act then $\rho = \Delta_S$, by [9, Lemma 3.9]. Hence $S/\rho = S/\Delta_S \cong S_S$. Then $S_S$ is $ES - STF$. So $S$ is eventually left $PSF$ by Lemma 2.4. Then each strongly faithful cyclic right $S$-act is $ES - STF$, by Theorem 4.14.

(4) $\Rightarrow$ (5). By Theorem 4.14 it is obvious.

(5) $\Rightarrow$ (2). Since $S$ is left cancellative, $S_S$ is strongly faithful by [9, Lemma 3.7]. Since $S/\rho(1,1) \cong S_S$, there exists at least a strongly faithful monocyclic $S$-act. Since $S$ is eventually left $PSF$, $S/\rho(1,1) \cong S_S$ is $ES - STF$, by Lemma 2.4. □

Recall, from [10], if $\rho$ be a right congruence on $S$ and $s \in S$, then by $\rho s$ we denote the right congruence on $S$ define by

$$x(\rho s)y \Leftrightarrow (sx)\rho(sy)$$

for $x, y \in S$.

If $\lambda$ be a left congruence on $S$ and $s \in S$, then by $s\lambda$ we denote the left congruence on $S$ define by

$$x(s\lambda)y \Leftrightarrow (xs)\lambda(ys)$$

for $x, y \in S$.

It is clear that if $\rho$ be a right congruence then $\rho s$ is right congruence and if $\lambda$ be a left congruence, $s\lambda$ is left congruence for $s \in S$.

**Lemma 4.16.** *Let $\rho \in Con(S_S)$. Then the following statments are equivalent:*

(1) *The cyclic right $S$-act $S/\rho$ is faithful.*

(2) *$\rho$ does not contain any left congruence $\tau$ on $S$ such that $\tau \neq \Delta_S$.*

(3) *$\bigcap_{u\in S} \rho u = \Delta_S$.*

*Proof.* (1) $\Rightarrow$ (2). It is obvious by [10, I, Proposition 5.24].

(2) $\Rightarrow$ (3). Let $\sigma = \bigcap_{u\in S} \rho u$. Since for each $u \in S$, $\rho u \in Con(S_S)$ so, it is clear that $\sigma \in Con(S_S)$. Now we show that $\sigma$ is a left congruence on $S$. Let $x, y \in S$, then we have:

$$(x, y) \in \sigma \Leftrightarrow (\forall u \in S)(x, y) \in \rho u \Leftrightarrow (\forall u \in S)(ux, uy) \in \rho$$

Now if $l \in S$, then we have:

$$\begin{aligned}(x, y) \in \sigma \Leftrightarrow (\forall u \in S)(ux, uy) \in \rho &\Rightarrow (\forall u \in S)(ulx, uly) \in \rho \\ &\Rightarrow (\forall u \in S)(lx, ly) \in \rho u \Rightarrow (lx, ly) \in \textstyle\bigcap_{u\in S} \rho u = \sigma.\end{aligned}$$

Therefore $\sigma$ is a left congruence on $S$ and clearly $\bigcap_{u\in S} \rho u \subseteq \rho$. On the other hand $\rho$ does not contain any nontrivial left congruence on $S$ by assumption. Hence $\sigma = \Delta_S$.

(3) $\Rightarrow$ (1). Let $S/\rho$ does not faithful. Then we have:

$$\begin{aligned}\exists x, y \in S, x \neq y, \forall u \in S, [u]_\rho x = [u]_\rho y &\Rightarrow (\forall u \in S)(ux, uy) \in \rho \Rightarrow (\forall u \in S)(x, y) \in \rho u \\ &\Rightarrow (x, y) \in \textstyle\bigcap_{u\in S} \rho u.\end{aligned}$$

Therefore $\sigma = \bigcap_{u\in S} \rho u \neq \Delta_S$, which is contradiction by assumption and so, $S/\rho$ is faithful. □

**Theorem 4.17.** *The following statments are equivalent:*

(1) *All faithful cyclic right $S$-acts are $ES-STF$.*

(2) *For any $\rho \in Con(S_S)$, $\rho$ contains any nontrivial left congruence $\tau$ on $S$ or the right act $S/\rho$ is $ES-STF$.*

(3) *For any $\rho \in Con(S_S)$, $\bigcap_{u\in S} \rho u \neq \Delta_S$ or the cyclic right $S$-act $S/\rho$ is $ES-STF$.*

*Proof.* (1) $\Rightarrow$ (2). Let $\rho$ be a right congruence on $S$ that does not contain any nontrivial left congruence on $S$. Then by Lemma 4.16, $S/\rho$ is faithful and so, $S/\rho$ is $ES-STF$ by assumption.

(2) $\Rightarrow$ (3). Let $\rho$ be a right congruence on $S$ such that $\bigcap_{u\in S} \rho u = \Delta_S$. Then by Lemma 4.16, and assumption $S/\rho$ is $ES-STF$.

(3) $\Rightarrow$ (1). Let $\rho$ be a right congruence on $S$ such that the cyclic right $S$-act $S/\rho$ is faithful. So by Lemma 4.16, $\bigcap_{u\in S} \rho u = \Delta_S$ and so, by assumption $S/\rho$ is $ES-STF$. □

Since

free $\Rightarrow$ projective generator $\Rightarrow$ projective $\Rightarrow$ strongly flat $\Rightarrow$ $WPF$ $\Rightarrow$ Condition (P)

we have the following theorem.

**Theorem 4.18.** *The following statments are equivalent:*

(1) *All right Rees factor acts of $S$ satisfying Condition $(P)$ are $ES-STF$.*

(2) *All $WPF$ right Rees factor acts of $S$ are $ES-STF$.*

(3) *All srtongly flat right Rees factor acts of $S$ are $ES-STF$.*

(4) *All projective right Rees factor acts of $S$ are $ES-STF$.*

(5) *All projective generator right Rees factor acts of $S$ are $ES-STF$.*

(6) *All free right Rees factor acts of $S$ are $ES-STF$.*

(7) *Either $S$ does not contain a left zero or it is eventually left $PSF$.*

*Proof.* Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7). Let $S$ contains a left zero such as $z$. Let $K_S = zS = \{z\}$ and so, $K_S$ is a right ideal such that $|K_S| = 1$. Therefore $S/K_S \cong S_S$ is free and so, by assumption $S/K_S \cong S_S$ is $ES-STF$. Hence $S$ is eventually left $PSF$ by Lemma 2.4.

(7) $\Rightarrow$ (1). Let $K$ be a right ideal of $S$ such that $S/K$ satisfies Condition $(P)$. If $K = S$, then $S/K = S/S_S \cong \Theta_S$. Hence by Theorem 2.5, $S/K \cong \Theta_S$ is $ES-STF$. If $K \neq S$ then by [10, III, Proposition 13.9], $|K| = 1$. If $z \in K$ then $K = zS = \{z\}$. Therefore $z$ is a left zero of $S$ and so, $S$ is eventually left $PSF$ by assumption. Hence $S_S$ is $ES-STF$ and so, $S/K \cong S_S$ is $ES-STF$. □

We will use $C_l(C_r)$ to denote the set of all left (right) cancellable elements of $S$.

**Lemma 4.19.** *Let $S \neq C_r$. Then the following statments hold:*

(1) *$I = S \setminus C_r$ is a proper right ideal of $S$.*

(2) *$S/I(I = S \setminus C_r)$ is a torsion free $S$-act.*

(3) *If $S$ be eventually left $PSF$, then $I = S \setminus C_r$ is a $GPW$-left stabilizing right ideal and so, $A_S = S\coprod^{I} S$ is $ES-STF$.*

*Proof.* For proof (1) and (2), we can refer to [9, Lemma 3.12].

(3). Let $s \in S$. Since $S$ is eventually left $PSF$, there exists $n \in \mathbb{N}$ such that $ts^n = t's^n$ for $t, t' \in S$, implies $tr = t'r$ and $rs^n = s^n$ for some $r \in S$. Let $r \in S \setminus I = C_r$ such that $rs^n \in I$. Since $I = S \setminus C_r$, $rs^n$ is not right cancellable. Thus there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1rs^n = l_2rs^n$. Now by assumption there exists $r' \in S$ such that $l_1rr' = l_2rr'$ and $r's^n = s^n$. Therefore $rr's^n = rs^n$. Since $l_1 \neq l_2$, so the equality $l_1rr' = l_2rr'$ implies that $rr' \in S \setminus C_r = I$. Let $i = rr' \in I$, thus $rs^n = is^n$ and so $I = S \setminus C_r$ is $GPW$-left stablizing right ideal. Hence by Theorem 2.14, $A_S = S\coprod^{I} S$ is $ES - STF$. □

**Lemma 4.20.** *Let $S$ be right cancellative. Then for every right $S$-act we have:*
*strongly torsion free ⇔ torsion free ⇔ $GP$-flat ⇔ principally weakly flat ⇔ Condition $(PWP)$ ⇔ $ES - STF$ ⇔ Condition $(P')$ ⇔ Condition $(PWP_E)$ ⇔ Condition $(PWP_{ssc})$ ⇔ TKF ⇔ PWKF.*

*Proof.* Implications strongly torsion free ⇔ torsion free ⇔ $GP$-flat ⇔ principally weakly flat ⇔ Condition (PWP) ⇔ Condition $(P')$ ⇔ Condition $(PWP_E)$ ⇔ Condition $(PWP_{ssc})$ ⇔ TKF ⇔ PWKF follow from [9, Lemma 3.13].

Condition (PWP) ⇒ $ES-STF$. Suppose that the right $S$-act $A_S$ satisfies Condition $(PWP)$ and $s \in S$. If $as = a's$, for $a, a' \in A_S$, then there exist $u, v \in S$ and $b \in A_S$ such that $a = bu, a' = bv$ and $us = vs$. Since $S$ is right cancellative, the equality $us = vs$ implies $u = v$. Hence $a = bu = bv = a'$, and so $A_S$ is $ES - STF$.

$ES - STF$ ⇒ Condition (PWP). Let $A_S$ be a $ES - STF$ right $S$-act. Assume that $as = a's$, for $s \in S$, $a, a' \in A_S$. Since $A_S$ is $ES - STF$, there exists $n \in \mathbb{N}$ such that $bs^n = b's^n$, for $b, b' \in A_S$, implies $br = b'r$ and $rs^n = s^n$ for some $r \in S$. We get from $as = a's$ the equality $as^n = a's^n$ which implies $ar = a'r$ and $rs^n = s^n$ for some $r \in S$. Since $S$ is right cancellative the equality $rs^n = s^n$ implies $r = 1$. Thus $a = a'$ and so $A_S$ satisfies Condition $(PWP)$. □

Since

PWKF ⇒ TKF ⇒ Condition $(PWP)$ and Condition $(P')$ ⇒ Condition $(PWP)$

we have the following two theorems.

**Theorem 4.21.** *Let (U) be a property on $S$-acts such that*

$$ES - STF \Rightarrow \textit{Property (U)} \Rightarrow \textit{torsion free}$$

*Then the following statments are equivalent:*

1. *$S$ is eventually left $PSF$ and property (U) implies $PWKF$.*
2. *$S$ is eventually left $PSF$ and property (U) implies TKF.*
3. *$S$ is eventually left $PSF$ and property (U) implies Condition $(PWP)$.*
4. *$S$ is eventually left $PSF$ and property (U) implies Codition $(P')$.*
5. *$S$ is right cancellative.*

*Proof.* Implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are obvious.

(3) ⇒ (5). Let $S$ does not be a right cancellative and $I = S \setminus C_r$. Then $I$ is a $GPW$-left stabilizing right ideal of $S$ by Lemma 4.19, and

$$A_S = S\coprod^{I} S = \{(l,x)|l \in S \setminus I\} \dot{\cup} \{(t,y)|t \in S \setminus I\} \dot{\cup} I = (1,x)S \cup (1,y)S,$$

is $ES-STF$. By assumption, $A_S$ satisfies Condition (PWP). If $i \in I$ then the equality $(1,x)i = (1,y)i$ implies that there exist $a \in A_S$ and $u, v \in S$ such that $(1,x) = au$, $(1,y) = av$ and $ui = vi$. Therefore there exist $t, l \in S \setminus I$ such that $(l,x) = a = (t,y)$ which is a contradiction. Hence $S$ is right cancellative, as required.

(5) ⇒ (1). Since $S$ is right cancellative, it is eventually left $PSF$. Also by Lemma 4.20, for every right $S$-act, properties torsion free, PWKF and $ES-STF$ are equivalent. Thus by assumption, every right $S$-act satisfying in property (U) is PWKF.

(5) ⇒ (4). Since $S$ is right cancellative, $S$ is eventually left $PSF$. Also by Lemma 4.20, torsion free, Condition $(P')$ and eventually semi strongly torsion freeness are equivalent. Thus by assumption every right $S$-act satisfying in property (U), satisfies Condition $(P')$. □

Notice that in the above theorem Property (U) can be replaced by $GP$-flat and $GPW$-flat.

**Theorem 4.22.** *Let (U) be a property on $S$-acts such that*

*$GPW$-flat ⇒ property (U) ⇒ torsion free*

*Then the following statments are equivalent:*

(1) *All right $S$-acts satisfying property (U) are PWKF and $ES-STF$.*

(2) *All right $S$-acts satisfying property (U) are TKF and $ES-STF$.*

(3) *All right $S$-acts satisfying property (U) are (PWP) and $ES-STF$.*

(4) *All right $S$-acts satisfying property (U) are $(P')$ and $ES-STF$.*

(5) *$S$ is right cancellative.*

*Proof.* Implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (3) are obviouse.

(3) ⇒ (5). By [12, Proposition 2.8], $S_S$ is GPW-flat and so satisfies property (U), by assumption. Therefore $S_S$ is $ES-STF$. Thus $S$ is eventually left $PSF$. Since by assumption every right $S$-act satisfying property (U) satisfies Condition (PWP), so by Theorem 4.21, $S$ is right cancellative.

(5) ⇒ (1). Let $S$ be right cancellative and so, by Theorem 4.21, every right $S$-act satisfying property (U) is PWKF. On the other hand since $S$ is right cancellative, $S$ is eventually left almost regular and so, by Theorem 4.5, every torsion free right $S$-act is $ES-STF$. Therefore every right $S$-act satisfying property (U), is $ES-STF$, by assumption.

(5) ⇒ (4). Let $S$ be right cancellative. Thus by Lemma 4.20, Conditions $(P')$ and PWKF are equivalent and so, by proof (5) ⇒ (1), the result follows. □

Notice that in the above theorem Property (U) can be replaced by $GP$-flat and $GPW$-flat.

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Received March 24, 2024

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