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# On $Q_r$-ordered semigroups

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**Abstract**. Ordered semigroups in which every proper right ideal is a power joined subsemigroup, namely $Q_r$-ordered semigroups, are investigated. We also give characterizations of archimedean weakly commutative $Q_r$-ordered semigroups.

## 1. Introduction and preliminaries

The concept of commutative $Q$-semigroups studied by T. E. Nordhl in [10] and his results were extended to quasi-commutative semigroups by C. S. H. Nagore in [9]. The Putcha's $Q$-semigroups were studied by A. Cherubini-Spoletini and A. Varisco in [4]. The concept of $Q_r$-semigroups was introduced by S. Bogdanović [1]. In this paper, we extend the notion of $Q_r$-semigroups to ordered semigroups. We prove that $S$ is an archimedean weakly commutative $Q_r$-ordered semigroup if and only if $S$ is a power joined or $S$ is an ideal extension of a power joined archimedean weakly commutative subsemigroup containing ordered idempotent by a nil ordered semigroup.

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that for any $x, y, z$ in $S$, $x \leqslant y$ implies $zx \leqslant zy$ and $xz \leqslant yz$, is called a *partially ordered semigroup*, or simply an *ordered semigroup*. Under the trivial relation, $x \leqslant y$ if and only if $x = y$, it is observed that every semigroup is an ordered semigroup.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For $A$, $B$ nonempty subsets of $S$, we write $AB$ for the set of all elements $xy$ in $S$ where $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements $x$ in $S$ such that $x \leqslant a$ for some $a$ in $A$, i.e.,

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$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, we write $Ax$ for $A\{x\}$, and $xA$ for $\{x\}A$. It was shown in [3] that the following hold: (1) $A \subseteq (A]$ and $((A]] = (A]$; (2) $A \subseteq B \Rightarrow (A] \subseteq (B]$; (3) $((A](B]] = ((A]B] = (A(B]] = (AB]$; (4) $(A](B] \subseteq (AB]$; (5) $(A]B \subseteq (AB]$ and $A(B] \subseteq (AB]$; (6) If $\{A_k\}_{k \in K}$ is a family of nonempty subsets of $S$, then $(\bigcup_{k \in K} A_k] = \bigcup_{k \in K}(A_k]$ and $(\bigcap_{k \in K} A_k] \subseteq \bigcap_{k \in K}(A_k]$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A nonempty subset $A$ of $S$ is called a *left* (resp., *right*) *ideal* of $S$ if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp., $AS \subseteq A$);

(ii) $A = (A]$, that is, for any $x$ in $A$ and $y$ in $S$, $y \leqslant x$ implies $y \in A$.

$A$ is called a (two-sided) ideal of $S$ if it is both a left and a right ideal of $S$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A left ideal $A$ of $S$ is said to be *proper* if $A \subset S$. A proper right and two-sided ideals are defined similarly. $S$ is *simple* if it does not contain proper ideals. A proper ideal $A$ of $S$ is said to be *maximal* if for any ideal $B$ of $S$ such that $A \subset B \subseteq S$, then $B = S$.

A nonempty subset $Q$ of $S$ is called a *quasi-ideal* of $S$ if it satisfies the following conditions:

(i) $(QS] \cap (SQ] \subseteq Q$;

(ii) $Q = (Q]$, that is, for any $x$ in $Q$ and $y$ in $S$, $y \leqslant x$ implies $y \in Q$ [2].

A nonempty subset $B$ of $S$ is called a *bi-ideal* of $S$ if it satisfies the following conditions:

(i) $BSB \subseteq B$;

(ii) $B = (B]$, that is, for any $x$ in $B$ and $y$ in $S$, $y \leqslant x$ implies $y \in B$ [5].

As it is easily to see, any one-sided ideal is a quasi-ideal and any quasi-ideal is a bi-ideal.

A subsemigroup $F$ is called a *filter* of $S$ if

(i) $a, b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$;

(ii) if $a \in F$ and $b$ in $S$, $a \leq b$, then $b \in F$ [6].

For an element $x$ of $S$, we denote by $N(x)$ the filter generated by $x$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal $I$ of $S$ is said to be *prime* if for any ideals $A, B$ of $S$, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal $I$ of $S$ is said to be *completely prime* if for any elements $a, b$ of $S$, $ab \in I$ implies $a \in I$ or $b \in I$. An ideal $I$ of $S$ is said to be *semiprime* if for any ideal $A$ of $S$, $A^2 \subseteq I$ implies $A \subseteq I$. An ideal $I$ of $S$ is said to be *completely* *semiprime* if for any element $a$ of $S$ and for any positive integer $n$, $a^n \in I$ implies $a \in I$ [12].

An element $e$ of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *ordered idempotent* if $e \leqslant e^2$. We call an ordered semigroup $S$ *idempotent ordered semigroup* if every element of $S$ is an ordered idempotent [8].

An element $a$ of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *left regular* (resp., *right regular*, *regular*, *intra-regular*) if there exist $x, y$ in $S$ such that $a \leqslant xa^2$ (resp., $a \leqslant a^2x$, $a \leqslant axa$, $a \leqslant xa^2y$) [12].

The *zero element* of an ordered semigroup $(S, \cdot, \leqslant)$, defined by Birkhoff, is an element 0 of $S$ such that $0 \leqslant x$ and $0x = 0 = x0$ for all $x \in S$. The set of all positive integers denoted by $\mathbb{N}$.

An element $a$ of an ordered semigroup $(S, \cdot, \leqslant)$ having a zero 0 is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$. An ordered semigroup $(S, \cdot, \leqslant)$ having a zero 0 is called *nil* if every element of $S$ is nilpotent, that is, for every $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n = 0$ [7].

Let $(S, \cdot, \leqslant_S)$, $(T, *, \leqslant_T)$ be an ordered semigroups, $f : S \to T$ a mapping from $S$ into $T$. The mapping $f$ is called *isotone* if $x, y \in S$, $x \leqslant_S y$ implies $f(x) \leqslant_T f(y)$ and *reverse isotone* if $x, y \in S$, $f(x) \leqslant_T f(y)$ implies $x \leqslant_S y$. The mapping $f$ is called a *homomorphism* if it is isotone and satisfies $f(xy) = f(x) * f(y)$ for all $x, y \in S$. The mapping $f$ is called a *isomorphism* if it is reverse isotone onto homomorphism. The ordered semigroups $S$ and $T$ are called *isomorphic*, in symbols $S \cong T$ if there exists an isomorphism between them.

An ordered semigroup $V$ is called an *ideal extension*(or just an *extension*) of an ordered semigroup $S$ by an ordered semigroup $Q$, if $Q$ has a zero 0, $S \cap (Q \setminus \{0\}) = \emptyset$, and there exists an ideal $K$ of $V$ such that $K \cong S$ and $V/K \cong Q$ [7].

Let $(S, \cdot, \leqslant)$ be an ordered semigroup and $K$ an ideal of $S$. $S/K$ is called the *Rees quotient ordered semigroup* of $S$, where 0 is an arbitrary element of $K$. It is observed that $K \cap [(S/K) \setminus \{0\}] = \emptyset$, $K \cong K$ and $S/K \cong S/K$ under the identity mapping and so $S$ is an ideal extension of $K$ by $S/K$.

## 2. Main results

We begin this section with the following definition.

**Definition 2.1.** An ordered semigroup $(S, \cdot, \leqslant)$ is called *right* (resp., *left*) *archimedean* if for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in (bS]$

(resp., $a^n \in (Sb]$). An ordered semigroup $(S, \cdot, \leqslant)$ is called *archimedean* if for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in (SbS]$.

**Definition 2.2.** An ordered semigroup $(S, \cdot, \leqslant)$ is called *weakly commutative* if for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $(ab)^n \in (bSa]$.

**Lemma 2.3.** *Let $(S, \cdot, \leqslant)$ be a weakly commutative ordered semigroup. The following statements are equivalent:*

(1) *$S$ is a left archimedean;*

(2) *$S$ is a right archimedean;*

(3) *$S$ is an archimedean.*

*Proof.* The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are obvious.

(1) $\Rightarrow$ (2). Let $a, b \in S$. Then there exists $n \in \mathbb{N}$ such that $a^n \leqslant xb$ for some $x \in S$. Since $S$ is a weakly commutative, $(xb)^m \in (bSx]$ for some $m \in \mathbb{N}$. We have $a^{nm} \leqslant (xb)^m \in (bSx] \subseteq (bS]$. Thus $S$ is a right archimedean.

Similarly, we have (3) $\Rightarrow$ (1). □

**Theorem 2.4.** *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $S$ is a weakly commutative and $S$ has no proper completely semiprime ideals if and only if $S$ is a left and right archimedean.*

*Proof.* Let $S$ be a weakly commutative and $S$ has no proper completely semiprime ideals. Let $x \in S$. Suppose that $S \setminus N(x) \neq \emptyset$. Since $N(x)$ is a subsemigroup of $S$, we have $S \setminus N(x)$ is proper completely prime ideal by Lemma 3.7 in [11]. It follows that $S \setminus N(x)$ is proper completely semiprime ideal. This is a contradiction. Thus $S \setminus N(x) = \emptyset$ and so $S = N(x)$. This implies that $S$ is a left and right archimedean by remark in [6]. Conversely, if $S$ is a left and right archimedean, then obviously $S$ is a weakly commutative. Let $A$ be any completely semiprime ideal of $S$, $a \in A$ and $b \in S$. Then there exists $n \in \mathbb{N}$ such that $b^n \in (Sa] \subseteq A$. This implies $b \in A$ and so $S = A$. □

**Lemma 2.5.** *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and $K$ an ideal of $S$. If $K$ is an archimedean weakly commutative subsemigroup of $S$ and $S/K$ is nil, then $S$ is a weakly commutative.*

*Proof.* Let $a, b \in S$. Since $S/K$ is nil, then there exists $h, k, t \in \mathbb{N}$ such that $a^h, b^k, (ab)^t \in K$. Since $K$ is an archimedean weakly commutative subsemigroup, $(ab)^{nt} \in (b^k K]$ and $(ab)^{mt} \in (Ka^h]$ for some $n, m \in \mathbb{N}$ by Lemma 2.3. We have $(ab)^{nt+mt} \in (b^k K](Ka^h] \subseteq (bKa] \subseteq (bSa]$. Thus $S$ is a weakly commutative. □

**Theorem 2.6.** *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If $S$ is an archimedean weakly commutative containing ordered idempotent, then $S$ is an ideal extension of an archimedean weakly commutative subsemigroup $K$ containing ordered idempotent by a nil ordered semigroup $S/K$. Conversely, if $S$ is an ideal extension of an archimedean weakly commutative subsemigroup $K$ by a nil ordered semigroup $S/K$, then $S$ is an archimedean weakly commutative.*

*Proof.* Assume that $S$ is an archimedean weakly commutative and $e$ an ordered idempotent element. We set $K = (SeS]$. Then $K$ is an ideal of $S$ and $e \in K$. Let $a, b \in K$. Since $S$ is an archimedean, then there exists $n \in \mathbb{N}$ such that $e \leqslant e^n \in (SbS]$. We have $a^3 \in K(SeS]K \subseteq (KeK] \subseteq (K(SbS]K] \subseteq (KbK]$. Thus $K$ is an archimedean subsemigroup. Since $S$ is an archimedean weakly commutative, then there exists $n, m \in \mathbb{N}$ such that $(ab)^n \in (b^2S] \subseteq (bK]$ and $(ab)^m \in (Sa^2] \subseteq (Ka]$ by Lemma 2.3. We have $(ab)^{n+m} \in (bK](Ka] \subseteq (bKa]$. Thus $K$ is a weakly commutative subsemigroup. Let $x \in S/K$. Since $S$ is an archimedean, then there exists $n \in \mathbb{N}$ such that $x^n \in (SeS] = K$. Thus $S/K$ is a nil ordered semigroup. Conversely, assume that $S$ is an ideal extension of an archimedean weakly commutative subsemigroup $K$ by a nil ordered semigroup $S/K$. We have $S$ is a weakly commutative by Lemma 2.5. Let $a, b \in S$. Since $S/K$ is nil, then there exists $h, k \in \mathbb{N}$ such that $a^h, b^k \in K$. Since $K$ is an archimedean subsemigroup, $a^{nh} \in (Kb^kK] \subseteq (KbK] \subseteq (SbS]$ for some $n \in \mathbb{N}$. Thus $S$ is an archimedean. □

**Corollary 2.7.** *An ordered semigroup $S$ is an archimedean weakly commutative containing ordered idempotent if and only if $S$ is an ideal extension of an archimedean weakly commutative subsemigroup containing ordered idempotent by a nil ordered semigroup.*

**Lemma 2.8.** *Let $(S, \cdot, \leqslant)$ be an archimedean weakly commutative ordered semigroup without ordered idempotent. Then for every $a \in S$, $a \notin (aS](a \notin (Sa])$.*

*Proof.* Let $a \in S$. If $a \in (aS]$. Then $a \leqslant ax$ for some $x \in S$. Since $S$ is an archimedean weakly commutative, we have $x^n \in (Sa]$ for some $n \in \mathbb{N}$ by Lemma 2.3. This implies $a \leqslant ax^n \in (aSa]$ and so $a$ is a regular. It follows that $S$ has an ordered idempotent. This is a contradiction. Thus $a \notin (aS]$. □

**Definition 2.9.** An ordered semigroup $(S, \cdot, \leqslant)$ is called a *power joined* if for every $a, b \in S$, there exists $n, m \in \mathbb{N}$ such that $a^n = b^m$.

**Example 2.10.** Let $S = \{a, b\}$, $\leqslant = \{(a, a), (b, b), (a, b)\}$ and $xy = b$ for all $x, y \in S$. It is clear that $S$ is a power joined ordered semigroup.

Obviously, a power joined ordered semigroup is an archimedean weakly commutative.

**Remark 2.11.** An ordered semigroup $S$ is a power joined if and only if for any two subsemigroups $A, B$ of $S$, $A \cap B \neq \emptyset$.

We immediately have the following:

**Lemma 2.12.** *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following statements are equivalent:*

(1) *$S$ is a power joined;*

(2) *every ideal of $S$ is a power joined subsemigroup;*

(3) *every left(right) ideal of $S$ is a power joined subsemigroup;*

(4) *every quasi-ideal of $S$ is a power joined subsemigroup;*

(5) *every bi-ideal of $S$ is a power joined subsemigroup.*

**Definition 2.13.** An ordered semigroup $(S, \cdot, \leqslant)$ is called *$Q$-ordered semigroup* if for every proper ideal of $S$ is a power joined subsemigroup.

**Definition 2.14.** An ordered semigroup $(S, \cdot, \leqslant)$ is called *$Q_r$-ordered semigroup* (resp., *$Q_l$-ordered semigroup*) if for every proper right(resp., left) ideal of $S$ is a power joined subsemigroup.

Clearly $Q_r(Q_l)$-ordered semigroup is $Q$-ordered semigroup. The converse is not true.

**Example 2.15.** Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by: $xy = x$ for $y = c$, and $xy = a$ for others, and $\leqslant = \{(a, a), (a, b), (a, c), (b, b), (c, c)\}$.

The ideals of $S$ are: $\{a\}, \{a, b\}$ and $S$. Obviously, $S$ is a $Q$-ordered semigroup. But the right ideal $\{a, c\}$ of $S$ is not power joined subsemigroup and so $S$ is not $Q_r$-ordered semigroup.

**Theorem 2.16.** *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $S$ is an archimedean weakly commutative $Q_r(Q_l)$-ordered semigroup if and only if one of the following conditions satisfied:*

(1) *$S$ is a power joined;*

(2) *$S$ is an ideal extension of a power joined archimedean weakly commutative subsemigroup $K$ containing ordered idempotent by a nil ordered semigroup $S/K$.*

*Proof.* Assume that $S$ is an archimedean weakly commutative $Q_r$-ordered semigroup. Suppose that $S$ does not contain an ordered idempotent element. Let $a, b \in S$. We have $a \notin (aS]$ by Lemma 2.8. Thus $(aS]$ is a proper right ideal of $S$. Since $S$ is $Q_r$-ordered semigroup, $(aS]$ is a power joined subsemigroup. Since $S$ is an archimedean weakly commutative, there exists $n \in \mathbb{N}$ such that $b^n \in (aS]$ and obviously there exists $m \in \mathbb{N}$ such that $a^m \in (aS]$. Then there exists $s, t \in \mathbb{N}$ such that $a^{ms} = b^{nt}$. Thus $S$ is a power joined. If $S$ has an ordered idempotent, then $S$ is an ideal extension of a power joined archimedean weakly commutative subsemigroup $K$ containing ordered idempotent by a nil ordered semigroup $S/K$ by Theorem 2.6. Conversely, it is clear, if $S$ is a power joined. Assume that $S$ is an ideal extension of a power joined archimedean weakly commutative subsemigroup $K$ containing ordered idempotent by a nil ordered semigroup $S/K$. We have $S$ is an archimedean weakly commutative by Theorem 2.6. Let $A$ be a proper right ideal of $S$ and $a, b \in A$. Since $S/K$ is nil, there exists $n, m \in \mathbb{N}$ such that $a^n, b^m \in K$. Since $K$ is a power joined subsemigroup, we have $a^{ns} = b^{mt}$ for some $s, t \in \mathbb{N}$. Thus $A$ is a power joined subsemigroup and so $S$ is a $Q_r$-ordered semigroup. □

**Definition 2.17.** An ordered semigroup $(S, \cdot, \leqslant)$ is called *$Q_q$-ordered semigroup* (resp., *$Q_b$-ordered semigroup*) if for every proper quasi-(resp., bi-)ideal of $S$ is a power joined subsemigroup.

The classes of all power joined ordered semigroups, will denoted by $\mathbf{P}$, the classes of all $Q_q$-ordered semigroups, will denoted by $\mathbf{Q_q}$, the classes of all $Q_b$-ordered semigroups, will denoted by $\mathbf{Q_b}$, the classes of all $Q_r$-ordered semigroups, will denoted by $\mathbf{Q_r}$, the classes of all $Q_l$-ordered semigroups, will denoted by $\mathbf{Q_l}$ and the classes of all $Q$-ordered semigroups, will denoted by $\mathbf{Q}$.

We have the following lemma:

**Lemma 2.18.** $\mathbf{P} \subset \mathbf{Q_b} \subset \mathbf{Q_q} \subset \mathbf{Q_l} \cup \mathbf{Q_r} \subset \mathbf{Q}$.

The following theorem can be obtained from Theorem 2.16 its dual theorem and Lemma 2.18.

**Theorem 2.19.** *Let $(S, \cdot, \leqslant)$ be an archimedean weakly commutative ordered semigroup without ordered idempotent. The following statements are equivalent:*

(1) *$S$ is a power joined;*

(2) *$S$ is $Q_b$-ordered semigroup;*

(3) *$S$ is $Q_q$-ordered semigroup;*

(4) *$S$ is $Q_r$-ordered semigroup;*

(5) *$S$ is $Q_l$-ordered semigroup.*

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