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# Sub-implicative filters in quasi-ordered relational systems

*Daniel Abraham Romano*

**Abstract.** The concept of sub-implicability of BCI-algebras was introduced in 2000. Then that concept was applied to the algebraic structures of residuated lattices (2011), KU-algebras (2017), BH-algebras (2018) and BCK-algebras (2021). In this paper, the idea of sub-implicability is applied to quasi-ordered residuated systems. A criterion for recognizing this type of filter in a quasi-ordered residuated system has been found and, in addition, some important features of this concept have been established.

## 1. Introduction

Residuated lattices were introduced by Ward and Dilworth in [24]. The filter theory of residuated lattice has been widely studied (see [2, 21, 23]), and some important results have been obtained (for example, see [22]).

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda ([1]). It should be noted that this algebraic system differs from the commutative residuated lattice (in the sense of [6]) ordered under a quasi-order (see Example 2.12):

- QRS does not have to be limited from below: and
- QRS, in the general case, does not have to be a lattice.

On the other hand, this class of algebraic structures differs from the class of hoop-algebras because the formula $(\forall x, y \in A)(x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$, which is one of the axioms in the class of hoop-algebras, is not a valid formula in the QRS class. Therefore, we can look at the QRS class as a generalization of these two mentioned classes of algebraic structures

The author introduced and developed the concepts of filters ([10]) in this algebraic structure as well as several types of filters such as associated

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([11]), implicative ([12]), comparative ([13]), weak implicative ([14]), fantastic ([15]), normal ([16]) and shift filters ([17]). In [14], Theorem 3.2, it was shown that any implicative filter in a QRS is a weak implicative filter and that the reverse need not be true.

The term 'sub-implicability' as the concept of sub-implicative ideas in BCI-algebras was first determined in 2000 in [7] by L. Y. Lin and J. Meng. Then this idea was extended to design sub-implicative filters in residuated lattices ([5]), sub-implicative ideals in KU-algebras ([8]), sub-implicative ideals in BH-algebras ([9]) and sub-implicative ideals in BCK-algebras ([19]).

In this paper, we introduce the concept of sub-implicative filters in quasi-ordered residuated systems as a new class of filters in quasi-ordered residuated systems. In addition to the previous one, a connection between (implicative) filter and sub-implicative filter was established. Thus, it was shown that any sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is (an implicative) a filter in $\mathfrak{A}$ and that the reverse need not be the case. A criterion to recognize whether a filter of a quasi-ordered residuated system $\mathfrak{A}$ is a sub-implicative filter in $\mathfrak{A}$ was found. Also, two important properties of this class of filters in quasi-ordered residuated systems are shown.

## 2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, and so on, are labels for the logical functions of conjunction, disjunction, implication, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark $A$ is the abbreviation for the formula $B$.

### 2.1 Concept of quasi-ordered residuated systems

In article [1], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

**Definition 2.1** ([1], Definition 2.1)**.** A *residuated relational system* is a structure $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and $R$ is a binary relation on $A$ and satisfying the following properties:

(1) $(A, \cdot, 1)$ is a commutative monoid;

(2) $(\forall x \in A)((x, 1) \in R)$;

(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following:

**Theorem 2.2** ([1], Proposition 2.1)**.** *Let* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, R\rangle$ *be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$*;*

(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*;*

(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$*;*

(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$*;*

(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set $A$ is a binary relation which is reflexive and transitive.

**Definition 2.3** ([1], Definition 3.1)**.** A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

**Example 2.4.** Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\rightarrow$' defined on $A$ as follows:

| $\cdot$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | a | a | d | c | d |
| b | b | d | b | d | d |
| c | c | c | d | c | d |
| d | d | d | d | d | d |

and

| $\rightarrow$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | b | c | d |
| b | 1 | a | 1 | c | c |
| c | 1 | 1 | b | 1 | b |
| d | 1 | 1 | 1 | 1 | 1 |

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq =: \{(1, 1), (a, 1), (b, 1), (c, 1), (d, 1), (b, b),$ $(a, a)$, $(c, c)$, $(d, d)$, $(c, a), (d, a)$, $(d, b), (d, c)\}$.

**Example 2.5.** For a commutative monoid $A$, let $\mathfrak{P}(A)$ denote the powerset of $A$ ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of $A$. Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X =: \{z \in A : Yz \subseteq X\}).$$

**Example 2.6.** Let $\mathbb{R}$ be a field of real numbers. Define a binary operations '$\cdot$' and '$\rightarrow$' on $A = [0,1] \subset \mathbb{R}$ by

$(\forall x, y \in [0,1])(x \cdot y =: max\{0, x+y-1\})$ and $x \rightarrow y =: min\{1, 1-x+y\})$.

Then, $A$ is a commutative monoid with the identity 1 and $\langle A, \cdot, \rightarrow, \leqslant, 1 \rangle$ is a quasi-ordered residuated system.

**Example 2.7.** Any commutative residuated lattice $\langle A, \cdot, \rightarrow, 0, 1, \sqcap, \sqcup, R \rangle$ where $R$ is a quasi-order lattice is a quasi-ordered residuated system.

The following proposition shows the basic properties of quasi-ordered residuated systems.

**Proposition 2.8** ([1], Proposition 3.1)**.** *Let $A$ be a quasi-ordered residuated system. Then the following holds:*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.
(12) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set $A$ generates an equivalence relation $\equiv_{\preccurlyeq} =: \preccurlyeq \cap \preccurlyeq^{-1}$ on $A$. Due to properties (9) and (10), this equivalence is compatible with the operations in $\mathfrak{A}$. Thus, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$. In connection with the previous one, it can be proved that it is valid:

**Lemma 2.9.** *Let $A =: \langle A, \cdot, 1, \rightarrow, \preccurlyeq \rangle$ be a quasi-ordered residuated system. Then the following holds:*

(13) $(\forall x \in A)(x \rightarrow 1 \equiv_{\preccurlyeq} 1)$
(14) $(\forall x \in A)(1 \rightarrow x \equiv_{\preccurlyeq} x)$.
(15) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \equiv_{\preccurlyeq} y \rightarrow (x \rightarrow z))$.

### 2.2 Concept of filters

In this subsection we give some notions that will be used in this article.

**Definition 2.10** ([10], Definition 3.1)**.** For a non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies the conditions

$(F2)$ $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
$(F3)$ $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown ([10], Proposition 3.4 and Proposition 3.2), that if a non-empty subset $F$ of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the conditions

(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([10], Theorem 3.1).

**Remark 2.11.** In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [3, 18]). It is obvious that our filter concept is also a filter in the sense of [3, 4, 18]. The commonly used term 'positive implication filter' is covered by the term 'implicative filter'. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [18] satisfies some additional condition. It is not a rare case (for example, in [20]) that the term 'positive implucative filter' is used instead of our term 'comparative filter'.

**Example 2.12.** Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\rightarrow$' as follows, $(\forall y, v \in A)(u \cdot v := min\{u, v\})$ and $u \rightarrow v := 1$ if $u \leqslant v$ and $u \rightarrow v := v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} := \langle A, \cdot, \rightarrow, 1, \leqslant \rangle$ is a quasi-ordered residuated system. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.

Terms covering some of the requirements used herein to identify various types of filters in the observed algebraic structure are mostly taken from papers on UP-algebras. In some other algebraic systems, different terms are used to cover the concepts of implicative and comparative filters mentioned herein.

**Definition 2.13** ([12], Definition 3.1)**.** For a non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ we say that an *implicative filter* of $\mathfrak{A}$ if (F2) and the following condition

(IF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \Longrightarrow u \to z \in F)$

are valid.

**Example 2.14.** ([12], Example 3.3)Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.4. Then the subset $F =: \{1, b\}$ is an implicative filter.

**Theorem 2.15** ([12], Theorem 3.5)**.** *A non-empty filter $F$ of a quasi-ordered residuates system $\mathfrak{A}$ is an implicative filter in $\mathfrak{A}$ if and only if $F$ satisfies the following condition*

(IFa) $(\forall u, v \in A)(u \to (u \to v) \in F \Longrightarrow u \to v \in F)$.

**Definition 2.16** ([14], Definition 3.1)**.** For a non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ we say that a *weak implicative filter* of $\mathfrak{A}$ if (F2) and the following condition

(WIF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \Rightarrow u \to (u \to z) \in F)$

are valid.

**Example 2.17.** Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on $A$ as follows:

| $\cdot$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | a | a | a | a | b |
| b | b | a | b | b | d |
| c | c | a | b | c | d |
| d | d | b | d | d | d |

and

| $\to$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | b | c | b |
| b | 1 | a | 1 | b | a |
| c | 1 | a | 1 | 1 | a |
| d | 1 | 1 | 1 | b | 1 |

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1),(a,1),(b,1),(c,1),(d,1),(a,a),(b,b),(c,c),(d,d),(d,a),(d,b),(c,b)\}.$$

Then the subset $\{1, a\}$ is a weak implicative filter in $\mathfrak{A}$.

Notions and notations that are used but not previously determined in this paper can be found in [1, 10, 11, 12, 14, 15].

## 3. Sub-implicative filters in QRS

The concept of sub-implicative ideas in BCI-algebras was first determined in 2000 in [7] by L. Y. Lin and J. Meng.

In this section, which is the main part of this paper, the concept of a sub-implicative filter in a quasi-ordered residuated system is presented and some of its important features are shown.

**Definition 3.1.** A non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ is called *sub-implicative filter* in $\mathfrak{A}$ if (F2) is valid and the following holds

(SIF) $(\forall x, y, z \in A)((z \to ((x \to y) \to ((y \to x) \to x)) \in F \,\wedge\, z \in F)$
$$\Longrightarrow\ (x \to y) \to y \in F).$$

The inter-relationship between the concept of filters and the concept of sub-implicative filters in a quasi-ordered residuated system $\mathfrak{A}$ is standard in the sense that every sub-implicative filter in $\mathfrak{A}$ is a filter in $\mathfrak{A}$ and the reverse need not be true. In our first theorem, we prove the previously stated statement.

**Theorem 3.2.** *Any sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a filter in $A$.*

*Proof.* If we put $z = x$ and $x = y$ in the valid formula (SIF), we get $x \to y \equiv_{\preccurlyeq} x \to ((y \to y) \to ((y \to y) \to y)) \in F$ and $x \in F$, from which it follows that $y \equiv_{\preccurlyeq} = (y \to y) \to y \in F$ according to (SIF). This proves that $F$ is a filter in $\mathfrak{A}$ since (F2) is present by assumption. $\square$

The second theorem provides an important criterion for determining the sub-implacability of filters in quasi-ordered residuated systems.

**Theorem 3.3.** *Let $F$ be a non-empty filter in a quasi-ordered residuated system $\mathfrak{A}$. Then, $F$ is a sub-implicative filter in $\mathfrak{A}$ if and only if the following holds*

(SIFa) $(\forall x, y \in A)((x \to y) \to ((y \to x) \to x) \in F \Longrightarrow (x \to y) \to y \in F)$.

*Proof.* Assume that a nonempty subset $F$ in a quasi-ordered residuated system $\mathfrak{A}$ satisfies conditions (F2), (F3) and (SIF). Then $F$ also satisfies the condition (F0): $1 \in F$ according to Propositions 3.4 and 3.2 in [10]. Let $x, y \in A$ be an arbitrary element such that $(x \to y) \to ((y \to x) \to x) \in F$. Taking into account the validity of (14), we have

$$1 \to ((x \to y) \to ((y \to x) \to x)) \in F \,\wedge\, 1 \in F.$$

From here, according to (SIF), we get $(x \to y) \to y \in F$, which proves the validity of the formula (SIFa).

Suppose, conversely, that a nonempty subset $F$ in a quasi-ordered residuated system $\mathfrak{A}$ satisfies the conditions (F2) and (SIFa). Let us prove (SIF). Let $x, y, z \in A$ be arbitrary elements such that $z \to ((x \to y) \to ((y \to x) \to x) \in F$ and $z \in A$. From here, we get $(x \to y) \to ((y \to x) \to x) \in F$ by (F3). It follows from (SIF) that $(x \to y) \to u \in F$. This means that $F$ is a sub-implicative filter in $\mathfrak{A}$. □

The next Example show that, in the general case, a filter in a quasi-ordered residuated system $\mathfrak{A}$ do not have to be a sub-implicative filter in $\mathfrak{A}$.

**Example 3.4.** Let $A = \{1, a, b, c, d, e, f\}$ and operations '$\cdot$' and '$\to$' defined on $A$ as follows:

| $\cdot$ | 1 | a | b | c | d | e | f |
|---|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | e | f |
| a | a | a | a | a | a | a | a |
| b | b | a | b | b | b | b | b |
| c | c | a | b | b | b | b | b |
| d | d | a | b | b | d | d | d |
| e | e | a | b | b | d | d | d |
| f | f | a | b | b | d | d | f |

and

| $\to$ | 1 | a | b | c | d | e | f |
|---|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | e | f |
| a | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| b | 1 | a | 1 | 1 | 1 | 1 | 1 |
| c | 1 | a | e | 1 | e | 1 | 1 |
| d | 1 | a | c | c | 1 | 1 | 1 |
| e | 1 | a | c | c | e | 1 | 1 |
| f | 1 | a | c | c | e | e | 1 |

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $a \preccurlyeq b \preccurlyeq\preccurlyeq c \preccurlyeq e \preccurlyeq f \preccurlyeq 1$ and $b \preccurlyeq d \preccurlyeq e$. Then the subsets $F_0 = \{1\}$, $F_1 = \{1, f\}$, $F_4 = \{1, f, e, d\}$ and $F_5 = \{1, f, e, c, d\}$ are filters in $\mathfrak{A}$. Subset $K = \{1, f, e\}$ is not a filter in $\mathfrak{A}$ because, for example, we have $c \in K$ and $c \to b = e \in K$ but $b \notin K$. The subset $L = \{1, f, e, c\}$ is not a filter in $\mathfrak{A}$ also, because, for example, we have $c \in L$ and $c \to b = e \in L$ but $b \notin L$. $F_5$ is a sub-implicative filter in $\mathfrak{A}$. On the other hand, the filter $F_0$ is not a sub-implicative filter in $\mathfrak{A}$ because, for example, we have

$$(a \to b) \to ((b \to a) \to a) = 1 \to (a \to a) = 1 \to 1 = 1 \in F_0$$

but $(a \to b) \to b = 1 \to b = b \notin F_0$. This means that the filter $F_0$ does not satisfy the condition (SIFa). It follows that the filter $F_0$ does not satisfy the condition (SIF) either. □

In what follows, we need the following Lemma:

**Lemma 3.5.** *In any quasi-ordered residuated system $\mathfrak{A}$ the following holds:*

(16) $(\forall x, y \in A)(((y \to x) \to x) \to x \equiv_{\preccurlyeq} y \to x)$.

*Proof.* For arbitrary elements $x, y \in A$, due to the reflexivity of the quasi-order relation $\preccurlyeq$, $y \to x \preccurlyeq y \to x$ holds. From here, we have $(y \to x)\cdot y \preccurlyeq x$ by (3). Thus $y \preccurlyeq (y \to x) \to x$ by (3) also. From here, applying (10), we get $((y \to x) \to x) \to x \preccurlyeq y \to x$.

On the other hand, for arbitrary elements $x, y \in A$, we have

$$1 \equiv_{\preccurlyeq} ((y \to x) \to x) \to ((y \to x) \to x) \equiv_{\preccurlyeq} (y \leftarrow x) \to (((y \to x) \to x) \to x).$$

Thus $y \to x \preccurlyeq (((y \to x) \to x) \to x)$ which together with the previously obtained inequality gives $y \to x \equiv_{\preccurlyeq} (((y \to x) \to x) \to x)$. □

**Theorem 3.6.** *Any sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter in $\mathfrak{A}$.*

*Proof.* Assume that $F$ is a sub-implicative filter in $\mathfrak{A}$. Hence, $F$ is a filter in $\mathfrak{A}$ by Theorem 3.2. To prove that $F$ is an implicative filter in $\mathfrak{A}$, it is sufficient to prove the validity of the formula (IFa) in accordance with Theorem 2.15. To this end, let $x, y \in A$ be arbitrary elements such that $x \to (x \to y) \in F$ (hypothesis). Since $F$ is a sub-implicative filter in $\mathfrak{A}$, the formula (SIFa) is a valid formula, that is, for arbitrary $u, v \in A$ it holds $(u \to v) \to ((v \to u) \to u) \in F \implies (u \to v) \to v \in F$. If we put $u \equiv_{\preccurlyeq} x \to y$ and $v \equiv_{\preccurlyeq} y$ in the previous implication, we get
vspace3pt

$$\begin{aligned}
&((x \to y) \to y) \to ((y \to (x \to (x \to y)) \to (x \to y)) \equiv_{\preccurlyeq} \\
&(y \to (x \to y)) \to ((x \to y) \to y) \to (x \to y)) \equiv_{\preccurlyeq} \quad \text{according to (15)} \\
&(x \to (y \to y)) \to (x \to (((x \to y) \to y) \to y)) \equiv_{\preccurlyeq} \quad \text{according to (15)} \\
&(x \to 1) \to (x \to (x \to y)) \equiv_{\preccurlyeq} \quad \text{according to (15)} \\
&1 \to (x \to (x \to y)) \equiv_{\preccurlyeq} \quad \text{according to (13) and (16)} \\
&x \to (x \to y) \in F \quad \text{according to (14) and and hypothesis.}
\end{aligned}$$

This means that $(u \to v) \to ((v \to u) \to u) \in F$. Then $(u \to v) \to v \in F$ by (SIFa) because $F$ is a sub-implicative filter in $\mathfrak{A}$. So, finally we have $F \ni (u \to v) \to v \equiv_{\preccurlyeq} ((x \to y) \to y) \to y \equiv_{\preccurlyeq} x \to y$. This shows that $F$ is an implicative filter in $\mathfrak{A}$. □

**Corollary 3.7.** *Any sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a weak implicative filter in $\mathfrak{A}$.*

*Proof.* Since, according to Theorem 3.2 in [14], every implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a weak implicative filter in $\mathfrak{A}$, it immediately follows that any sub-implicative filter in $\mathfrak{A}$ is a weak implicative filter in $\mathfrak{A}$. $\square$

The following example shows that the converse of the Theorem 3.9 does not have to be true.

**Example 3.8.** Let $A = \{1, a, b, c, d\}$ as in Example 2.4. As shown in Example 2.14, $F =: \{1, b\}$ is an implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ but it is not a sub-implicative filter in $\mathfrak{A}$ because, for example, $(c \to a) \to ((a \to c) \to c) \equiv_{\preccurlyeq} 1 \in F$ but $(c \to a) \rightharpoonup a \equiv_{\preccurlyeq} a \notin F$. This shows that filter $F$ does not satisfy the condition (SIFa).

In the following two theorems, we give some interesting properties of sub-implicative filters in quasi-ordered residuated systems.

**Theorem 3.9.** *Let $F$ be a sub-implicative filter in a quasi-ordered reisduated system $\mathfrak{A}$. Then the following holds*

(SIFb) $(\forall x, y, z \in A)(((y \to z) \to (x \to y) \in F \wedge x \in F) \implies y \in F)$.

*Proof.* Let $F$ be a sub-implicative filter of a quasi-ordered residuated system $\mathfrak{A}$. Then, $F$ is a filter in $\mathfrak{A}$ according to Theorem 3.2. Let $x, y, z \in A$ be arbitrary elements such that $(y \to z) \to (x \to y) \in F$ and $x \in F$.

(i) Since $F \ni (y \to z) \to (x \to y) \equiv_{\preccurlyeq} x \to ((y \to z) \to y) \in F$ according to (12), we conclude that $(y \to z) \to y \in F$ due to (F3).

(ii) On the other hand, from the valid formula (11) $z \cdot y \preccurlyeq z$, it follows $z \preccurlyeq y \to z$ according to (3). Thus $(y \to z) \to y \preccurlyeq z \to y$ in accordance with (10). From here we get $z \to y \in F$ according to (F2).

(iii) From $y \cdot (z \to y) \preccurlyeq y$, which is a valid formula in $\mathfrak{A}$ (see (11)), we have $y \preccurlyeq (z \to y) \to y$ according to (3). From here, according to (10), we get

$$(y \to z) \to y \preccurlyeq (y \to z) \to ((z \to y) \to y).$$

Since $(y \to z) \to y \in F$ according to already shown in (i), from the last inequality it follows $(y \to z) \to ((z \to y) \to y) \in F$ in accordance with (F2). As $F$ is a sub-implicative filter in $\mathfrak{A}$, it satisfies the condition (SIFa) according to Theorem 3.3. Hence, we have $(y \to z) \to z \in F$.

(iv) Now, from $(y \to z) \to z \in F$ and from

$$(y \to z) \to z \preccurlyeq (z \to y) \to ((y \to z) \to y),$$

which is obtained by applying (10) to the left side of this inequality, it follows $(z \to y) \to ((y \to z) \to y) \in F$ according to (F2). If again applying (SIFa), we get $(z \to y) \to y \in F$. Finally, from $z \to y \in F$ and from $(z \to y) \to y \in F$ it follows $y \in F$ according to (F3). This proves the validity of the formula (SIFb). □

**Theorem 3.10.** *Let $F$ be a sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$. Then the following holds*

(SIFc) $(\forall x, y \in A)((x \to y) \to x \in F \implies x \in F)$.

*Proof.* Let $F$ be a sub-implicative filter in a quasi-ordered residuated system $\mathfrak{A}$. This means that $F$ satisfies the condition (SIFb). Let $x, y \in a$ be arbitrary elements such that $(x \to y) \to x \in F$. Thus, in accordance with (14), we have $(x \to y) \to (1 \to x) \in F$. Since $1 \in F$, then $x \in F$ by (SIFb). So, the formula (SIFc) is a valid formula. □

## 4. Final comments and another class of filters

The concept of filters in quasi-ordered residuated systems is an important notion. Thus, implicative, weak implicative, comparative, associated, fantastic, normal, shift and interior filters were determined and investigated. In this report, one more class of filters was added to that family of filters in quasi-ordered residuated systems.

It is to be expected that this does not exhaust the possibility of determining new types of filters in this algebraic structure. To illustrate the previous observation, a determination of one such possibilities can be given by the following definition:

**Definition 4.1.** A non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ is called *new type of filter* in $\mathfrak{A}$ if (F2) is valid and the following holds

$$(*)\ (\forall x, y, z \in A)((((x \to y) \to z) \to ((y \to x) \to x) \in F \wedge z \in F) \implies (x \to y) \to y \in F).$$

In what follows, we need the following lemma

**Lemma 4.2** ([12], Lemma 3.1)**.** *Let a subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition* (*F*2)*. Then the following holds*

(16) $(\forall u \in A)(u \in F \iff 1 \to u \in F)$.

It can be shown without much difficulty that a new type of filter in a quasi-ordered residuated system $\mathfrak{A}$, determined in this way, is a filter in $\mathfrak{A}$.

**Theorem 4.3.** *Any new type of filter $F$ in a quasi-ordered reduced system $\mathfrak{A}$ is a filter in $\mathfrak{A}$.*

*Proof.* Let $F$ be a new type of filter in $\mathfrak{A}$. This means that $F$ satisfies the conditions (F2) and $(*)$. Since $F$ is a nonempty subset in $\mathfrak{A}$, $F$ also satisfies the condition (F0) according to Propositions 3.4 and 3.2 in [10]. Let us prove the validity (F3). If we put $x \equiv y$ and $z \equiv x$ in $(*)$, we get

$$(((y \to y) \to x) \to ((y \to y) \to y) \in F \ \wedge \ x \in F) \implies (y \to y) \to y \in F.$$

Thus, by Lemma 4.2, we have

$$(x \to y \equiv (1 \to x) \to (1 \to y) \in F \ \wedge \ x \in F) \implies y \in F.$$

This proves (F3). So, $F$ is a filter in $\mathfrak{A}$. $\square$

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International Mathematical Virtual Institute
Kordunaška Street 6
78000 Banja Luka
Bosnia and Herzegovina
E-mail: daniel.a.romano@hotmail.com