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Variations of some inverse properties in Cheban loops

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Abstract. A Cheban loop $(G, \odot)$ is characterized by the identities $(z \odot yx)x = zx \odot xy$ and $x(xy \odot z) = yx \odot xz$. It was shown that a Cheban loop $(G, \odot)$ satisfies the cross inverse property if and only if $(G, \odot)$ is an abelian group or all the left (right) translations of $(G, \odot)$ are right (left) regular. Additionally, it was proved that in a Cheban loop, the following algebraic properties are equivalent: flexibility, left inverse property, right inverse property, inverse property, left alternativity, right alternativity and alternativity. This study also revealed that a Cheban loop could be categorized as a weak inverse property loop if it is flexible and the middle inner mapping belongs to a permutation group; as an automorphic inverse property loop if it has a semi-commutative law and the middle inner mapping is contained in a permutation group; as an anti-automorphic inverse property loop if every element has a two-sided inverse and the middle inner mapping is contained in a permutation group; and as a semi-automorphic inverse property loop if it is flexible, the middle inner mapping is contained in a permutation group, and a semi-cross inverse property holds. Finally, the study established the necessary and sufficient conditions for a Cheban loop to have an exponent of 2 or to be a centrum square.

1. Introduction

Let G be a non-empty set. Define a binary operation " $\odot$ " on G. If $x \odot y \in G$ for all $x, y \in G$, then the pair $(G, \odot)$ is called a groupoid. If $a \odot x = b$ and $y \odot a = b$ have unique solutions $x, y \in G$ for all $a, b \in G$ then $(G, \odot)$ is called a quasigroup. Let $(G, \odot)$ be a quasigroup and let there exist a unique element $e \in G$ called the identity element such that for all $x \in G, x \odot e = e \odot x = x$, then $(G, \odot)$ is called a loop. At times, we

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write xy instead of $x \odot y$ and stipulate that " $\odot$ " has lower priority than juxtaposition among factors to be multiplied. Let $(G, \odot)$ be a groupoid and a be a fixed element in G, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \odot x$ and $xR_a = x \odot a$ for all $x \in G$. It can now be seen that a groupoid $(G, \odot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_a^{-1} and R_a^{-1} exist. Let

$$a\backslash b = bL_a^{-1} = aM_b \text{ and } a/b = aR_b^{-1} = bM_a^{-1} \text{ and note that}$$
$$a\backslash b = c \Longleftrightarrow a \odot c = b \text{ and } a/b = c \Longleftrightarrow c \odot b = a.$$

Thus, for any quasigroup $(G, \odot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$. M_a is called the middle translation for any fixed $a \in G$. Consequently, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1.1. A *loop* $(G, \odot, /, \backslash, e)$ is a set G together with three binary operations $(\odot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i) $a \odot (a\backslash b) = b$, $(b/a) \odot a = b$ for all $a, b \in G$.

(ii) $a\backslash a = b/b$ or $e \odot a = a \odot e = a$ for all $a, b \in G$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\odot)$ among factors to be multiplied. For instance, $a \odot b/c$ and $a \odot b\backslash c$ stand for $a(b/c)$ and $a(b\backslash c)$ respectively.

In a loop $(G, \odot)$ with identity element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that $x^\lambda \odot x = e$ while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that $x \odot x^\rho = e$.

For more study on quasigroup and loop theory, reader should check [7, 9, 10, 11, 12, 14, 15].

2. Preliminaries

The following concepts are important for this study. Let G be a non-empty set, the set of all permutations on G forms a group $SYM(G)$ called the symmetric group of G.

Definition 2.1. Let $(G, \odot)$ be a quasigroup and let $A, B, C \in SYM(G)$. If

$$xA \odot yB = (x \odot y)C \qquad \forall\, x, y \in G$$

then the triple (A, B, C) is called an autotopism and such triples form a group $AUT(G, \odot)$ called the autotopism group of $(G, \odot)$. If $A = B = C$, then A is called an automorphism of $(G, \odot)$ whose set form a group $AUM(G, \odot)$ called the automorphism group of $(G, \odot)$.

The left and right multiplication groups $\mathcal{M}_\lambda(G, \odot) = \langle\{L_x : x \in G\}\rangle$ and $\mathcal{M}_\rho(G, \odot) = \langle\{R_x : x \in G\}\rangle$ respectively, are subgroups of the multiplication group $\mathcal{M}(G, \odot) = \langle\{R_x, L_x \ : x \in G\}\rangle \leqslant SYM(G)$ of a loop $(G, \odot)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping. The middle inner mapping is defined as $T_x = R_x L_x^{-1}$ for any $x \in G$.

Definition 2.2. Let $(G, \odot)$ be a quasigroup. Then,

1. $A \in SYM(G)$ is called λ-regular if there exists $(A, I, A) \in AUT(G, \odot)$; the set of such mappings form a group $\Lambda(G, \odot)$.
2. $A \in SYM(G)$ is called ρ-regular if there exists $(I, A, A) \in AUT(G, \odot)$; the set of such mappings form a group $\mathbf{P}(G, \odot)$.

Definition 2.3. A quasigroup $(G, \odot)$ is said to have the

1. left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \odot (x \odot y) = y$ for all $x, y \in G$.
2. right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $(y \odot x) \odot x^\rho = y$ for all $x, y \in G$.
3. inverse property (IP) if it has both the LIP and RIP.
4. right alternative property (RAP) if $y \odot xx = yx \odot x$ for all $x, y \in G$.
5. left alternative property (LAP) if $xx \odot y = x \odot xy$ for all $x, y \in G$.
6. flexible or elastic property if $(x \odot y) \odot x = x \odot (y \odot x)$ holds for all $x, y \in G$.
7. cross inverse property (CIP) if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \odot x^\rho = y$ or $x \odot yx^\rho = y$ or $x^\lambda \odot yx = y$ or $x^\lambda y \odot x = y$ for all $x, y \in G$.
8. weak inverse property (WIP) if it obeys $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ for all $x, y \in G$.

Definition 2.4. A loop $(G, \odot)$ is said to be

1. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$.
2. an anti-automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in G$.
3. a semi-automorphic inverse property loop (SAIPL) if it obeys any of the identities $(x \odot yx)^{\rho} = x^{\rho} \odot y^{\rho}x^{\rho}$ or $(x \odot yx)^{\lambda} = x^{\lambda} \odot y^{\lambda}x^{\lambda}$ for all $x, y \in G$.
3. a power associative loop if $< x >$ is a subgroup of $(G, \odot)$ for all $x \in G$ and a diassociative loop if $< x, y >$ is a subgroup $(G, \odot)$ for all $x, y \in G$.

Definition 2.5. A loop $(G, \odot)$ is called

1. RChL if it satisfies the identity $\underbrace{(z \odot yx) \odot x = zx \odot xy}_{\text{RChI}}$ for all $x, y, z \in G$.
2. LChL if it satisfies the identity $\underbrace{x(xy \odot z) = yx \odot xz}_{\text{LChI}}$ for all $x, y, z \in G$.

The identity of a Cheban loop was introduced by Cheban in [1] as a generalization of the Bol-Moufang type. A loop is a Cheban loop if and only if it is a left Cheban loop (LChL) and a right Cheban loop (RChL). It was reported by the author that the identity of Cheban loop falls under the class of generalized Moufang loop. Characterization of left Cheban loops was studied by Cote et al. [4] in 2011 while Phillips and Shcherbacov [13] in (2010), announced the following structural properties of Cheban loops; they are power associative and conjugacy closed loops (CCL). Research on Cheban loops appeared again in the literature in 2022 when Chinaka et al. [2] constructed a right Cheban loop of small order while the holomorphic structure of RChL was studied by the same authors [3] in 2023. Jaiyéọlá et al. [8] linked Cheban loops to Frute loops (both loops of generalized Bol-Moufang type) in 2021. In 2022, George and Jaiyéọlá [6] studied some new classes of loops which are also power associative and conjugacy closed, discovered by George et al. [5].

This current paper studies some algebraic properties of a Cheban loop and the variation of certain inverse properties in it.

Remark 2.6. In this study, our focus is on the left and right Cheban loops, specifically the identities outlined in Definition 2.5. Our results feature Cheban loop whenever left and right Cheban loops are considered, with the exception being when only one of them is utilized, in which case it will be explicitly specified.

3. Main Results

Lemma 3.1. *In a Cheban loop $(G, \odot)$, the following hold for all $x, y, z \in G$:*

1. $(z \odot y)x = zx \odot x(y/x)$,
2. $x(y \odot z) = (x\backslash y)x \odot xz$,
3. $yx = x \odot x(y/x)$,
4. $xy = (x\backslash y)x \odot x$,
5. $x = y^\rho x \odot x(y/x)$,
6. $x = (x\backslash y)x \odot xy^\lambda$,
7. $x = zx \odot x(z^\rho/x)$,
8. $x = (x\backslash z^\lambda)x \odot x$,
9. $(x^\lambda y)x = x(y/x)$ *or* $R_x^{-1}L_x = L_{x^\lambda}R_x$,
10. $x(yx^\rho) = (x\backslash y)x$ *or* $L_x^{-1}R_x = R_{x^\rho}L_x$,
11. $x((x^\lambda \odot y)x) = yx$ *or* $R_xL_x^{-1} = L_{x^\lambda}R_x$,
12. $xy = (x(yx^\rho))x$ *or* $L_xR_x^{-1} = R_{x^\rho}L_x$,
13. $T_x = R_x^{-1}L_x$,
14. $T_x^{-1} = L_x^{-1}R_x$,
15. $T_x = L_{x^\lambda}R_x$,
16. $T_x^{-1} = R_{x^\rho}L_x$.

Proof. Let $(G, \odot)$ be a Cheban loop. Then, it is a RChL and LChL.

Do the following substitutions in $(z \odot yx) \odot x = zx \odot xy$ and $x(xy \odot z) = yx \odot xz$ respectively:

Put $y = y/x$ and $y = x\backslash y$ to obtain items 1 and 2 respectively. Set $z = e$ in items 1 and 2 to obtain items 3 and 4 respectively. Set $z = y^\rho$ and $z = y^\lambda$ in items 1 and 2 to obtain items 5 and 6 respectively. Set $y = z^\rho$ and $y = z^\lambda$ in items 1 and 2 to obtain item 7 and 8 respectively. Put $z = x^\lambda$ in RChI to get $(x^\lambda \cdot yx)x = x^\lambda x \odot xy \Rightarrow (x^\lambda \cdot yx)x = xy \Rightarrow R_x^{-1}L_x = L_{x^\lambda}R_x$ which is item 9 and put $z = x^\rho$ in LChI to get $x(yx^\rho) = (x\backslash y)x \Rightarrow R_{x^\rho}L_x = L_x^{-1}R_x$ which is item 10.

Use item 9 in item 3 to get $yx = x((x^\lambda y \odot x) \Rightarrow R_xL_x^{-1} = L_{x^\lambda}R_x$ which is item 11. Use item 10 in item 4 to get $xy = (x(yx^\rho))x \Rightarrow L_xR_x^{-1} = R_{x^\rho}L_x$ which is item 12. From item 3, get item 13, and from item 4, get item 14. Items 15 and 16 follow from items 12 and 14. $\square$

Lemma 3.2. *A loop $(G, \odot)$ is a Cheban loop if and only if*

$$(R_x, R_x^{-1}L_x, R_x),\ (L_x^{-1}R_x, L_x, L_x) \in AUT(G, \odot) \text{ for all } x \in G.$$

Hence, $(R_x, T_x, R_x), (T_x^{-1}, L_x, L_x) \in AUT(G, \odot)$ for all $x \in G$.

Proof. Use Definition 2.5 by putting the identities in autotopic forms. □

Theorem 3.3. *Let $(G,\odot)$ be a Cheban loop. The following are equivalent:*

1. $(G,\odot)$ *is a cross inverse property.*
2. $(G,\odot)$ *is commutative,*
3. $(G,\odot)$ *is an abelian group,*
4. $R_x \in \Lambda(G,\odot)\ \forall\ x \in G$,
5. $L_x \in \mathbf{P}(G,\odot)\ \forall\ x \in G$,
6. $R_x \in \mathcal{M}_\lambda(G,\odot)\ \forall\ x \in G$,
7. $L_x \in \mathcal{M}_\rho(G,\odot)\ \forall\ x \in G$.

Proof. Let $(G,\odot)$ be a Cheban loop.

$1 \Leftrightarrow 2$. Going by Lemma 3.1, we have $x((x^\lambda \odot y)x) = yx$. If $(G,\odot)$ is a cross inverse property, then $x \odot y = y \odot x$ which is commutativity. The converse is also true by direct implication.

$1 \Leftrightarrow 3$. Using item 9 in item 1 of Lemma 3.1, we have $zy \odot x = zx \odot (x^\lambda y)x$. So, $(G,\odot)$ has a CIP if and only if $(z \odot y)x = zx \odot y \Leftrightarrow (G,\odot)$ is an abelian group.

$3 \Leftrightarrow 4$. By Lemma 3.2, $(T_x^{-1}, L_x, L_x) \in AUT(G,\odot)$ for all $x \in G$. So, $(G,\odot)$ is commutative if and only if $(T_x^{-1}, L_x, L_x) = (I, L_x, L_x) \Leftrightarrow L_x \in \mathbf{P}(G,\odot)$.

$2 \Leftrightarrow 5$. By Lemma 3.2, $(R_x, T_x, R_x) \in AUT(G,\odot)$ for all $x \in G$. $(G,\odot)$ is commutative if and only if $(R_x, T_x, R_x) = (R_x, I, R_x) \in AUT(G,\odot) \Leftrightarrow R_x \in \Lambda(G,\odot)$ for all $x \in G$.

$1 \Leftrightarrow 6$. Going by item 3 of Lemma 3.1, $x^\lambda \odot yx = x^\lambda \odot [x \odot x(y/x)]$. So, CIP holds if and only if $y = x^\lambda \odot [x \odot x(y/x)] \Leftrightarrow R_x = L_x^2 L_{x^\lambda}$.

$1 \Leftrightarrow 7$. Going by item 4 of Lemma 3.1, $xy \odot x^\rho = [(x\backslash y)x \odot x] \odot x^\rho$. So, CIP holds if and only if $y = [(x\backslash y)x \odot x] \odot x^\rho \Leftrightarrow L_x = R_x^2 R_{x^\rho}$. □

Theorem 3.4. *Let $(G,\odot)$ be a Cheban loop. The following are equivalent:*

1. *Flexibility.*
2. *Left inverse property.*
3. *Right inverse property.*
4. *Inverse property.*
5. *Left alternative property.*
6. *Right alternative property.*
7. *Alternative property.*
8. $R_{x^\rho}^{-1} = R_{x^\rho(x/x^\rho)}$.
9. $L_{x^\lambda}^{-1} = L_{(x^\lambda\backslash x)x^\lambda}$.
10. $x^\lambda \odot (yx \odot x) = xy$.
11. $(x \odot xy)x^\rho = yx$.

Proof. Let $(G,\odot)$ be a Cheban loop, then it is a RChL and LChL. Thus, by Lemma 3.1, $(G,\odot)$ satisfies both RChI and LChI.

$1 \Leftrightarrow 2$. By item 13 of Lemma 3.1, $T_x = R_x^{-1}L_x$ which means $R_xL_x^{-1} = R_x^{-1}L_x$. Also, item 9 of Lemma 3.1 is $R_x^{-1}L_x = L_{x^\lambda}R_x$. So, $(G,\odot)$ has LIP if and only if $L_{x^\lambda} = L_x^{-1} \Leftrightarrow L_{x^\lambda}R_x = L_x^{-1}R_x \Leftrightarrow R_x^{-1}L_x = L_x^{-1}R_x \Leftrightarrow R_xL_x^{-1} = L_x^{-1}R_x \Leftrightarrow R_xL_x = L_xR_x$ if and only if $(G,\odot)$ is flexible.

$1 \Leftrightarrow 3$. By items 10 and 13 of Lemma 3.1, $(G,\odot)$ has RIP $\Leftrightarrow L_x^{-1}R_x = R_x^{-1}L_x \Leftrightarrow L_x^{-1}R_x = R_xL_x^{-1} \Leftrightarrow R_xL_x = L_xR_x$ if and only if $(G,\odot)$ is flexible.

$1 \Leftrightarrow 4$. This is based on $1 \Leftrightarrow 2$ and $1 \Leftrightarrow 3$.

$1 \Leftrightarrow 5$. Put $z = x$ in item 1 of Lemma 3.1 to get $xy \odot x = x^2 \odot x(y/x)$. Flexibility holds $\Leftrightarrow x \odot yx = x^2 \odot x(y/x) \Leftrightarrow x \odot (yx)x = x^2 \odot xy \Leftrightarrow R_x^2L_x = L_xL_{x^2} \Leftrightarrow L_x^2L_x = L_xL_{x^2} \Leftrightarrow L_x^2 = L_{x^2} \Leftrightarrow$ LAP holds.

$1 \Leftrightarrow 6$. Put $z = x$ in item 2 of Lemma 3.1 to get $x \odot yx = (x\backslash y)x \odot x^2$. Flexibility holds $\Leftrightarrow xy \odot x = (x\backslash y)x \odot x^2 \Leftrightarrow x(xy) \odot x = yx \odot x^2 \Leftrightarrow L_x^2R_x = R_xR_{x^2} \Leftrightarrow R_x^2R_x = R_xR_{x^2} \Leftrightarrow R_x^2 = R_{x^2} \Leftrightarrow$ RAP holds.

$1 \Leftrightarrow 7$. This is based on $1 \Leftrightarrow 5$ and $1 \Leftrightarrow 6$.

$1 \Leftrightarrow 8$. In item 1 of Lemma 3.1, put $x = y^\rho$ to get $(z \odot y)y^\rho = zy^\rho \odot y^\rho(y/y^\rho)$. RIP holds $\Leftrightarrow z = zy^\rho \odot y^\rho(y/y^\rho) \Leftrightarrow R_{y^\rho}^{-1} = R_{y^\rho(y/y^\rho)}$.

$2 \Leftrightarrow 9$. In item 2 of Lemma 3.1, put $x = y^\lambda$ to get $y^\lambda(y \odot z) = (y^\lambda\backslash y)y^\lambda \odot y^\lambda z$. LIP holds $\Leftrightarrow z = (y^\lambda\backslash y)y^\lambda \odot y^\lambda z \Leftrightarrow L_{y^\lambda}^{-1} = L_{(y^\lambda\backslash y)y^\lambda}$.

$2 \Leftrightarrow 10$. Going by item 3 of Lemma 3.1, $x^\lambda \odot yx = x^\lambda \odot [x \odot x(y/x)] \Leftrightarrow x^\lambda \odot (yx \odot x) = x^\lambda \odot [x \odot xy] \Leftrightarrow x^\lambda \odot [(x\backslash y)x \odot x] = x^\lambda \odot [x \odot y]$. So, LIP holds if and only if $x^\lambda \odot [(x\backslash y)x \odot x] = y \Leftrightarrow x^\lambda \odot (yx \odot x) = xy$.

$3 \Leftrightarrow 11$. Going by item 4 of Lemma 3.1, $xy \odot x^\rho = [(x\backslash y)x \odot x] \odot x^\rho \Leftrightarrow x(xy) \odot x^\rho = [yx \odot x] \odot x^\rho \Leftrightarrow x(x \odot y/x) \odot x^\rho = yx \odot x^\rho$. So, RIP holds if and only if $y = x(x \odot y/x) \odot x^\rho \Leftrightarrow (x \odot xy)x^\rho = yx$. □

Lemma 3.5. *Let $(G,\odot)$ be a Cheban loop. The following hold for all $x \in G$.*

1. $J_\lambda R_x M_x = T_x$.
2. $T_x^{-1} = J_\rho L_x M_x^{-1}$.

Proof. 1. Setting $y = y/x$ and $z = y^\lambda$ in a RChI, we obtain

$$x = y^\lambda x \odot x(y/x) \Rightarrow (y^\lambda x)\backslash x = x \odot (y/x) \Rightarrow yJ_\lambda R_x M_x = yR_x^{-1}L_x$$

for all $x \in G$. We get $J_\lambda R_x M_x = R_x^{-1}L_x = T_x$.

2. Setting $y = x\backslash y$ and $z = y^\rho$ in the identity of LChL, to get

$$x = (x\backslash y)x \odot xy^\rho \Rightarrow x/(xy^\rho) = (x\backslash y) \odot x \Rightarrow yJ_\rho L_x M_x^{-1} = yL_x^{-1}R_x \Rightarrow J_\rho L_x M_x^{-1} = L_x^{-1}R_x = T_x^{-1}.$$ □

Theorem 3.6. *Let $(G, \odot)$ be a Cheban loop. Define a set*

$$G_\lambda^\rho = \{\Delta \in SYM(G) : J_\rho \Delta J_\lambda = J_\lambda \Delta J_\rho = \Delta\}$$

1. *Then, any two of the following implies the third:*
   - (a) *$(G, \odot)$ is a WIPL.*
   - (b) $T_x \in G_\lambda^\rho \leqslant SYM(G)$.
   - (c) *$(G, \odot)$ is flexible.*
2. *Then, any two of the following implies the third:*
   - (a) *$(G, \odot)$ is an AIPL.*
   - (b) $T_x \in G_\lambda^\rho \leqslant SYM(G)$.
   - (c) $yx^\rho = x^\lambda y$.
3. *Then, any two of the following implies the third:*
   - (a) *$(G, \odot)$ is an AAIPL.*
   - (b) $T_x \in G_\lambda^\rho \leqslant SYM(G)$.
   - (c). $x^\rho = x^\lambda$.
4. *Then, any three of the following implies the forth:*
   - (a) *$(G, \odot)$ is a SAIPL.*
   - (b) $T_x^{-1} L_x \leqslant G_\lambda^\rho$.
   - (c) *$(G, \odot)$ is flexible.*
   - (d) $x \odot yx = y$.

Proof. Let $(G, \odot)$ be a Cheban loop. Recall in Lemma 3.5 that $J_\lambda R_x M_x = T_x$ and $T_x^{-1} = J_\rho L_x M_x^{-1}$.

$$T_x J_\rho L_x = M_x \text{ and } T_x^{-1} J_\lambda R_x = M_x^{-1}. \text{ So,}$$
$$M_x^{-1} M_x = T_x^{-1} J_\lambda R_x T_x J_\rho L_x = I \Rightarrow J_\lambda R_x T_x J_\rho L_x = T_x. \qquad (1)$$

1. $(G, \odot)$ is a weak inverse property loop if and only if

$$(xy)^\lambda x = y^\lambda \Leftrightarrow y L_x J_\lambda R_x = y J_\lambda \Leftrightarrow L_x J_\lambda R_x = J_\lambda. \qquad (2)$$

From (1), we get $L_x J_\lambda R_x T_x J_\rho L_x = L_x T_x$. Assume that $(G, \odot)$ is a WIPL and using (2) in the last equation, we get $J_\lambda T_x J_\rho L_x = L_x T_x$. If $(G, \odot)$ is flexible, then

$$L_x T_x L_x^{-1} = L_x R_x L_x^{-2} = R_x L_x L_x^{-2} = R_x L_x^{-1} = T_x.$$

So, $J_\lambda T_x J_\rho L_x = L_x T_x \Rightarrow L_x T_x L_x^{-1} = J_\lambda T_x J_\rho \Rightarrow T_x = J_\lambda T_x J_\rho \Rightarrow T_x \in G_\lambda^\rho$. Thus, (a) and (c) give (b). Similarly, we can deduce the other two implications

2. $(G,\odot)$ has the automorphic inverse property $\Leftrightarrow (y\odot x)^\rho = y^\rho \odot x^\rho \Leftrightarrow J_\lambda R_x J_\rho = R_{x^\rho}$. From (1), we get $J_\lambda R_x J_\rho J_\lambda T_x J_\rho L_x = T_x$. If $(G,\odot)$ is AIPL, we obtain $R_{x^\rho} J_\lambda T_x J_\rho L_x = T_x$. In addition, if $T_x \in G_\lambda^\rho$, we get $R_{x^\rho} T_x L_x = T_x \Rightarrow R_{x^\rho} R_x L_x^{-1} L_x = T_x \Rightarrow R_{x^\rho} R_x = L_{x^\lambda} R_x \Rightarrow R_{x^\rho} = L_{x^\lambda}$. The other two implications can be similarly obtained.

3. $(G,\odot)$ is an AAIPL $\Leftrightarrow (x\odot y)^\rho = y^\rho \odot x^\rho \Leftrightarrow L_{x^\rho} = J_\lambda R_x J_\rho$. Recall (1) to get $J_\lambda R_x J_\rho J_\lambda T_x J_\rho L_x = T_x$. If $(G,\odot)$ is an AAIPL, then the last equation becomes $L_{x^\rho} J_\lambda T_x J_\rho L_x = T_x$. If $T_x \in G_\lambda^\rho$, then $L_{x^\rho} T_x L_x = T_x \Rightarrow L_{x^\rho} R_x L_x^{-1} L_x = R_x^{-1} L_x \Rightarrow L_{x^\rho} R_x = T_x \Rightarrow L_{x^\rho} R_x = L_{x^\lambda} R_x \Rightarrow L_{x^\rho} = L_{x^\lambda}$. Hence, (a) and (b) imply (c). The other implications can be obtained in a similar way.

4. $(G,\odot)$ is a SAIPL $\Leftrightarrow (x\odot yx)^\rho = x^\rho \odot y^\rho x^\rho \Leftrightarrow R_{x^\rho} L_{x^\rho} = J_\lambda R_x L_x J_\rho$. From (1), we have

$$J_\lambda R_x = T_x L_x^{-1} J_\lambda T_x^{-1} \Rightarrow J_\lambda R_x L_x J_\rho = T_x L_x^{-1} J_\lambda T_x^{-1} L_x J_\rho. \tag{3}$$

If $T_x^{-1} L_x \in G_\rho^\lambda$ and $(G,\odot)$ is a semi-automorphic inverse property loop, then (3) becomes $J_\lambda R_x L_x J_\rho = T_x L_x^{-1} T_x^{-1} L_x$. In addition, if $(G,\odot)$ is flexible, we get $R_{x^\rho} L_{x^\rho} = T_x L_x^{-1} T_x^{-1} L_x \Rightarrow R_{x^\rho} L_{x^\rho} = R_x L_x^{-1} L_x^{-1} T_x^{-1} L_x = R_x L_x^{-1} L_x^{-1} L_x R_x^{-1} L_x = R_x L_x^{-1} R_x^{-1} L_x = R_x L_x^{-1} L_x R_x^{-1} = I$. Thus, $R_{x^\rho} L_{x^\rho} = I$. Therefore, (a), (b) and (c) imply (d). The other implications can be obtained similarly. □

Theorem 3.7. *Let $(G,\odot)$ be a Cheban loop.*

1. *If $xJ_\rho = xJ_\lambda$ and $|T_x| = |M_x| = 2$, then $(G,\odot)$ is commutative.*
2. *Any two of the following give the third for all $x \in G$:*
   - *(a)* $L_x = R_x$.
   - *(b)* $J_\rho = J_\lambda$.
   - *(c)* $T_x^2 = M_x^2$.
3. *If $L_x = R_x$, then any two of the following give the third:*
   - *(a)* $xJ_\rho = xJ_\lambda$.
   - *(b)* $|M_x| = 2$.
   - *(c)* $|T_x| = 2$.
4. *$(G,\odot)$ is centrum square if and only if $yx \odot x(y/x) = (x\backslash y)x \odot xy$.*

5. *The following are equivalent for all* $x \in G$*:*
   (a) $|x| = 2$.
   (b) $R_x = M_x$.
   (c) $M_x = L_x^{-1}$.

Proof. 1. If $|T_x| = |M_x| = 2$ and $x^\lambda = x^\rho$, then from Lemma 3.5, $T_x^{-1} = J_\lambda R_x M_x \Rightarrow J_\rho T_x^{-1} M_x^{-1} = R_x$ and $T_x = J_\rho L_x M_x^{-1} \Rightarrow J_\lambda T_x M_x = L_x$. So that $R_x = J_\rho T_x^{-1} M_x^{-1} = J_\lambda T_x M_x = L_x$. Thus, $(G, \odot)$ is commutative.

2. If $(G, \odot)$ is commutative, then $L_x = R_x$ and so $J_\rho T_x M_x^{-1} = R_x = J_\lambda T_x M_x$. If $x^\lambda = x^\rho$, then $T_x M_x^{-1} = T_x^{-1} M_x \Rightarrow T_x^2 = M_x^2$. So, (a) and (b) imply (c). The other implications can be obtained with similar steps.

3. Similarlt as 2.

4. By item 1 of Lemma 3.1, $(z \odot y)x = zx \odot x(y/x)$. Now, put $z = y$ to get $y^2 \odot x = yx \odot x(y/x)$. By item 1 of Lemma 3.1, $x(y \odot z) = (x \backslash y)x \odot xz$. Now, put $z = y$ to get $x \odot y^2 = (x \backslash y)x \odot xy$. Thus, $x \odot y^2 = y^2 \odot x \Leftrightarrow yx \odot x(y/x) = (x \backslash y)x \odot xy$.

5. By item 1 of Lemma 3.5, $J_\lambda R_x M_x = T_x$. So, $(G, \odot)$ is of exponent 2 if and only if $R_x M_x = T_x \Leftrightarrow R_x M_x = R_x L_x^{-1} \Leftrightarrow M_x = L_x^{-1}$. By item 2 of Lemma 3.5, $T_x^{-1} = J_\rho L_x M_x^{-1}$. So, $(G, \odot)$ is of exponent 2 if and only if $T_x^{-1} = L_x M_x^{-1} \Leftrightarrow L_x R_x^{-1} = L_x M_x^{-1} \Leftrightarrow M_x = R_x$. □

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