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Semi-cross product and extension theory of gyrogroups

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Abstract. We present the concept of a semi-cross product between a group and a gyrogroup, offering a means to construct a broader range of gyrogroups. Also, we develop the Schrier's extension theory of gyrogroups. Consequently, we prove that a 2-fold extension $\{e\} \to H \to G \to K \to \{e\}$ of a group H by a gyrogroup K splits if and only if G is a semi-cross product of H and K. Furthermore, we establish an equivalence between the category GEXT of group-gyro extensions, and the category GFAC of group-gyro factor systems.

1. Introduction

The relativistic velocities do not follow the usual vector addition. Einstein addition of relativistically admissible velocities was introduced by Einstein in 1905 paper ([4], p. 141]). Let c denote the speed of light in vacuum, and let $\mathbb{R}^3(c) = \{v \in \mathbb{R}^3 \mid \|v\| \leq c\}$ be the c-ball of relativistically admissible velocities in $\mathbb{R}^3$. The Einstein addition $\oplus_E$ in $\mathbb{R}^3(c)$ is given by

$$u \oplus_E v = \frac{1}{1 + \frac{\langle u,v \rangle}{c^2}} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \frac{\gamma_u}{1+\gamma_u} \langle u, v \rangle u \right\},$$

where γ_u is the Lorentz factor or gamma factor given by $\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}$. This addition is neither associative nor commutative in $\mathbb{R}^3(c)$. Ungar showed that $(\mathbb{R}^3(c), \oplus_E)$ is not a group but holds nice algebraic properties. In 1988, Ungar [13, 14] introduced a non-associative algebraic structure, *gyrogroup,* which is a generalization of groups. In [10, 11, 12], authors have

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shown that most of the results of the group theory like Lagrange theorem, the fundamental isomorphism theorem, Cayley theorem are still hold for gyrogroups. In a similar vein to group theory, Foguel and Ungar [5] introduced the concept of a normal subgroup within the context of gyrogroups. In this framework, a normal subgroup of a gyrogroup is defined such that the quotient of a gyrogroup by a normal subgroup remains a gyrogroup.

It is evident that while every group qualifies as a gyrogroup, the converse is not always true. Such gyrogroups that are not groups are known as non-degenerate gyrogroups. This prompts the following question: for any given natural number n does there exist a non-degenerate gyrogroup of order n? In [9], the authors showed that gyrogroups are Bol loops. Given that every Bol loop of order p, p^2, and $2p$ (as per [2]) is a group, it follows that there are no non-degenerate gyrogroups of order p, p^2, and $2p$, where p is a prime.

In this paper, we introduce the concept of a semi-cross product between a group and a gyrogroup, which extends the notion of a semi-direct product of groups. Specifically, we provide examples of gyrogroups of order 24 and 32, as well as gyrogroups of infinite order. While the existing notion of a gyrosemidirect product in [14] of a group and a gyrogroup results in a group only, our novelty lies in demonstrating that semi-cross products yield non-degenerate gyrogroups. Thus, the outcomes presented in this paper offer a method for constructing non-degenerate gyrogroups. Furthermore, we endeavor to develop an extension theory for gyrogroups, analogous to the extension theory established for groups. Schreier's extension theory for groups was devised to classify all groups G having H as a normal subgroup, such that G/H is isomorphic to K, for any two groups H and K (for further elucidation, refer to [6]). Given that gyrogroups constitute a generalization of groups and considering the well-established theory of Schreier's extension for groups, it is intriguing to explore Schreier's extension theory for gyrogroups.

In this context, we present the following questions regarding gyrogroups:

1. Given a group H, how can we classify all gyrogroups G such that G/H forms a gyrogroup?

2. Given a gyrogroup H, how can we classify all gyrogroups G such that G/H becomes a group?

3. Given a gyrogroup H, how can we classify all gyrogroups G such that G/H remains a gyrogroup?

In [1], Bruck explored extensions of loops, providing a framework for obtaining additional examples of gyrocommutative gyrogroups. In 2000, Rozga [8] investigated central extensions for gyrocommutative gyrogroups, identifying a natural emergence of a cocycle equation for a subset of these gyrogroups. More recently, Lal and Kakkar [7] discussed several results on extensions for group-based gyrogroups. In section 3, we endeavor to extend Schreier's extension theory to categorize all gyrogroups G with H as a normal subgroup, such that G/H is isomorphic to K, for any arbitrary group H and gyrogroup K. Additionally, we observe that for an extension

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

of a group H by gyrogroup K, the extension splits if and only if G is a semi-cross product of H and K.

Now, we revisit some fundamental definitions and terminology related to gyrogroups, which will be used later in this paper.

Definition 1.1. [14] A pair $(G, \cdot)$ consisting of a non-empty set G and a binary operation "$\cdot$" on G is called a *gyrogroup* if the binary operation "$\cdot$" satisfies the following axioms.

1. There exists an element $e \in G$, called a left identity, such that
$$e \cdot a = a \quad \text{for all } a \in G.$$
2. For each $a \in G$, there exists an element $a^{-1} \in G$, called a left inverse of a, such that $a^{-1} \cdot a = e$.
3. For $a, b \in G$, there exists an automorphism $\mathrm{gyr}[a, b]$ such that
$$a \cdot (b \cdot c) = (a \cdot b) \cdot \mathrm{gyr}[a, b](c)$$
for all $a, b, c \in G$. This is called the left gyroassociative law.
4. $\mathrm{gyr}[a \cdot b, b] = \mathrm{gyr}[a, b]$ for all $a, b \in G$. This is called the left loop property.

Definition 1.2. [5, 12] Let $(G, \cdot)$ be a gyrogroup. Then

1. A gyrogroup which not a group is called a *non-degenerate gyrogroup*.
2. G is said to be *gyrocommutative* if $a \cdot b = \mathrm{gyr}[a, b](b \cdot a)$ (gyrocommutative law) holds for all $a, b \in G$.
3. A nonempty subset H of G is a *subgyrogroup* of G if H forms a gyrogroup under the operation inherited from G, and the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$.

4. A subgyrogroup H of G is said to be a *L-subgyrogroup* of G if it satisfies, $\text{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.
5. A subgroup H of G is said to be *normal* in G if the following axioms hold:
   (a) $\text{gyr}[g, h] = I_G$ for all $h \in H$ and $g \in G$;
   (b) $\text{gyr}[g, g'](H) \subseteq (H)$ for all $g, g' \in G$;
   (c) $gH = Hg$ for all $g \in G$.

Note: By (a) and (c) of Definition 1.2 (4), we have $(gh)g^{-1} = g(hg^{-1}) \in H$. Also, $\text{gyr}[gh, g^{-1}] = I_G$.

Lemma 1.3 ([5]). *If H is a normal subgroup of a gyrogroup G, then $\frac{G}{H}$ forms a factor gyrogroup.*

Definition 1.4.

1. A short exact sequence

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

   of gyrogroups is called an extension of H by K, where i is the inclusion map. Here, $\ker(\beta) = i(H) = H$. A map $t : K \longrightarrow G$ is called a section of E if $\beta \circ t = I_K$ and $t(e) = e$.

2. A morphism from an extension of gyrogroups

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i_1} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

   to another extension of gyrogroups

$$E' \equiv \{e\} \longrightarrow H' \xrightarrow{i_2} G' \xrightarrow{\beta'} K' \longrightarrow \{e\}$$

   is a triple (λ, μ, ν), where $\lambda : H \to H'$, $\mu : G \to G'$ and $\nu : K \to K'$ are gyrogroup homomorphisms such that the following diagram

$$\begin{tikzcd}
E \equiv \{e\} \arrow[r] & H \arrow[r, "i_1"] \arrow[d, "\lambda"] & G \arrow[r, "\beta"] \arrow[d, "\mu"] & K \arrow[r] \arrow[d, "\nu"] & \{e\} \\
E' \equiv \{e\} \arrow[r] & H' \arrow[r, "i_2"] & G' \arrow[r, "\beta'"] & K' \arrow[r] & \{e\}
\end{tikzcd}$$

   is commutative.

Now, we present several gyrogroup identities here for further reference. Note that we use the notation ab instead of $a \cdot b$ for all $a, b \in G$. Throughout this paper, G denotes a gyrogroup unless otherwise specified.

Proposition 1.5 ([14]). *Let $(G, \cdot)$ be a gyrogroup. Then, for all $a, b, c \in G$,*

1. $(a^{-1})(ab) = b$ *(left cancellation);*
2. $ab = e \Leftrightarrow ba = e$*;*
3. $\mathrm{gyr}[ab, a^{-1}] = \mathrm{gyr}[a, b]$*;*
4. $\mathrm{gyr}[a, b](c) = (ab)^{-1}(a(bc))$ *(gyrator identity);*
5. $\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$*;*
6. $(ab)^{-1} = \mathrm{gyr}[a, b](b^{-1}a^{-1})$*;*
7. $(ab)c = a(b\ \mathrm{gyr}[b, a](c))$*.*

2. Semi-cross product and examples of gyrogroups

In this section, we introduce the concept of a semi-cross product between a group and a gyrogroup. Leveraging this semi-cross product, we construct several examples of gyrogroups with both finite and infinite orders.

Proposition 2.1. *Let H be a group and K be a gyrogroup. Suppose that $\sigma : K \longrightarrow Aut(H)$ is a map such that*

1. $\sigma_e = I_H$*, where e denotes the identity element of K and I_H denotes the identity map on H,*
2. $\sigma_{x^{-1}} = \sigma_x^{-1}$*;*
3. $\sigma_{((xy)y)^{-1}} \circ \sigma_{xy} = \sigma_{(xy)^{-1}} \circ \sigma_x$*.*

for all $x, y \in K$, where σ_x denotes the image of x under σ. Then $H \times K$ is a gyrogroup with the binary operation $*$ given by

$$(h, x) * (k, y) = (h\sigma_x(k), xy)$$

and the gyroautomorphism

$$\mathrm{gyr}[(h, x), (k, y)](l, z) = \big((\sigma_{(xy)^{-1}} \circ \sigma_x \circ \sigma_y)(l), \mathrm{gyr}[x, y](z)\big)\,.$$

Proof. 1. Identity: (e, e) is the identity element.
2. Inverse: Let $(h, x) \in H \times K$. Then $(\sigma_{x^{-1}}(h^{-1}), x^{-1})$ is the inverse.
3. Left gyroassociative law: Let $(h, x), (k, y), (l, z) \in H \times K$. Then

$$(h, x) * ((k, y) * (l, z)) = (h, x) * (k\sigma_y(l), yz) = (h\sigma_x(k\sigma_y(l)), x(yz))$$

On the other hand,

$$\begin{aligned}
&((h, x) * (k, y)) * \mathrm{gyr}[(h, x), (k, y)](l, z) \\
&= (h\sigma_x(k), xy) * ((\sigma_{(xy)^{-1}} \circ \sigma_x \circ \sigma_y)(l), \mathrm{gyr}[x, y](z)) \\
&= (h\sigma_x(k)(\sigma_{(xy)} \circ \sigma_{(xy)^{-1}} \circ \sigma_x \circ \sigma_y)(l), (xy)\mathrm{gyr}[x, y](z)) \\
&= (h\sigma_x(k\sigma_y(l)), x(yz)).
\end{aligned}$$

Therefore, $(h, x) * ((k, y) * (l, z)) = ((h, x) * (k, y)) * \mathrm{gyr}[(h, x), (k, y)](l, z)$.
4. Left loop property: Let $(h, x), (k, y), (l, z) \in H \times K$. Then

$$\begin{aligned}
\mathrm{gyr}[(h, x) * (k, y), (k, y)](l, z) &= \mathrm{gyr}[(h\sigma_x(k), xy), (k, y)](l, z) \\
&= ((\sigma_{((xy)y)^{-1}} \circ \sigma_{xy} \circ \sigma_y)(l), \mathrm{gyr}[xy, y](z)) \\
&= ((\sigma_{(xy)^{-1}} \circ \sigma_x \circ \sigma_y)(l), \mathrm{gyr}[x, y](z)) \\
&= \mathrm{gyr}[(h, x), (k, y)](l, z).
\end{aligned}$$

□

Remark 2.2. We refer to this structure as the semi-cross product of H and K (defined in Proposition 2.1), denoted by $H \bowtie K$.

Let $G = H \bowtie K$. Then, we have the following observations:

1. The set $\tilde{H} = \{(h, e) \mid h \in H\}$ is a normal subgroup of G and the set $\tilde{K} = \{(e, k) \mid k \in K\}$ is a L-subgyrogroup of G.
2. $H \cong \tilde{H}$ and $K \cong \tilde{K}$.
3. $H \cap K = \{e\}$.
4. $\sigma_x(k) = xhx^{-1}$, for all $x \in K$ and $h \in H$.

By Proposition 2.1, one can deduce the following results analogous to group extension.

Theorem 2.3. *Let G be a gyrogroup with H as a normal subgroup and K as an L-subgyrogroup. Suppose $\sigma : K \longrightarrow Aut(H)$ is a map defined by $\sigma_x(h) = xhx^{-1}$ with $H \cap K = \{e\}$ and $G = HK$. Then, $G \cong H \bowtie K$.*

Proof. Since H is a normal subgroup, $H \cap K = \{e\}$ and $G = HK$, every element of G can be uniquely written in the form hk, for some $h \in H$ and $k \in K$.

Now, since H is normal, $\mathrm{gyr}[g, h] = I_G$, for all $g \in G$ and $h \in H$. By using this, we have $\sigma_{x^{-1}} = \sigma_x^{-1}$, for all $x \in K$.

$$\begin{aligned}
\sigma_{(xy)^{-1}} \circ \sigma_x(h) &= (xy)^{-1}((xhx^{-1})(xy)) \\
&= (xy)^{-1}(x((hx^{-1})(xy))) \; (\because \; \mathrm{gyr}[hx^{-1}, x] = I_G) \\
&= (xy)^{-1}(x(hy)) = (xy)^{-1}(x(y(y^{-1}hy))) \; (\because \; H \text{ is normal}) \\
&= \mathrm{gyr}[x, y](y^{-1}hy) \text{ (by gyrator identity)} \\
&= \mathrm{gyr}[xy, y](y^{-1}hy) \text{ (by loop identity)} \\
&= ((xy)y)^{-1}((xy)(y(y^{-1}hy))) = ((xy)y)^{-1}((xy)(hy))) \\
&= ((xy)y)^{-1}(((xy)h)y) \; (\because \text{ since } H \text{ is normal}) \\
&= ((xy)y)^{-1}((((xy)h(xy)^{-1})(xy))y) \; (\because \;\; H \text{ is normal}) \\
&= ((xy)y)^{-1}(((\sigma_{(xy)}(h))(xy))y) \\
&= ((xy)y)^{-1}((\sigma_{(xy)}(h))((xy)y)) \; (\because \; H \text{ is normal}) \\
&= \sigma_{((xy)y)^{-1}} \circ \sigma_{xy}(h).
\end{aligned}$$

Now, it is evident that the map $\phi : HK \longrightarrow H \bowtie K$ defined by $\phi(hk) = (h, k)$ is a gyrogroup isomorphism. Therefore, $G \cong H \bowtie K$. □

Remark 2.4. 1. If K is a group and σ is a group homomorphism, then the semi-cross product is equivalent to the semi-direct product of H and K.

2. Suppose $Aut(H)$ is an abelian group where every element is its own inverse. Let $\sigma : K \longrightarrow \mathrm{Aut}(H)$ be a map satisfying the first two conditions of Proposition 2.1. Then, the third condition becomes $\sigma_{\big((xy)y\big)} = \sigma_x$ for all $x, y \in K$. Moreover, if K is a group, then the third condition becomes $\sigma_{xy^2} = \sigma_x$ for all $x, y \in K$.

3. Suppose $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is the gyrogroup given in Example 3.2 in [3]. The table for the element $(xy)y$ is as follows:

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	1	1	6	1	1	6	1	1
2	2	5	1	2	2	2	5	2
3	3	4	4	3	3	4	4	3
4	4	3	3	4	4	3	3	4
5	5	2	5	5	5	5	2	5
6	6	6	1	6	6	1	6	6
7	7	7	7	3	7	7	7	7

The first column and the first row stand for the values of x and y, respectively.

Suppose K is the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. The table for the element $(xy)y = xy^2$ is as follows:

	1	-1	i	-i	j	-j	k	-k
1	1	1	-1	-1	-1	-1	-1	-1
-1	-1	-1	1	1	1	1	1	1
i	i	i	-i	-i	-i	-i	-i	-i
-i	-i	-i	i	i	i	i	i	i
j	j	j	-j	-j	-j	-j	-j	-j
-j	-j	-j	j	j	j	j	j	j
k	k	k	-k	-k	-k	-k	-k	-k
-k	-k	-k	k	k	k	k	k	k

The first column and the first row stand for the values of x and y, respectively.

Example 2.5. [Gyrogroups of order 24]
1. Let $H = \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ be the cyclic group of order 3 and $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the gyrogroup given in Example 3.2 in [3]. Then $\mathrm{Aut}(H) = \{I_H, f\}$, where $f : H \longrightarrow H$ defined by $f(\bar{0}) = \bar{0}, f(\bar{1}) = \bar{2}, f(\bar{2}) = \bar{1}$.

(a) If $\sigma : K \longrightarrow \mathrm{Aut}(H)$ is the trivial map ($\sigma(x) = I_H$, for all x), then $G = H \bowtie K$ is a gyrogroup of order 24 with the given binary operation and gyroautomorphisms:

$$(h, x) * (k, y) = (hk, xy)$$

and

$$\mathrm{gyr}[(h, x), (k, y)] = (I_H, \mathrm{gyr}[x, y]).$$

(b) Let $\sigma : K \longrightarrow \text{Aut}(H)$ be a map defined by

$$\sigma(x) = \begin{cases} f, & \text{if } x = 7 \\ I_H, & \text{otherwise.} \end{cases}$$

Clearly, $\sigma(0) = I_H$ and $\sigma(x^{-1}) = \sigma^{-1}(x)$, for all x. By the definition of K, every element of K is its own inverse. Since f is its own inverse, $\sigma(x^{-1}) = \sigma(x)$. Hence, by Remark 2.4 (2) and (3), $\sigma_{((xy)y)} = \sigma_x \; \forall\, x, y \in K$. Thus $G = H \bowtie K$ is a gyrogroup of order 24. The table for the element $\sigma_{xy} \circ \sigma_x \circ \sigma_y$ (we need to calculate this for gyroautomorphims) is as follows:

	0	1	2	3	4	5	6	7
0	I_H	I_H	I_H	I_H	I_H	I_H	I_H	I_H
1	I_H	I_H	I_H	I_H	I_H	I_H	f	f
2	I_H	I_H	I_H	I_H	I_H	f	I_H	f
3	I_H	I_H	I_H	I_H	f	I_H	I_H	f
4	I_H	I_H	I_H	f	I_H	I_H	I_H	f
5	I_H	I_H	f	I_H	I_H	I_H	I_H	f
6	I_H	f	I_H	I_H	I_H	I_H	I_H	f
7	I_H	f	f	f	f	f	f	I_H

The first column and the first row stand for the values of x and y, respectively. More precisely, the binary operation on G and gyroautomorphisms are given below:

$$(h, x) * (k, y) = \begin{cases} (hf(k), 7y), & \text{if } x = 7 \\ (hk, xy), & \text{otherwise} \end{cases}$$

and

$$\text{gyr}[(h, x), (k, y)] = \begin{cases} (f, I), & \text{if } (x, y) \in X \\ (I_H, A), & (x, y) \in Y \\ (I_H, I), & \text{otherwise} \end{cases}$$

where $X = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3),(1,7),(7,1),(2,7),$
$(7,2),(3,7),(7,3),(7,6),(6,7),(7,5),(5,7),(7,4),(4,7)\}$ and
$Y = \{(1,2),(1,3),(1,4),(1,5),(2,1),(2,3),(2,4),(2,6),(3,1),$
$(3,2),(3,5),(3,6),(4,1),(4,2),(4,5),(4,6),(5,1),(5,3),(5,4),$
$(5,6),(6,2),(6,3),(6,4),(6,5)\}$.

2. Let $H = \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ be the cyclic group of order 3 and $K = Q_8$, where Q_8 is the quaternion group of order 8. Then $\mathrm{Aut}(H) = \{I_H, f\}$, where $f : H \longrightarrow H$ defined by $f(\bar{0}) = \bar{0}, f(\bar{1}) = \bar{2}, f(\bar{2}) = \bar{1}$. Define a map $\sigma : K \longrightarrow \mathrm{Aut}(H)$ by

$$\sigma(x) = \begin{cases} f, & \text{if } x = \pm i \\ I_H, & \text{otherwise.} \end{cases}$$

Clearly, $\sigma(0) = I_H$ and $\sigma(x^{-1}) = \sigma^{-1}(x)$, for all x. Since f is its own inverse, $\sigma(x^{-1}) = \sigma(x)$. Hence by Remark 2.4 (2) and (3), $\sigma_{xy^2} = \sigma_x \ \forall\, x, y \in K$. Thus $G = H \bowtie K$ is a gyrogroup of order 24. The table for the element $\sigma_{xy} \circ \sigma_x \circ \sigma_y$ (we need to calculate this for gyroautomorphims) is as follows:

	1	-1	i	-i	j	-j	k	-k
1	I_H	I_H	I_H	I_H	I_H	I_H	I_H	I_H
-1	I_H	I_H	I_H	I_H	I_H	I_H	I_H	I_H
i	I_H	I_H	I_H	I_H	f	f	f	f
-i	I_H	I_H	I_H	I_H	f	f	f	f
j	I_H	I_H	f	f	I_H	I_H	f	f
-j	I_H	I_H	f	f	I_H	I_H	f	f
k	I_H	I_H	f	f	f	f	I_H	I_H
-k	I_H	I_H	f	f	f	f	I_H	I_H

The first column and the first row stand for the values of x and y, respectively. More precisely, the binary operation on G and gyroautomorphisms are given below:

$$(h, x) * (k, y) = \begin{cases} (hf(k), xy), & \text{if } x = \pm i \\ (hk, xy), & \text{otherwise} \end{cases}$$

and

$$\mathrm{gyr}[(h, x), (k, y)] = (\sigma_{xy} \circ \sigma_x \circ \sigma_y, I_K) = \begin{cases} (f, I_K), & \text{if } (x, y) \in A \\ (I_H, I_K), & \text{otherwise} \end{cases}$$

where A is

$$\{(j,k),(j,-k),(-j,k),(-j,-k),(k,j),(k,-j),(-k,j),(-k,-j),(i,k),\\ (i,-k),(-i,k),(-i,-k),(i,j),(i,-j),(-i,j),(-i,-j),(j,k),(j,-k),\\ (-j,k),(-j,-k),(k,j),(k,-j),(-k,j),(-k,-j)\}.$$

Example 2.6. [Gyrogroups of order 32]
Let $H = \mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be the cyclic group of order 4. Consider the gyrogroup $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ given in Example 3.2 in [3]. Then $\mathrm{Aut}(H) = \{I_H, f\}$, where $f : H \longrightarrow H$ defined by $f(\bar{0}) = \bar{0}, f(\bar{1}) = \bar{3}, f(\bar{2}) = \bar{2}, f(\bar{3}) = \bar{1}$.

The maps σ from K to $\mathrm{Aut}(H)$ given in Examples 2.5 1(a) and 1(b) will give two gyrogroups of order 32. Also, the map σ from Q_8 to $\mathrm{Aut}(H)$ given in Example 2.5 (2) will give another gyrogroup of order 32.

Example 2.7. [Gyrogroup of infinite order]
Let $H = \mathbb{Z}$ be the group of integers and $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the gyrogroup given in Example 3.2 in [3]. Then $\mathrm{Aut}(H) = \{I_H, f\}$, where $f : H \longrightarrow H$ defined by $f(x) = -x \ \forall \ x \in H$. The maps σ from K to $\mathrm{Aut}(H)$ given in Examples 2.5 1(a) and 1(b) will give two gyrogroups of infinite order. Also, the map σ from Q_8 to $\mathrm{Aut}(H)$ given in Example 2.5 (2) will give another gyrogroup of infinite order.

3. Extension theory of a group by a gyrogroup

The primary objective of this section is to establish Schreier's extension theory for gyrogroups under specific conditions.

Let $E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$ be a gyrogroup extension of a gyrogroup H by by another gyrogroup K. Let t be a section of E and let $g \in G$. Then

$$\beta(t(\beta(g))g^{-1}) = e \Rightarrow t(\beta(g))g^{-1} \in H.$$

Thus, there exists a unique $h \in H$ such that

$$\begin{aligned}
&t(\beta(g))g^{-1} = h^{-1} \\
&\Rightarrow h(t(\beta(g))g^{-1}) = e \\
&\Rightarrow (ht(x))\mathrm{gyr}[h, t(x)](g^{-1}) = e, \text{ where } \beta(g) = x \in K \\
&\Rightarrow (\mathrm{gyr}[h, t(x)](g^{-1}))((ht(x))) = e \text{ (since } ab = e \Leftrightarrow ba = e) \\
&\Rightarrow ht(x) = \mathrm{gyr}[h, t(x)](g) \\
&\Rightarrow g = \mathrm{gyr}[t(x), h](ht(x)).
\end{aligned}$$

Therefore, given a gyrogroup extension

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

of H by K with a choice of section t, every $g \in G$ can be written as

$$g = \text{gyr}[t(x), h](ht(x)), \text{for some } h \in H \text{ and } x \in K.$$

Now, suppose $g = \text{gyr}[t(x), h](ht(x)) = \text{gyr}[t(x'), h](ht(x'))$. Then applying β on both the sides,

$$\begin{aligned}
&\beta(\text{gyr}[t(x), h](ht(x))) = \beta(\text{gyr}[t(x'), h](ht(x')))\\
\Rightarrow &\text{gyr}[\beta(t(x)), \beta(h)](\beta(h)\beta(t(x))) = \text{gyr}[\beta(t(x')), \beta(h)](\beta(h)\beta(t(x')))\\
\Rightarrow &\text{gyr}[\beta(t(x)), e](\beta(t(x))) = \text{gyr}[\beta(t(x')), e](\beta(t(x')))\\
\Rightarrow &x = x' \text{ (since gyr}[g, e] = I_G \text{ and } \beta t = I_K).
\end{aligned}$$

This shows that every $g \in G$ has a unique representation of the type $\text{gyr}[t(x), h](ht(x))$.

Proposition 3.1. *Let* $E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$ *be a gyrogroup extension of* H *by* K *with* $\text{gyr}[h, g] = I_G$*, for all* $h \in H$ *and* $g \in G$*. Then* H *is a normal subgroup of* G*.*

Proof. Since $H = \ker(\beta)$ and $\text{gyr}[h, g] = I_G$, H is a subgroup. Let $g, g' \in G$ and $h \in H$. Then $\beta(\text{gyr}[g, g'](h)) = e$. Therefore, $\text{gyr}[g, g'](h) \in H$. Thus, $\text{gyr}[g, g'](H) \subseteq H$ for all $g, g' \in G$.

Now, let $ah \in aH$. Then $\beta((ah)a^{-1}) = e$. This implies, $(ah)a^{-1} = h'$ for some $h' \in H$. The following steps are followed from identities (3), (5) and (7) of Proposition 1.5.

$$\begin{aligned}
&((ah)a^{-1})a = h'a\\
\Rightarrow &(ah)(a^{-1}\text{gyr}[a^{-1}, ah](a)) = h'a\\
\Rightarrow &(ah)(a^{-1}\text{gyr}[h, a](a)) = h'a\\
\Rightarrow &(ah)(a^{-1}a) = h'a\\
\Rightarrow &ah = h'a
\end{aligned}$$

Thus, $ah \in Ha$, that is, $aH \subseteq Ha$. Similarly, $Ha \subseteq aH$. Hence, $Ha = aH$. Therefore, H is a normal subgroup. □

Now, we study gyrogroup extensions under a special condition.

Definition 3.2. *A gyrogroup extension*

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

of H by K is said to be a group-gyro extension if $\mathrm{gyr}[h,g] = I_G$, *for all* $h \in H$ *and* $g \in G$.

It can be easily seen that if $E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$ is a group-gyro extension of H by K and t is a section of E, then any $g \in G$ can be uniquely expressed as $g = ht(x)$.

Also, for each $h \in H$ and $x \in K$, we have $(t(x)h)t(x)^{-1} = t(x)(ht(x)^{-1}) \in H$. Thus, for $x \in K$, there exists a map $\sigma_x^t : H \to H$ given by

$$\sigma_x^t(h) = (t(x)h)t(x)^{-1}. \tag{1}$$

In fact, $\sigma_x^t \in Aut(H)$ for each $x \in K$. Let $x, y \in K$. Then

$$\begin{aligned}
&\beta(t(xy)(t(x)t(y))^{-1}) = e\\
&\Rightarrow t(xy)(t(x)t(y))^{-1} = f^t(x,y)^{-1} \text{ for some } f^t(x,y) \in H\\
&\Rightarrow f^t(x,y)(t(xy)(t(x)t(y))^{-1}) = e\\
&\Rightarrow (f^t(x,y)t(xy))(t(x)t(y))^{-1}) = e\\
&\Rightarrow (t(x)t(y))^{-1})(f^t(x,y)t(xy)) = e\\
&\Rightarrow t(x)t(y) = f^t(x,y)t(xy).
\end{aligned}$$

Thus, there exists a map $f^t : K \times K \to H$ such that

$$t(x)t(y) = f^t(x,y)t(xy). \tag{2}$$

Since $t(e) = (e)$,

$$f^t(x,e) = e = f^t(e,y) \text{ for each } x, y \in K. \tag{3}$$

Also, for $h \in H$ and $x, y \in K$

$$\begin{aligned}
t(x)(ht(y)) &= (t(x)h)\mathrm{gyr}[t(x),h](t(y))\\
&= (t(x)ht(x)^{-1}t(x))t(y)\\
&= ((t(x)ht(x)^{-1})t(x))t(y) \quad (\text{since } \mathrm{gyr}[t(x)h, t(x)^{-1}] = I_G)\\
&= (\sigma_x^t(h)t(x))t(y)\\
&= \sigma_x^t(h)(t(x)\mathrm{gyr}[t(x),\sigma_x^t(h)](t(y)))\\
&= \sigma_x^t(h)(t(x)t(y))\\
&= \sigma_x^t(h)(f^t(x,y)t(xy)) = (\sigma_x^t(h)f^t(x,y))t(xy).
\end{aligned}$$

Similarly,

$$(t(x)h)t(y) = (\sigma_x^t(h)f^t(x,y))t(xy) = t(x)(ht(y)).$$

Further,

$$\begin{aligned}(ht(x))t(y) &= h(t(x)\text{gyr}[t(x),h](t(y))) \\ &= h(t(x)t(y)) = h(f^t(x,y)t(xy)) = (hf^t(x,y))t(xy).\end{aligned}$$

Now, it is easy to observe that

$$(ht(x))(kt(y)) = (h\sigma_x^t(k)f^t(x,y))t(xy). \tag{4}$$

By Equation 2, for $x \in K$ we have

$$\begin{aligned}&t(x^{-1})t(x) = f^t(x^{-1},x) \\ &\Rightarrow (f^t(x^{-1},x))^{-1}(t(x^{-1})t(x)) = e \\ &\Rightarrow t(x)(f^t(x^{-1},x)^{-1}t(x^{-1})) = e \\ &\Rightarrow t(x)^{-1} = f^t(x^{-1},x)^{-1}t(x^{-1}).\end{aligned}$$

So, we have the following equation

$$t(x)^{-1} = f^t(x^{-1},x)^{-1}t(x^{-1}). \tag{5}$$

Now, let $h,k,l \in H$ and $x,y,z \in K$. Then by the gyrator identity, we have

$$\begin{aligned}&\text{gyr}[ht(x),kt(y)](lt(z)) = ((ht(x))(kt(y))^{-1}((ht(x))((kt(y)(lt(z))) \\ &= (h\sigma_x^t(k)f^t(x,y)t(xy))^{-1}((ht(x))(k\sigma_y^t(l)f^t(y,z)t(yz))) \\ &= (t(xy)^{-1}f^t(x,y)^{-1}\sigma_y^t(k)^{-1}h^{-1})(h\sigma_x^t(k\sigma_y^t(l)f^t(y,z))f^t(x,yz)t(x(yz))) \\ &= f^t((xy)^{-1},xy)^{-1}t((xy)^{-1})f^t(x,y)^{-1}\sigma_x^t(\sigma_y^t(l)f^t(y,z))f^t(x,yz)t(x(yz)) \\ &= f^t((xy)^{-1},xy)^{-1}(\sigma^t_{(xy)^{-1}}(f^t(x,y)^{-1}\sigma_x^t(\sigma_y^t(l)f^t(y,z))f^t(x,yz))t((xy)^{-1})) \\ &.t(x(yz)) \\ &= f^t((xy)^{-1},xy)^{-1}(\sigma^t_{(xy)^{-1}}(f^t(x,y)^{-1}\sigma_x^t(\sigma_y^t(l)f^t(y,z))f^t(x,yz))) \\ &.f^t((xy)^{-1},(x(yz))t(\text{gyr}[x,y](z))\end{aligned}$$

Here, for $x,y \in K$, we define a function $F^t_{(x,y)} : H \times K \longrightarrow H$ such that

$$\begin{aligned}&F^t_{(x,y)}(l,z) = \\ &\quad f^t((xy)^{-1},xy)^{-1}(\sigma^t_{(xy)^{-1}}(f^t(x,y)^{-1}\sigma_x^t(\sigma_y^t(l)f^t(y,z))f^t(x,yz))) \\ &\quad f^t((xy)^{-1},x(yz)).\end{aligned} \tag{6}$$

Therefore,

$$\operatorname{gyr}[ht(x), kt(y)](lt(z)) = F^t_{(x,y)}(l,z)t(\operatorname{gyr}[x,y](z)). \tag{7}$$

Now we see the properties of the function F^t which is defined in (6).

Proposition 3.3. *For $x, y \in K$, $F^t_{(x,y)}$ satisfies the following properties:*

1. $F^t_{(x,e)}(l,z) = l = F^t_{(e,y)}(l,z)$, $\forall\ l \in H$ *and* $z \in K$.
2. *For* $l_1, l_2 \in H$ *and* $z_1, z_2 \in K$, $F^t_{(x,y)}(l_1\sigma^t_{z_1}(l_2)f^t(z_1,z_2), z_1z_2)$
$= F^t_{(x,y)}(l_1,z_1)\sigma^t_{(\operatorname{gyr}[x,y](z_1))}(F^t_{(x,y)}(l_2,z_2))f^t(\operatorname{gyr}[x,y](z_1), \operatorname{gyr}[x,y](z_2))$.
3. $F^t_{(xy,y)} = F^t_{(x,y)}$.

Proof. 1. Since $\operatorname{gyr}[h, kt(y)] = I_G = \operatorname{gyr}[ht(x), k]$, we have

$$F^t_{(x,e)}(l,z) = l = F^t_{(e,y)}(l,z), \ \forall\ l \in H \text{ and } x, y, z \in H.$$

2. Note that $\operatorname{gyr}[x,y]$ is an automorphism, hence

$$\begin{aligned}
&\operatorname{gyr}[ht(x), kt(y)]((l_1t(z_1))(l_2t(z_2)))\\
&= \operatorname{gyr}[ht(x), kt(y)](l_1t(z_1))\operatorname{gyr}[ht(x), kt(y)](l_2t(z_2))\\
&= (F^t_{(x,y)}(l_1,z_1)t(\operatorname{gyr}[x,y]z_1))(F^t_{(x,y)}(l_2,z_2)t(\operatorname{gyr}[x,y](z_2)))\\
&= F^t_{(x,y)}(l_1,z_1)\sigma^t_{\operatorname{gyr}[x,y](z_1))}(F^t_{(x,y)}(l_2,z_2))f^t(\operatorname{gyr}[x,y](z_1), \operatorname{gyr}[x,y](z_2))\\
&.t(\operatorname{gyr}[x,y](z_1z_2)).
\end{aligned}$$

Also,

$$\begin{aligned}
&\operatorname{gyr}[ht(x), kt(y)]((l_1t(z_1))(l_2t(z_2)))\\
&= \operatorname{gyr}[ht(x), kt(y)](l_1\sigma^t_{z_1}(l_2)f^t(z_1,z_2)t(z_1z_2))\\
&= F^t_{(x,y)}(l_1\sigma^t_{z_1}(l_2)f^t(z_1,z_2), z_1z_2)t(\operatorname{gyr}[x,y](z_1z_2)).
\end{aligned}$$

On comparing both the expressions for the gyromap, we get

$$\begin{aligned}
&F^t_{(x,y)}\left(l_1\sigma^t_{z_1}(l_2)f^t(z_1,z_2), z_1z_2\right) =\\
&F^t_{(x,y)}(l_1,z_1)\sigma^t_{(\operatorname{gyr}[x,y](z_1))}\left(F^t_{(x,y)}(l_2,z_2)\right)f^t(\operatorname{gyr}[x,y](z_1), \operatorname{gyr}[x,y](z_2)).
\end{aligned}$$

3. Also,

$$\begin{aligned}
\operatorname{gyr}[ht(x)kt(y), kt(y)](lt(z)) &= \operatorname{gyr}[h\sigma^t_x(k)f^t(x,y)t(xy), kt(y)](lt(z))\\
&= F^t_{(xy,y)}(l,z)t(\operatorname{gyr}[xy,y](z))
\end{aligned}$$

$$\operatorname{gyr}[ht(x), kt(y)](lt(z)) = F^t_{(x,y)}(l,z)t(\operatorname{gyr}[x,y](z)).$$

Since $\operatorname{gyr}[ht(x)kt(y), kt(y)] = \operatorname{gyr}[ht(x), kt(y)]$ and $\operatorname{gyr}[xy,y] = \operatorname{gyr}[x,y]$, $F^t_{(xy,y)} = F^t_{(x,y)}$. ☐

Also, let $h, k, l \in H$ and $x, y, z \in K$,

$$\begin{aligned}
&ht(x)((kt(y))(lt(z))) = ht(x)(k\sigma_y^t(l)f^t(y,z)t(yz))\\
&= h\sigma_x^t(k\sigma_y^t(l)f^t(y,z))(t(x)t(yz))\\
&= h\sigma_x^t(k\sigma_y^t(l)f^t(y,z))f^t(x,yz)t(x(yz)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(ht(x))(kt(y)))\text{gyr}[ht(x), kt(y)](lt(z))\\
&= (h\sigma_x^t(k)f^t(x,y)t(xy))(F^t_{(x,y)}(l,z)t(\text{gyr}[x,y](z)))\\
&= h\sigma_x^t(k)f^t(x,y)\sigma_{xy}^t(F^t_{(x,y)}(l,z))f^t(xy, \text{gyr}[x,y](z))t((xy)\text{gyr}[x,y](z)).
\end{aligned}$$

Since $ht(x)((kt(y))(lt(z))) = (ht(x))(kt(y)))\text{gyr}[ht(x), kt(y)](lt(z))$ and $x(yz) = (xy)\text{gyr}[x,y](z)$,

$$\sigma_x^t(\sigma_y(l)f^t(y,z))f^t(x,yz) = f^t(x,y)\sigma_{xy}^t(F^t_{(x,y)}(l,z))f^t(xy, \text{gyr}[x,y](z)). \tag{8}$$

Remark 3.4. By using the assumption $\text{gyr}[h,g] = I_G$, it is easy to see the following:

1. If $ht(x) = h't(x)$, then $h = h'$.
2. ${\sigma_x^t}^{-1}(h) = f^t(x^{-1},x)^{-1}\sigma^t_{x^{-1}}(h)f^t(x^{-1},x)$, for all $x \in K$ and $h \in H$.
3. $\sigma_x^t(f^t(x^{-1},x)^{-1})f^t(x,x^{-1}) = e$.
4. $\text{gyr}[t(x)h, t(x)^{-1}] = I_G$, for all $x \in K$ and $h \in H$.

Definition 3.5. Let H be a group and K be a gyrogroup. Then a *group-gyro factor system* is a quintuplet (K, H, σ, f, F), where $\sigma : K \longrightarrow Aut(H)$ is a map, f is a map from $K \times K$ to H and F is a map from $K \times K$ to $H^{H\times K}$ satisfying the following relations:

1. $\sigma_e = I_H$;
2. $f(x,e) = e = f(e,y)$ for each $x, y \in K$;
3. $f(x,y)\sigma_{xy}(F_{(x,y)}(l,z))f(xy, \text{gyr}[x,y](z)) = \sigma_x(\sigma_y(l)f(y,z))f(x,yz)$ for each $x, y, z \in K$;
4. $F_{(x,e)}(l,z) = l = F_{(e,y)}(l,z),\ \forall\ l \in H$ and $x, y, z \in H$;
5. $F_{(x,y)}(l_1\sigma_{z_1}(l_2)f(z_1,z_2), z_1z_2)$
   $= F_{(x,y)}(l_1,z_1)\sigma_{\text{gyr}[x,y](z_1))}(F_{(x,y)}(l_2,z_2))f(\text{gyr}[x,y](z_1), \text{gyr}[x,y](z_2))$;
6. $F_{(xy,y)} = F_{(x,y)},\ \forall\ x, y \in K$.

Proposition 3.6. *Every group-gyro extension E of H by K with a choice of a section t determines a group-gyro factor system $(K, H, \sigma^t, f^t, F^t)$, where σ^t, f^t and F^t are described by Proposition* 3.3 *and equations* (1), (3), (8).

Conversely, given a group-gyro factor system (K, H, σ, f, F), there exists a group-gyro extension E of H by K, and a section t of E such that the group-gyro factor system associated with E is (K, H, σ, f, F).

Proof. The proof of the direct implication stems from the preceding discussion that inspired Definition 3.5. For the converse implication, consider $G = H \times K$. Define a binary operation on G as follows:

$$(a, x) \cdot (b, y) = (a\sigma_x(b)f(x, y), xy)$$

and gyro morphism by

$$\mathrm{gyr}[(a, x), (b, y)](c, z) = (F_{(x,y)}(c, z), \mathrm{gyr}[x, y](z)).$$

Then G is a gyrogroup and H is a normal subgroup of G. By defining section t by $t(x) = (e, x)$, we have $\sigma^t = \sigma, f^t = f$ and $F^t = F$. $\square$

3.1. Equivalence between Category GEXT of group-gyro extension and Category GFAC of group-gyro factor systems

Let (λ, μ, ν) be a morphism from the group-gyro extension E_1 to the group-gyro extension E_2. Then we have the following commutative diagram:

$$\begin{tikzcd}
E_1 \equiv \{e\} \arrow[r] & H_1 \arrow[r, "i_1"] \arrow[d, "\lambda"] & G_1 \arrow[r, "\beta_1"] \arrow[d, "\mu"] & K_1 \arrow[r] \arrow[d, "\nu"] & \{e\} \\
E_2 \equiv \{e\} \arrow[r] & H_2 \arrow[r, "i_2"] & G_2 \arrow[r, "\beta_2"] & K_2 \arrow[r] & \{e\}
\end{tikzcd}$$

Let t_1 and t_2 be sections of E_1 and E_2, respectively. Consider the corresponding group-gyro factor systems $(K_1, H_1, f^{t_1}, \sigma^{t_1}, F^{t_1})$ and $(K_2, H_2, f^{t_2}, \sigma^{t_2}, F^{t_2})$ of E_1 and E_2, respectively. Then for $x \in K_1$, $\mu(t_1(x)) \in G_2$ and $\beta_2(\mu(t_1(x))) = \nu(\beta_1(t_1(x))) = \nu(x) = \beta_2(t_2(\nu(x)))$. Thus $t_2(\nu(x))\mu(t_1(x))^{-1} \in H_2$. In turn, we have a unique $g(x) \in H_2$ such that

$g(x)^{-1} = t_2(\nu(x))\mu(t_1(x))^{-1}$
$\Rightarrow g(x)(t_2(\nu(x))\mu(t_1(x))^{-1}) = e$ (by left cancellation)
$\Rightarrow (g(x)t_2(\nu(x)))\mathrm{gyr}[g(x), t_2(\nu(x))](\mu(t_1(x))^{-1}) = e$
$\Rightarrow (g(x)t_2(\nu(x)))\mu(t_1(x))^{-1} = e$
$\Rightarrow \mu(t_1(x))^{-1}(g(x)t_2(\nu(x))) = e.$

This implies

$$\mu(t_1(x)) = g(x)t_2(\nu(x)). \tag{9}$$

Since $t_1(e) = e = t_2(e)$, it follows that

$$g(e) = e. \tag{10}$$

Since $\mu(t_1(x)t_1(y)) = \mu(t_1(x))\mu(t_1(y))$, we have the following equation

$$\lambda(f^{t_1}(x,y))g(xy) = g(x)\sigma^{t_2}_{\nu(x)}(g(y))f^{t_2}(\nu(x),\nu(y)). \tag{11}$$

Also,

$$\mu((t_1(x)h)t_1(x)^{-1}) = \lambda(\sigma^{t_1}_x(h)). \tag{12}$$

By using the fact that μ is a gyrogroup homomorphism and Equation 9,

$$\begin{aligned}\mu((t_1(x)h)t_1(x)^{-1}) &= \mu(t_1(x))\lambda(h)\mu(t_1(x)^{-1})\\ &= (g(x)t_2(\nu(x)))\lambda(h)(t_2(\nu(x))^{-1}g(x)^{-1})\\ &= g(x)\sigma^{t_2}_{\nu(x)}(\lambda(h))g(x)^{-1}\end{aligned}$$

Finally we have

$$\mu((t_1(x)h)t_1(x)^{-1}) = g(x)\sigma^{t_2}_{\nu(x)}(\lambda(h))g(x)^{-1}. \tag{13}$$

On comparing Equation (12) and (13), we have

$$\lambda(\sigma^{t_1}_x(h)) = g(x)\sigma^{t_2}_{\nu(x)}(\lambda(h))g(x)^{-1}. \tag{14}$$

We know that,

$$\mathrm{gyr}[ht_1(x), kt_1(y)](lt_1(z)) = F^{t_1}_{(x,y)}(l,z)t_1(\mathrm{gyr}[x,y](z)),$$

$$\begin{aligned}\mu(\mathrm{gyr}[ht_1(x), kt_1(y)](lt_1(z))) &= \mathrm{gyr}[\lambda(h)\mu(t_1(x)),\lambda(k)\mu(t_1(y))](\lambda(l)\mu(t_1(z)))\\ &= \mathrm{gyr}[\lambda(h)g(x)t_2(\nu(x)), \lambda(k)g(y)t_2(\nu(y))](\lambda(l)g(z)t_2(\nu(z)))\\ &= \mathrm{gyr}[(\lambda(h)g(x))t_2(\nu(x)), (\lambda(k)g(y))t_2(\nu(y))]((\lambda(l)g(z))t_2(\nu(z)))\\ &= F^{t_2}_{(\nu(x),\nu(y))}(\lambda(l)g(z)), \nu(z))t_2(\mathrm{gyr}[\nu(x),\nu(y)](\nu(z))).\end{aligned}$$

We have the following equation

$$\begin{aligned}&\mu(\mathrm{gyr}[ht_1(x), kt_1(y)](lt_1(z)))\\ &\qquad = F^{t_2}_{(\nu(x),\nu(y))}(\lambda(l)g(z)), \nu(z))t_2(\mathrm{gyr}[\nu(x),\nu(y)](\nu(z))).\end{aligned} \tag{15}$$

Also we have,

$$\begin{aligned}\mu(\text{gyr}[ht_1(x),kt_1(y)](lt_1(z))) &= \lambda(F^{t_1}_{(x,y)}(l,z))\mu(t_1(\text{gyr}[x,y](z)))\\ &= \lambda(F^{t_1}_{(x,y)}(l,z))g(gyr[x,y](z))t_2(\nu(\text{gyr}[x,y](z)).\end{aligned}$$

Finally, we have

$$\begin{aligned}&\mu(\text{gyr}[ht_1(x),kt_1(y)](lt_1(z)))\\ &\quad= \lambda(F^{t_1}_{(x,y)}(l,z))g(gyr[x,y]z)t_2(\text{gyr}[\nu(x),\nu(y)](\nu(z))).\end{aligned} \tag{16}$$

On comparing Equation (15) and (16),

$$F^{t_2}_{(\nu(x),\nu(y))}(\lambda(l)g(z),\nu(z)) = \lambda(F^{t_1}_{(x,y)}(l,z))g(gyr[x,y](z)). \tag{17}$$

Thus a morphism (λ,μ,ν) between two group-gyro extensions E_1 and E_2 together with choices of sections t_1 and t_2 of the corresponding extensions, induces a map g from K_1 to H_2 such that the triple (ν,g,λ) satisfies equations (10), (11), (14) and (17). It can be seen as a morphism from the factor system $(K_1,H_1,f^{t_1},\sigma^{t_1},F^{t_1})$ to $(K_2,H_2,f^{t_2},\sigma^{t_2},F^{t_2})$.

Let (λ_1,μ_1,ν_1) be a morphism from the group-gyro extension

$$E_1 \equiv \{e\} \longrightarrow H_1 \xrightarrow{i_1} G_1 \xrightarrow{\beta_1} K_1 \longrightarrow \{e\}$$

to the group-gyro extension

$$E_2 \equiv \{e\} \longrightarrow H_2 \xrightarrow{i_2} G_2 \xrightarrow{\beta_2} K_2 \longrightarrow \{e\}$$

and (λ_2,μ_2,ν_2) be another morphism from the group-gyro extension E_2 to the group-gyro extension E_3

$$E_3 \equiv \{e\} \longrightarrow H_3 \xrightarrow{i_3} G_3 \xrightarrow{\beta_3} K_3 \longrightarrow \{e\}\ .$$

Let $t_1,\ t_2,$ and t_3 be corresponding choices of sections. Then $\mu_1(t_1(x)) = g_1(x)\ t_2(\nu_1(x))$ and $\mu_2(t_2(x)) = g_2(x)t_3(\nu_2(x))$, where $g_1 : K_1 \longrightarrow H_2$ and $g_2 : K_2 \longrightarrow H_3$ are uniquely determined maps same as g in Equation (9). In turn, we have

$$\begin{aligned}\mu_2(\mu_1(t_1(x))) &= \lambda_2(g_1(x))\mu_2(t_2(\nu_1(x)))\\ &= \lambda_2(g_1(x)))g_2(\nu_1(x))t_3(\nu_2(\nu_1(x)))\\ &= \lambda_2(g_1(x))g_2(\nu_1(x))t_3(\nu_2(\nu_1(x)))\\ &= g_3(x)t_3(\nu_2(\nu_1(x))),\end{aligned}$$

where $g_3(x) = \lambda_2(g_1(x))g_2(\nu_1(x))$, for each $x \in K_1$. Thus the composition $(\lambda_2 \circ \lambda_1, \mu_2 \circ \mu_1, \nu_2 \circ \nu_1)$ induces the triple $(\nu_2 \circ \nu_1, g_3, \lambda_2 \circ \lambda_1)$.

Now we introduce the category GFAC whose objects are group-gyro factor system, and a morphism from $(K_1, H_1, f^1, \sigma^1, F^1)$ to $(K_2, H_2, f^2, \sigma^2, F^2)$ is a triple (ν, g, λ), where $\nu : K_1 \longrightarrow K_2$ is gyrogroup homomorphism, $\lambda : H_1 \longrightarrow H_2$ is group homomorphism, and $g : K_1 \longrightarrow H_2$ is a map such that

1. $g(e) = e$;
2. $\lambda(f^1(x, y))g(xy) = g(x)\sigma^2_{\nu(x)}(g(y))f^2(\nu(x), \nu(y))$;
3. $\lambda(\sigma^1_x(h)) = g(x)\sigma^2_{\nu(x)}(\lambda(h))g(x)^{-1}$;
4. $F^2_{(\nu(x),\nu(y))}(\lambda(l)g(z), \nu(z)) = \lambda(F^1_{(x,y)}(l, z))g(gyr[x, y](z))$.

The composition of two morphisms:

(ν_1, g_1, λ_1) from $(K_1, H_1, f^1, \sigma^1, F^1)$ to $(K_2, H_2, f^2, \sigma^2, F^2)$ and

(ν_2, g_2, λ_2) from $(K_2, H_2, f^2, \sigma^2, F^2)$ to $(K_3, H_3, f^3, \sigma^3, F^3)$

is the triple $(\nu_2\ o\ \nu_1, g_3, \lambda_2\ o\ \lambda_1)$, where $g_3(x) = \lambda_2(g_1(x))g_2(\nu_1(x))$ for each $x \in K_1$.

So, finally from the above discussion, we have the following Theorem:

Theorem 3.7. *There is an equivalence between the category* **GEXT** *of group-gyro extensions to the category* **GFAC** *of group-gyro factor systems.*

3.2. Dependency of an extension on sections

Let s and t be two sections of an extension

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}\ .$$

Then there exists an identity preserving map $g : K \to H$ such that $s(x) = g(x)t(x)$ (see Equation (9)). Hence by taking $\lambda = I_H$, $\mu = I_G$ and $\nu = I_K$ in equations (11), (14), (17), we have the following identities:

1. $f^s(x, y)g(xy) = g(x)\sigma^t_x(g(y))f^t(x, y)$.
2. $\sigma^s_x(h)) = g(x)\sigma^t_x(h)g(x)^{-1}$.
3. $F^t_{(x,y)}(lg(z), z) = F^s_{(x,y)}(l, z))g(gyr[x, y](z))$.

Definition 3.8. A group-gyro extension

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

of H by K is called an *split extension* if there is a section t which is a gyrogroup homomorphism. Such a section t is called a splitting of the extension. The corresponding factor system $(K, H, f^t, \sigma^t, F^t)$ is such that f^t is trivial in the sense that $f^t(x, y) = e$, for all $x, y \in K$.

Theorem 3.9. *Let H be a group, K be a gyrogroup and*

$$E \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$$

of H by K be an split group-gyro extension. Then $G \cong H \bowtie K$. Conversely, if $G = H \bowtie K$, then there is a natural projection p from G to K such that

$$E \equiv \{e\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow \{e\}$$

is a split group-gyro extension of H by K.

Proof. Since E is an split extension, we have a section t which is gyrogroup homomorphism. This implies that $f^t(x, y) = e$, for all $x, y \in K$. Therefore, by Remark 3.4 and Proposition 3.3, we have $\sigma_{x^{-1}} = \sigma_x^{-1}$ and $\sigma_{((xy)y)^{-1}} \circ \sigma_{xy} = \sigma_{(xy)^{-1}} \circ \sigma_x$, for all $x, y \in K$. The remaining part of the proof is easy to see. □

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