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Almost groupoids and morphisms

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Abstract. We introduce a new algebraic concept of morphism that generalizes the notion of group morphism, called almost groupoid morphism. The aim of this paper is to prove some main properties of almost groupoid morphisms that extend to almost groupoids the corresponding well-known results for groupoids.

1. Introduction

The notion of groupoid from an algebraic point of view was first introduced by H. Brandt in a 1926 paper [3]. The Brandt groupoids were generalized by C. Ehresmann in [5]. C. Ehresmann added further structures (topological and differentiable) to groupoids, thereby introducing them as a tool in topology and differential geometry. After the introduction of topological and differentiable groupoids by Ehresmann in the 1950's, they have been studied by many mathematicians with different approaches ([5]). The topological and differentiable groupoids endowed with supplementary structures: topological and Lie groupoids ([6, 10, 11]), play an essential role by their applications in analysis, differential geometry and physics ([13, 16]) and so on.

Another approach to the notion of a groupoid is that of a structured groupoid. The concept of a structured groupoid is obtained by adding another algebraic structure such that the composition of groupoid is compatible with the operation of the added algebraic structure. The most important types of structured groupoids are: group-groupoid, vector groupoid and vector space-groupoid ([4, 12, 15]).

The notion of almost groupoid from the algebraic point of view is defined in this paper. It is very important to note that the category of almost groupoids is a subcategory of groupoids category.

The paper is organized as follows. In Section 2 we recall the concept of groupoid. In Section 3 we give the definition of the almost groupoid from the algebraic point of view and recall some of their properties. We also present the notions of almost groupoid substructures such as: almost subgroupoid, wide and normal almost subgroupoid. In Section 4, some basic results on almost groupoid

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morphisms are presented. Furthermore, we prove correspondence theorems for almost groupoids.

2. Preliminaries on groupoids

This section deals with the groupoids in the sense of Brandt. For more details about groupoids, we refer the reader to ([1, 7, 8]).

A groupoid (in the sense of Brandt) can be thought of as a generalization of a group, that is, a set in which only certain multiplications are defined and it contains several units elements.

Definition 2.1. ([13]) A *groupoid* G *over* G_0 is a pair (G, G_0) of nonempty sets such that $G_0 \subseteq G$ endowed with two surjective maps $\alpha, \beta : G \to G_0$ (called the *source*) and (*target*), respectively), a partially binary operation $\mu : G_{(2)} \to G$, $(x, y) \rightarrowtail \mu(x, y) := xy$, $G_{(2)} = \{(x, y) \in G \times G | \beta(x) = \alpha(y)\}$ and an injective map $\iota : G \to G, x \rightarrowtail \iota(x) := x^{-1}$ satisfying the following properties:

(1) (*associativity*) $(xy)z = x(yz)$, in the sense that, if one side of the equation is defined so is the other one and then they are equal;

(2) (*identities*) $(\alpha(x), x),\ (x, \beta(x)) \in G_{(2)}$ and $\alpha(x)x = x\beta(x) = x$;

(3) (*inverses*) $(x^{-1}, x),\ (x, x^{-1}) \in G_{(2)}$ and $x^{-1}x = \beta(x),\ xx^{-1} = \alpha(x)$.

The elements of $G_{(2)}$ are called *composable pairs* of G. The element $\alpha(x)$ (resp. $\beta(x)$)] is the *left unit* (resp. *right unit*) of $x \in G$. The subset $G_0 = \alpha(G) = \beta(G)$ of G is called the *unit set* of G.

Remark 2.2. The basic concepts from groupoid theory has been defined and studied in a series of papers, for instance: subgroupoids, morphisms of groupoids, quotient groupoid, semidirect product and general constructions of Brandt groupoids ([1, 7, 14]). Many works are devoted to the study of important problems related to finite groupoids ([2, 9]).

3. Almost groupoids

Firstly, we will give the definition of an almost groupoid from a purely algebraic point of view.

Definition 3.1. *An almost groupoid G over G_0 (in the sense of Brandt) is a pair (G, G_0) of nonempty sets such that $G_0 \subseteq G$, endowed with a surjective map $\theta : G \to G_0$, a partially binary operation $m : G_{(2)} \to G$, $(x, y) \longmapsto m(x, y) := x \cdot y$, where $G_{(2)} := \{(x, y) \in G \times G | \theta(x) = \theta(y)\}$ and a map $\iota : G \to G$, $x \longmapsto \iota(x) := x^{-1}$, satisfying the following properties:*

$(AG1)$ *(associativity)I $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if one of two products $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ is defined, then the other product* *is also defined and they are equals;*

$(AG2)$ *(units): for each* $x \in G$, $(\theta(x), x)$, $(x, \theta(x)) \in G_{(2)}$ *and* $\theta(x) \cdot x = x \cdot \theta(x) = x$;

$(AG3)$ *(inverses): for each* $x \in G$, (x, x^{-1}), $(x^{-1}, x) \in G_{(2)}$ *and* $x \cdot x^{-1} = x^{-1} \cdot x = \theta(x)$.

If G is an almost groupoid over G_0, we will sometimes write $x \cdot y$ or xy for $m(x, y)$. Also, the set $G_{(2)}$ is called the *set of composable pairs* of G.

An almost groupoid G over G_0 with the *structure functions* θ (*units map*),m (*multiplication*) and ι (*inversion*), is denoted by $(G, \theta, m, \iota, G_0)$, (G, θ, m, G_0) or (G, G_0). G_0 is called the *units set* of G. Whenever we write a product in a given almost groupoid, we are assuming that it is defined.

In view of Definition 3.1, if $x, y, z \in G$, then:

(i) the product $x \cdot y \cdot z := (x \cdot y) \cdot z$ is true if $\theta(x \cdot y) = \theta(x)$ and also $\theta(y \cdot z) = \theta(y)$.

(ii) $\theta(x) \in G_0$ is *the unit* of x and $x^{-1} \in G$ is the *inverse* of x.

The basic properties of almost groupoids are given in Propositions 3.2 and 3.3.

Proposition 3.2. *If* $(G, \theta, m, \iota, G_0)$ *is an almost groupoid, then:*

(i) $\theta(u) = u$, $(\forall) u \in G_0$.

(ii) $u \cdot u = u$ *and* $\iota(u) = u$, $(\forall) u \in G_0$.

(iii) (*uniqueness of the units*):

  (a) $(x, y) \in G_{(2)}$ *and* $x \cdot y = y$ *then* $x = \theta(y)$;

  (b) $(x, y) \in G_{(2)}$ *and* $x \cdot y = x$ *then* $y = \theta(x)$.

(iv) (*uniqueness of the inverse*):

  if $(x, y) \in G_{(2)}$, $x \cdot y = \theta(x)$ *and* $y \cdot x = \theta(x)$ *then* $y = x^{-1}$.

(v) $\theta(x \cdot y) = \theta(x)$, $(\forall)(x, y) \in G_{(2)}$.

(vi) $\theta(x^{-1}) = \theta(x)$, $(\forall) x \in G$.

(vii) *If* $(x, y), (y, z) \in G_{(2)}$ *then* $(x \cdot y, z), (x, y \cdot z) \in G_{(2)}$ *and* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Proposition 3.3. *If* $(G, \theta, m, \iota, G_0)$ *is an almost groupoid, then:*

(i) *if* $(x, y) \in G_{(2)}$, *then* $(y^{-1}, x^{-1}) \in G_{(2)}$ *and* $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

(ii) $(x^{-1})^{-1} = x$, $(\forall) x \in G$.

(iii) *For each* $u \in G_0$, *the set* $G(u) := \theta^{-1}(u) = \{x \in G | \theta(x) = u\}$ *is a group, called the isotropy group of* G *at* u.

Corollary 3.4. *If* $(G, \theta, m, \iota, G_0)$ *is an almost groupoid, then:*

(i) $\theta(\theta(x)) = \theta(x)$, $(\forall) x \in G$.

(ii) *If* $x, y \in G$, *then:* $x \cdot y$ *is defined if and only if* $y \cdot x$ *is defined.*

(iii) *For* $a \in G$, *the products* $a^2 := a \cdot a$, $a^3 := a^2 \cdot a$ *are defined.*

Proof. (i). By Proposition 3.2 (iii) we have $\theta(\theta(x)) \overset{(AG3)}{=} \theta(x \cdot x^{-1}) = \theta(x)$.
$(ii) - (iii)$. These statements follow from definition and Proposition 3.2. □

Proposition 3.5. *Let* $(G, \theta, m, \iota, G_0)$ *be an almost groupoid. Then*

$$\theta \circ \iota = \theta \quad \text{and} \quad \iota \circ \iota = Id_G.$$

Proof. The first relation is a consequence of equality (iv) in Proposition 3.2. Indeed, we have $\theta(x^{-1}) = \theta(\iota(x)) = (\theta \circ \iota)(x)$. But, $\theta(x^{-1}) = \theta(x)$. Hence, $(\theta \circ \iota)(x) = \theta(x), (\forall)x \in G$. Also, we have $(\iota \circ \iota)(x) = \iota(\iota(x)) = \iota(x^{-1}) = (x^{-1})^{-1} = x$. Then $\iota \circ \iota = Id_G$, since $(\iota \circ \iota)(x) = Id_G(x), (\forall)x \in G$. □

Example 3.6. (i). A group G having e as unity, is an almost groupoid over $\{e\}$ with respect to structure functions: $\theta(x) := e, (\forall)x \in G$; $G_{(2)} = G \times G$, $m(x,y) := xy$, $(\forall)x, y \in G$ and $\iota : G \to G$, $\iota(x) := x^{-1}, (\forall)x \in G$.
(ii). A nonempty set G_0 may be regarded to be an almost groupoid over G_0, called the *null almost groupoid* associated to G_0. For this, we take $\theta = \iota = Id_{G_0}$ and $u \cdot u = u$, $(\forall)u \in G_0$.

Remark 3.7. Each almost groupoid $(G, \theta, m, \iota, G_0)$ is a groupoid for which the structure functions α (source) and β (target) are equal to θ. Clearly, it is not true in general that every groupoid is an almost groupoid.

Example 3.8. For $a, k \in \mathbb{R}$ consider the matrix $A(a,k) = \begin{pmatrix} a & ka \\ 0 & 1 \end{pmatrix}$. Let $G = \{ A(a,k) \mid a, k \in \mathbb{R}, a \neq 0\}$ and $G_0 = \{ A(1,k) \mid k \in \mathbb{R}\}$. Then $(G, \theta, \odot, \iota, G_0)$ is an almost groupoid.

The structure functions θ and ι are defined by:

$$\theta : G \to G_0, A(a,k) \rightarrowtail \theta(A(a,k)) := A(1,k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix};$$

$$\iota : G \to G, A(a,k) \rightarrowtail \iota(A(a,k)) := A(a^{-1},k) = \begin{pmatrix} a^{-1} & k \\ 0 & 1 \end{pmatrix}.$$

The set of composable pairs $G_{(2)}$ is given by:

$$\begin{aligned} G_{(2)} &= \{(A(a_1,k_1), A(a_2,k_2)) \in G \times G \mid \theta(A(a_1,k_1)) = \theta(A(a_2,k_2))\} \\ &= \{(A(a_1,k_1), A(a_2,k_2)) \in G \times G \mid k_1 = k_2\}. \end{aligned}$$

The product $\odot$ is defined by: $A(a_1,k_1) \odot A(a_2,k_1) := A(a_1a_2,k_1)$, that is:

$$\begin{pmatrix} a_1 & k_1a_1 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} a_2 & k_1a_2 \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} a_1a_2 & k_1a_1a_2 \\ 0 & 1 \end{pmatrix}, (\forall)\ k_1 \in \mathbb{R}, a_1a_2 \neq 0.$$

It is easy to verify that the conditions $(AG1)-(AG3)$ of Definition 3.1 hold. For this, we consider $A, B, C \in G$, where $A = A(a,k), B = A(a_1,k_1), C = A(a_2,k_2)$. The product $A \odot B \odot C$ is defined if and only if $k = k_1 = k_2$. We have:

$(AG1)$ (*associativity*) $A \odot B) \odot C = (A(a,k) \odot A(a_1,k)) \odot A(a_2,k) = A(aa_1,k) \odot A(a_2,k) = A(aa_1a_2,k) = A \odot (B \odot C)$;

$(AG2)$ (*units*) $A \odot \theta(A) = A(a,k) \odot \theta(A(a,k)) = A(a,k) \odot A(1,k) =$

$A(a,k) = A = \theta(A) \odot A;$

$(AG3)$ $(inverses)$ $A \odot \iota(A) = A(a,k) \odot \iota(A(a,k)) = A(a,k)) \odot A(a^{-1},k) = A(1,k) = \theta(A(a,k)) = \theta(A) = \iota(A) \odot A.$

Let us we present some substructures in almost groupoids and prove several properties of them which generalize well-known results in group theory.

Definition 3.9. Let $(G, \theta, m, \iota, G_0)$ be an almost groupoid. A pair of nonempty subsets (H, H_0) where $H \subseteq G$ and $H_0 \subseteq G_0$, is called *almost subgroupoid* of G, if:

(i) $\theta(H) = H_0$;

(ii) H is closed under multiplication and inversion, that is: if $x, y \in H$ and $(x,y) \in G_{(2)}$ then $x \cdot y \in H$;

(iii) if $x \in H$ then $x^{-1} \in H$.

By Definition 3.9, if (H, H_0) is an almost sugroupoid of $(G, \theta, m, \iota, G_0)$, then (H, H_0) endowed with the restrictions of θ and ι at H and the restriction of multiplication m at $H_{(2)} = (H \times H) \cap G_{(2)}$, is an almost groupoid over H_0.

Definition 3.10. Let (H, H_0) be an almost subgroupoid of (G, G_0).

(i) (H, H_0) is said to be a *wide almost subgroupoid* of (G, G_0), if $H_0 = G_0$, that is H and G have the same units.

(ii) A wide almost subgrupoid (H, H_0) of (G, G_0), is called *normal almost subgroupoid,* if for all $g \in G$ and for all $h \in H$ such that $g \cdot h \cdot g^{-1}$ is defined, we have $g \cdot h \cdot g^{-1} \in H$.

Example 3.11. (1). If (G, G_0) is an almost groupoid, then G_0 is a wide almost subgroupoid of G. We shall check that the conditions in Definition 3.9 hold.

(i). Clearly, $\theta(G_0) = G_0$.

(ii). Let $u, v \in G_0$ such that $(u,v) \in G_{(2)}$. Then $\theta(u) = \theta(v)$ and it follows that $u = v$, since $u, v \in G_0$. Therefore, $u \cdot v$ is defined if and only if $u = v$. By Proposition 3.2(i) and (ii), we have $u \cdot u = u$. Hence $u \cdot u \in G_0$. Also, from $u \cdot u = u$ implies that $u^{-1} = u$ and $u^{-1} \in G_0$. Therefore, (iii) holds.

(2). G_0 is a normal almost subgroupoid of the almost groupoid (G, G_0), called the *null almost subgroupoid* of G. For this, we consider $g \in G$ and $u \in G_0$ such that the product $g \cdot u \cdot g^{-1}$ is defined in G. Then $\theta(g) = \theta(u) = u$, since $u \in G_0$. But, $\theta(g \cdot u \cdot g^{-1}) = \theta(g) = u$. Hence, $g \cdot u \cdot g^{-1} \in G_0$.

Proposition 3.12. *Let (G, G_0) be an almost groupoid. Then:*

(i) *The isotropy group $G(u), u \in G_0$ is an almost subgroupoid of G.*

(ii) *$Is(G) = \cup_{u \in G_0} G(u) \subset G$ is a normal almost subgroupoid of G.*

Proof. (i). By Example 3.6, every group is an almost groupoid.

(ii). Clearly, $\theta(Is(G)) = G_0$. We observe that $Is(G)$ is a union disjoint of the family $\{(G(u), \{u\})\}_{u \in G_0}$ of almost subgroupoids of G. According to Corollary 3.4, we get that $Is(G)$ is wide almost subgroupoid over G_0.

Consider now the elements $g \in G$ and $a \in Is(G)$ such that $(\exists) g \cdot a \cdot g^{-1}$ in G. Then $(\exists) u \in G_0$ with $\theta(a) = u$. Hence, $g \cdot a \cdot g^{-1} \in Is(G)$ since $\theta(g \cdot a \cdot g^{-1}) = \theta(g) = \theta(a) = u$. Therefore, $Is(G)$ is a normal almost subgroupoid of G. □

The almost groupoid $(Is(G), G_0)$ is called *isotropy almost subgroupoid* of G.

The following proposition extends to the context of almost groupoids several elementary results for groups.

Proposition 3.13. *Let* $(H_i, H_{0,i})_{i \in I}$ *be a family of almost subgroupoids of almost groupoid* (G, G_0) *such that* $\cap_{i \in I} H_{0,i} \neq \emptyset$. *Then:*

(*i*) $(\cap_{i\in I} H_i, \cap_{i\in I} H_{0,i})$ *is an almost subgroupoid of* (G, G_0).

(*ii*) *If* $(H_i, H_{0,i})$ *is a wide almost subgroupoid for each* $i \in I$, *then* $(\cap_{i\in I} H_i, H_{0,i})$ *is a wide almost subgroupoid of* (G, G_0).

(*iii*) *If* $(H_i, H_{0,i})$ *is a normal almost subgroupoid for each* $i \in I$, *then* $(\cap_{i\in I} H_i, H_{0,i})$ *is a normal almost subgroupoid of* (G, G_0).

Proof. To prove this proposition we introduce the notations: $H := \cap_{i\in I} H_i$ and $H_0 = \cap_{i\in I} H_{0,i}$.

(*i*). We prove that (H, H_0) satisfies the conditions of Definition 3.9.

• $\theta(H) = H_0$. Indeed, $\theta(H) = \theta(\cap_{i\in I} H_i) = \cap_{i\in I} \theta(H_i) = \cap_{i\in I} H_{0,i} = H_0$, since $\theta(H_i) = H_{0,i}$ for each $i \in I$.

• Let $x, y \in H$ such that $(x, y) \in G_{(2)}$. From $x, y \in \cap_{i\in I} H_i$ it follows that $x, y \in H_i$ for each $i \in I$. Then $x \cdot y \in H_i$ for each $i \in I$, since H_i is an almost groupoid. Hence $x \cdot y \in \cap_{i\in I} H_i = H$.

• Let $x \in H$. Then $x \in H_i$ for each $i \in I$. It follows that $x^{-1} \in H_i$ for each $i \in I$, since H_i is an almost groupoid. Thus, $x^{-1} \in \cap_{i\in I} H_i = H$. Therefore, the conditions (*i*) and (*ii*) of Definition 3.9 hold.

(*ii*). In view of (*i*) follows that (H, H_0) is an almost subgroupoid of (G, G_0). We have $H_0 = \cap_{i\in I} H_{0,i} = G_0$ since $H_{0,i} = G_0$ for each $i \in I$. Therefore, (H, H_0) is a wide almost subgroupoid of (G, G_0).

(*iii*). In view of (*ii*) we have that (H, H_0) is a wide almost subgroupoid of (G, G_0). It remains to prove that the condition (*ii*) from Definition 3.10 holds.

Let $h \in H$ and $g \in G$ such that $\theta(h) = \theta(g)$ and prove that $g \cdot h \cdot g^{-1} \in H$. Indeed, we have $h \in H_i$ for each $i \in I$. Then the product $g \cdot h \cdot g^{-1}$ is defined in G. Taking account that H_i is normal in G for each $i \in I$ implies that $g \cdot h \cdot g^{-1} \in H_i$ for each $i \in I$. Hence $g \cdot h \cdot g^{-1} \in H$. Therefore H is normal in G. □

Example 3.14. Let $G = \mathbb{R}^* \times \mathbb{R} = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ and $G_0 = \{1\} \times \mathbb{R}$. Then $(G, \theta, \otimes, \iota, G_0)$ is an almost groupoid over G_0, where the set of composable pairs $G_{(2)}$ and its structure functions and are given by:

$\theta : G \rightarrow G_0, (a, b) \rightarrowtail \theta(a, b) := (1, \frac{b}{a})$;

$G_{(2)} = \{((a, b), (c, d)) \in G^2 \mid \theta(a, b) = \theta(c, d)\} = \{((a, b), (c, d)) \in G^2 \mid bc = ad\}$;

$(a, b) \otimes (c, d) := (ac, bc)$ and $\iota(a, b) := (\frac{1}{a}, \frac{b}{a^2}), (\forall)\ a, b, c, d \in \mathbb{R},\ a \neq 0$.

It is easy to verify that the conditions of Definition 3.1. For this, we consider the elements $x, y, z \in G$, where $x = (a, b), y = (a_1, b_1), z = (a_2, b_2)$. The product $x \otimes y \otimes z$ is defined if and only if $\frac{b}{a} = \frac{b_1}{a_1} = \frac{b_2}{a_2}$. We have:

$(AG1)$ $(associativity)$ $(x \otimes y) \otimes z = ((a, b) \otimes (a_1, b_1)) \otimes (a_2, b_2) = (aa_1, ba_1) \otimes (a_2, b_2) = (aa_1a_2, ba_1a_2) = (x \otimes y) \otimes z$;

$(AG2)$ $(units)$ $x \otimes \theta(x) = (a, b) \otimes \theta(a, b) = (a, b) \otimes (1, \frac{b}{a}) = (a, b) = x = \theta(x) \otimes x$;

$(AG3)$ $(inverse)$ $x \otimes \iota(x) = (a, b) \otimes \iota(a, b) = (a, b) \otimes (\frac{1}{a}, \frac{b}{a^2}) = (1, \frac{b}{a}) = \theta(a, b) = x = \iota(x) \otimes x$, $(\forall) x \in G$.

4. Almost groupoid morphisms

Let us we present the notion of *almost groupoid morphism* and prove several properties of them which generalize well-known results in group theory.

Definition 4.1. Let $(G, \theta, m, \iota, G_0)$ and $(G', \theta', m', \iota', G'_0)$ be two almost groupoids. A *morphism of almost groupoids* or *almost groupoid morphism* from (G, G_0) into (G', G'_0) is a pair (f, f_0) of maps $(f, f_0) : (G, G_0) \to (G', G'_0)$ such that the following conditions hold:

(i) $f(m(x, y)) = m'(f(x), f(y))$ for all $(x, y) \in G_{(2)}$;
(ii) $\theta' \circ f = f_0 \circ \theta$.

An almost groupoid morphism $(f, f_0) : (G, G_0) \to (G', G'_0)$ such that f and f_0 are bijective maps, is called *isomorphism of almost groupoids.*

Note that if $(f, f_0) : (G, G_0) \to (G', G'_0)$ is an almost groupoid morphism then f commutes with the structure functions ι and ι', that is $f \circ \iota = \iota' \circ f$.

Proposition 4.2. *If* $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ *is an almost groupoid morphism, then:*

(i) $f(u) \in G'_0$, *for all* $u \in G_0$.
(ii) $f(x^{-1}) = (f(x))^{-1}$, *for all* $x \in G$.

Proof. (i). Let $u \in G_0$. Then $\theta(u) = u$. From $(\theta(u), u) \in G_{(2)}$ it follows that $(f(\theta(u)), f(u)) \in G'_{(2)}$ and $f(\theta(u)) \cdot f(u) = f(\theta(u) \cdot u) = f(u)$, since f is an almost groupoid morphism. But, $\theta'(f(u)) \cdot f(u) = f(u)$. From $\theta'(f(u)) \cdot f(u) = f(u)$ and $f(\theta(u)) \cdot f(u) = f(u)$ it follows $\theta'(f(u)) = f(\theta(u))$. Hence $\theta'(f(u)) = f(u)$. Therefore, $f(u) \in G'_0$.

(ii). Let $x \in G$. According to (AG3), we have $x \cdot x^{-1} = x^{-1} \cdot x = \theta(x)$. Then $f(x \cdot x^{-1}) = f(\theta(x))$. It follows $f(x) \cdot f(x^{-1}) = f(\theta(x))$, since f is an almost groupoid morphism. Applying (i), since $\theta(x) \in G_0$, we have $f(\theta(x)) \in G'_0$ and $f(\theta(x)) = f_0(\theta(x))$. From the relations $f(x) \cdot f(x^{-1}) = f(\theta(x))$ and $f(\theta(x)) = f_0(\theta(x))$ it is obtained that $f(x) \cdot f(x^{-1}) = f_0(\theta(x))$. Taking account that $f_0 \circ \theta = \theta' \circ f$, we have $f(x) \cdot f(x^{-1}) = \theta'(f(x))$, whence it follows that $f(x^{-1})$ is the inverse of $f(x)$ in G', that is, $f(x^{-1}) = (f(x))^{-1}$. $\square$

Proposition 4.3. *The pair* $(f, f_0) : (G, G_0) \to (G', G'_0)$, *where* $f : G \to G'$, $f_0 : G_0 \to G'_0$ *is an almost groupoid morphism if and only if the following conditions are verified for all* $(x, y) \in G_{(2)}$:
(i) $(f(x), f(y)) \in G'_{(2)}$;
(ii) $f(m(x, y)) = m'(f(x), f(y))$.

Proof. Firstly, we suppose that (f, f_0) is an almost groupoid morphism. Then, the conditions (ii) and (iii) from Definition 4.1 are satisfied. The condition (ii) from Proposition 4.3 is clearly verified. It remains to prove that the condition (i) of Proposition 4.3 holds. For this, let $(x, y) \in G_{(2)}$. Then $\theta(x) = \theta(y)$. We have $f_0(\theta(x)) = f_0(\theta(y)$, that is, $(f_0 \circ \theta)(x) = (f_0 \circ \theta)$. Applying the condition (iii) from Definition 4.1, it follows that $(\theta' \circ f)(x) = (\theta' \circ f)(y)$. Thus, $\theta'(f(x)) = \theta'(f(y))$, and so $(f(x), f(y)) \in G'_{(2)}$. Hence, the condition (i) of Proposition 4.3 is verified.

Conversely, let $f : G \to G'$ verifying the conditions (i) and (ii) from Proposition 4.3. The conditions (ii) of Definition 4.1 is verified. It remains to prove that $\theta' \circ f = f_0 \circ \theta$. For this, we define the map $f_0 : G_0 \to G'_0$ by $f_0(u) := \theta'(f(u))$, for all $u \in G_0$. For all $x \in G$ we have $(x, \theta(x)) \in G_{(2)}$. By hypothesis we have $(f(x), f(\theta(x))) \in G'_{(2)}$ and $f(x) \cdot f(\theta(x)) = f(x \cdot \theta(x)) = f(x)$. Since $f(x) \cdot \theta'(f(x)) = f(x)$, then $f(x) \cdot f(\theta(x)) = f(x) \cdot \theta'(f(x))$ and so $f(\theta(x)) = \theta'(f(x))$. But $f(\theta(x)) = f_0(\theta(x))$, since $\theta(x) \in G_0$. Hence $\theta'(f(x)) = f_0(\theta(x))$ for all $x \in G$, that is, $\theta' \circ f = f_0 \circ \theta$. Therefore, the condition (iii) from Definition 4.1 holds. □

Proposition 4.4. *An almost groupoid morphism* (f, f_0) *is linked with the structure functions by the following commutative diagrams:*

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ \theta \downarrow & & \downarrow \theta' \\ G_0 & \xrightarrow{f_0} & G'_0 \end{array} \qquad \begin{array}{ccc} G_{(2)} & \xrightarrow{f \times f} & G'_{(2)} \\ m \downarrow & & \downarrow m' \\ G & \xrightarrow{f} & G' \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{f} & G' \\ \iota \downarrow & & \downarrow \iota' \\ G & \xrightarrow{f} & G' \end{array}$$

where $(f \times f)(x, y) := (f(x), f(y))$, $\forall\ (x, y) \in G \times G$. *More precisely:*

$$\theta' \circ f = f_0 \circ \theta, \quad m' \circ (f \times f) = f \circ m, \quad \iota' \circ f = f \circ \iota \tag{1}$$

Proof. We apply the Propositions 4.2 and 4.3. □

The *kernel* of the almost groupoid morphism $(f, f_0) : (G, G_0) \to (G', G'_0)$ is defined by: $Ker(f) := \{x \in G \mid f(x) \in G'_0\} \subseteq G.$

Example 4.5. We consider the almost groupoid $(G, \theta, \otimes, \iota, G_0)$ given in Example 3.6, where $G = \mathbb{R}^* \times \mathbb{R} = \{(a, b) \in \mathbb{R}^2 | a \neq 0\}$ and $G_0 = \{(1, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$. Also, let the multiplicative group $G' = \mathbb{R}^*$. This group is regarded as groupoid over $G'_0 = \{1\}$ with $\theta' : G' \to G'_0$ given by $\theta'(a) = 1, (\forall) a \in \mathbb{R}^*$ (see Example 3.6(i)).

• Define the function $\varphi : G \to G'$ by $\varphi(a,b) := a$, for all $(a,b) \in G$. Then the function $\varphi_0 : G_0 \to G'_0$ is given by $\varphi_0(1,b) = 1,\ (\forall) b \in \mathbb{R}$.

The pair $(\varphi, \varphi_0) : (G, G_0) \to (G', G'_0)$ is an almost groupoid morphism. It is easy to verify that the conditions (ii) and (iii) of Definition 4.1(i) hold.

For this, we consider $x, y \in G$ such that the product $x \otimes y$ is defined. Then $x = (a,b)$ and $y = (c, \frac{bc}{a})$ with $a, c \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Then $\varphi(x \otimes y) = \varphi(a,b) \otimes (c, \frac{bc}{a})) = \varphi(ac, bc) = ac$ and $\varphi(x) \cdot \varphi(y) = \varphi(a,b) \cdot \varphi(c, \frac{bc}{a}) = ac$, Then $\varphi(x \otimes y) = \varphi(x) \cdot \varphi(y)$.

Let $x = (a,b) \in G$. We have $(\theta' \circ \varphi)(x) = \theta'(\varphi(a,b)) = \theta'(a) = 1$ and $(\varphi_0 \circ \theta)(x) = \varphi_0(\theta(a,b)) = \varphi_0(1, \frac{b}{a}) = 1$. Hence, $\theta' \circ \varphi = \varphi_0 \circ \theta$.

• We have $Ker(\varphi) = G_0$. indeed, $Ker(\varphi) = \{(a,b) \in G | \varphi(a,b) \in G'_0\} = \{(1,b) | b \in \mathbb{R}\} = G_0$, since $G'_0 = \{1\}$ and $\varphi(a,b) = 1$ if and only if $a = 1$.

Also, the almost groupoid morphism φ is not injective, since there exists $x_1, x_2 \in G$ such that $\varphi(x_1) = \varphi(x_2)$ and $x_1 \neq x_2$. For example, for $x_1 = (a_1, b_1)$ and $x_2 = (a_1, b_2)$ with $b_1 \neq b_2$ we have $\varphi(a_1, b_1) = a_1 = \varphi(a_1, b_2)$.

Proposition 4.6. *Let $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ be an almost groupoid morphism. Then:*

(i) If (H', H'_0) is an almost subgroupoid of (G', G'_0), then $(f^{-1}(H'), f_0^{-1}(H'_0))$ is an almost subgroupoid of (G, G_0).

(ii) If (N', G'_0) is a normal almost subgroupoid of (G', G'_0), then $(f^{-1}(N'), G_0)$ is a normal almost subgroupoid of (G, G_0) such that $Ker(f) \subseteq f^{-1}(N')$.

Proof. (i). We prove that $(f^{-1}(H'), f_0^{-1}(H'_0))$ satisfies the conditions of Definition 3.9.

$\theta(f^{-1}(H')) = f_0^{-1}(H'_0)$. Indeed, if $u \in \theta(f^{-1}(H'))$ it follows that $u = \theta(x)$, with $x \in f^{-1}(H')$. Then $f_0(u) = f_0(\theta(x)) = \theta'(f(x)) \in H'_0$, since $f(x) \in H'$ and $\theta'(H') = H'_0$. Hence, $u \in f_0^{-1}(H'_0))$ and $\theta(f^{-1}(H')) \subseteq f_0^{-1})(H'_0)$. Clearly, $f_0^{-1}(H'_0) \subseteq \theta(f^{-1}(H'))$.

Let $x, y \in f^{-1}(H')$ such that $x \cdot y$ is defined, i.e. $\theta(x) = \theta(y)$. It follows that $f(x), f(y) \in H'$ and $\theta'(f(x)) = f_0(\theta(x)) = f_0(\theta(y)) = \theta'(f(y))$. Thus, $f(x) \cdot f(y)$ is defined in G' and $f(x) \cdot f(y) \in H'$ since H' is an almost subgroupoid. Then, $f(x \cdot y) = f(x) \cdot f(y) \in H'$, that is, $x \cdot y \in f^{-1}(H')$. Therefore, the conditions (i) and (ii) of Definition 3.9 hold.

(ii). In view of (i) follows that $(f^{-1}(N'), G_0)$ is an almost subgroupoid of (G, G_0).

It is easy to prove that $\theta(f^{-1}(N')) = G_0$, since $\theta'(N') = G'_0$.

Let $h \in f^{-1}(N')$ and $g \in G$ such that $\theta(h) = \theta(g)$ and prove that $g \cdot h \cdot g^{-1} \in f^{-1}(N')$. Indeed, we have $f(h) \in N'$ and $\theta'(f(g)) = f_0(\theta(g)) = f_0(\theta(h)) = \theta'(f(h))$. Then the product $f(g) \cdot f(h) \cdot (f(h))^{-1}$ is defined in G'. Taking account that N' is normal in G' implies that $f(g) \cdot f(h) \cdot (f(g))^{-1} \in N'$. Then $f(g) \cdot f(h) \cdot (f(g))^{-1} = fg \cdot h \cdot g^{-1}) \in N'$. Hence $g \cdot h \cdot g^{-1} \in f^{-1}(N')$. Therefore $f^{-1}(N')$ is normal in G.

We have $Ker(f) \subseteq f^{-1}(N')$. Indeed, for $x \in Ker(f)$, $(\exists)u' \in N'_0$ such that $f(x) = \theta'(u')$. By assertion (i) of Proposition 3.2, follows $f(x) = u' \in N'_0 \subseteq N'$, i.e. $x \in f^{-1}(N')$. □

According to Proposition 4.6, if $(f, f_0) : (G, G_0) \to (G', G'_0)$ is an almost groupoid morphism, then the *inverse image* of an almost subgroupoid (resp., normal) of G' by (f, f_0) is an almost subgroupoid (resp., normal) of G.

Proposition 4.7. *Let* (f, f_0) *be an almost groupoid morphism.*

(i) *The kernel of* (f, f_0) *is a normal almost subgroupoid of* G.

(ii) *If* (f, f_0) *is an injective almost groupoid morphism, then* $Ker(f) = G_0$.

Proof. (i). G'_0 is a normal almost subgroupoid of G' (see Example 3.11(ii)). We observe that $ker(f) = f^{-1}(G'_0)$. Taking $N' = G'_0$ in Proposition 4.6(ii), we deduce $Ker(f)$ is a normal almost subgroupoid of G.

(ii). We have $G_0 \subseteq Ker(f)$. Indeed, if $u \in G_0$ then $f(u) \in G'_0$ (according to Proposition 4.2(i)) and $u \in Ker(f)$.
Let now $x, y \in Ker(f)$ such that $x \cdot y$ is defined. Then $f(x), f(y) \in G'_0$ and $\theta(x) = \theta(y)$. We have $f(x) = \theta'(f(x)) = f_0(\theta(x)) = f_0(\theta(y)) = \theta'(f(y)) = f(y)$ and how f is injective, we deduce $x = y$. Therefore, the only products of $Ker(f)$ are those of the form $x \cdot x$. Suppose that $x \cdot x = z$, for $x \in Ker(f)$. Then $f(z) = f(x \cdot x) = f(x) \cdot f(x) = f(x)$. From $f(z) = f(x)$ implies $z = x$, because f is injective. So $x \cdot x = x$. Therefore, $Ker(f) \subseteq G_0$. Hence, $Ker(f) = G_0$. □

Next, we introduce a special type of almost groupoid morphism, called the *strong almost groupoid morphism.* Using this new type of morphism, we prove the correspondence theorem for almost groupoids.

Definition 4.8. A *strong almost groupoid morphism* is a morphism of almost groupoids $(f, f_0) : (G, G_0) \to (G', G'_0)$ satisfying the following condition:

for all $x, y \in G$ *such that* $f(x) \cdot f(y)$ *is defined in* G' *implies* $(x, y) \in G_{(2)}$.

Theorem 4.9. *Let* $(f, f_0) : (G, G_0) \to (G', G'_0)$ *be a strong almost groupoid morphism. The morphism* (f, f_0) *is injective if and only if* $Ker(f) = G_0$.

Proof. By Proposition 4.7(ii), if (f, f_0) is injective then $Ker(f) = G_0$. Conversely, suppose that $Ker(f) = G_0$. Let now $x, y \in G$ such that $f(x) = f(y)$. Then $\theta'(f(x)) = \theta'(f(y))$ and the product $f(x) \cdot f(y)$ is defined. Taking account that (f, f_0) is strong, it follows $(x, y) \in G_{(2)}$, that is, $\theta(x) = \theta(y)$. By Proposition 3.4(v), $x \cdot y^{-1}$ and $f(x) \cdot (f(y))^{-1}$ are defined. If $f(x) = f(y)$ then $f(x) \cdot (f(y))^{-1} = f(y) \cdot (f(y))^{-1}$. As, $f(x) \cdot f(y^{-1}) = \theta'(f(y))$, then $f(x \cdot y^{-1}) \in G'_0$. From $f(x \cdot y^{-1}) \in G'_0$ follows $x \cdot y^{-1} \in Ker(f) = G_0$ and $(\exists)z \in G$ such that $x \cdot y^{-1} = \theta(z)$. Hence, $\theta(x \cdot y^{-1}) = \theta(\theta(z))$ implies $\theta(x) = \theta(z)$. We have $x \cdot y^{-1} = \theta(y)$ and $x \cdot y^{-1} = \theta(y)$ then $x \cdot y^{-1} \cdot y = \theta(y) \cdot y$. That is $x \cdot \theta(y) = y$. Hence $x \cdot \theta(x) = y$ and thus $x = y$. Therefore, the function f is injective. □

Proposition 4.10. *Let* $(f, f_0) : (G, G_0) \to (G', G'_0)$ *be a strong almost groupoid morphism. Then:*

(i) *If* (H, H_0) *is an almost subgroupoid of* (G, G_0), *then* $(f(H), f_0(H_0))$ *is an alamost subgroupoid of* (G', G'_0). *In particular,* $(Im(f) = f(G), f_0(G_0))$ *is an almost subroupoid of* (G', G'_0).

(ii) *If the function* f *is surjective and* (N, G_0) *is a normal almost subgroupoid of* (G, G_0), *then* $(f(N), G'_0)$ *is a normal almost subgroupoid of* (G', G'_0).

Proof. (i). $(f(H), f_0(H_0))$ satisfies the conditions of Definition 3.10. In fact:

$\theta'(f(H)) \subseteq f_0(H_0)$. Indeed, for any $u' \in \theta'(f(H))$, $(\exists) y' \in f(H)$ such that $u' = \theta'(y')$. For $y' \in f(H)$, $(\exists) y \in H$ such that $f(y) = y'$. Then $u' = \theta'(y') = \theta'(f(y)) = f_0(\theta(y)) \in f_0(H_0)$, since $\theta(y) \in H_0$. Also, $f_0(H_0) \subseteq \theta'(f(H))$. For this, consider $z' \in f_0(H_0)$. Then $(\exists) u \in H_0$ such that $z' = f_0(u)$ and $u = \theta(x)$ for some $x \in H$. We have $z' = f_0(u) = f_0(\theta(x)) = \theta'(f(x)) \in \theta'(f(H))$, since $x \in H$. Hence, $\theta'(f(H)) = f_0(H_0)$.

Let $x', y' \in f(H)$ such that $x' \cdot y'$ is defined. We prove that $x' \cdot y' \in f(H)$. Indeed, $x' = f(x), y' = f(y)$ with $x, y \in H$. Since $x' \cdot y'$ is defined it implies that $(f(x), f(y)) \in G'_{(2)}$ and follows that $(\exists) x \cdot y \in G$, since f is strong. Then $x \cdot y \in H$, since H is an almost subgroupoid. We have $x' \cdot y' = f(x) \cdot f(y) = f(x \cdot y) \in f(H)$.

For any $x' \in f(H)$, we prove that $(x')^{-1} \in f(H)$. For this, let $x \in H$ such that $x' = f(x)$. Then $(x')^{-1} = (f(x))^{-1} = f(x^{-1}) \in f(H)$, since $x^{-1} \in H$. Hence, $(f(H), f_0(H_0))$ is an almost subgroupoid.

(ii). In view of (i) follows that $(f(N), G'_0)$ is an almost subgroupoid of (G', G'_0).

Since $\theta(N) = G_0$ it follows $\theta'(f(N)) = G'_0$.

Let $h' \in f(N)$ and $g' \in G'$ such that the product $g' \cdot h' \cdot (g')^{-1}$ is defined in G'. We prove that $g' \cdot h' \cdot (g')^{-1} \in f(N)$.

Indeed, for $h' \in f(N)$, $(\exists) h \in N$ with $f(h) = h'$. Also, for $g' \in G'$, $(\exists) g \in G$ with $f(g) = g'$, since f is surjective. We have $g' \cdot h' \cdot (g')^{-1} = f(g) \cdot f(h) \cdot (f(g))^{-1} = f(g) \cdot f(h) \cdot f(g^{-1})$. Then $f(g) \cdot f(h)$ and $f(h) \cdot f(g^{-1})$ are defined in G'. Since f is strong, implies that $(\exists) g \cdot h$ and $h \cdot g^{-1}$. Then $g \cdot h \cdot g^{-1}$ is defined in G and $g \cdot h \cdot g^{-1} \in N$, since N is normal in G. Then $g' \cdot h' \cdot (g')^{-1} = f(g) \cdot f(h) \cdot f(g^{-1}) = f(g \cdot h \cdot g^{-1}) \in f(N)$. Therefore, $f(N)$ is normal. □

According to Proposition 4.10, if $(f, f_0) : (G, G_0) \to (G', G'_0)$ is a strong almost groupoid morphism, then the *direct image* of an almost subgroupoid (resp., normal) of G by (f, f_0) is an almost subgroupoid (resp., normal) of G', if f is surjective.

Let $(f, f_0) : (G, G_0) \to (G', G'_0)$ be an almost groupoid morphism.

(i) Denote by $\mathcal{S}(G)$ (resp. $\mathcal{N}(G)$) the set of almost subgroupoids (resp., the set of normal almost subgroupoids) of G which contains the kernel of f, that is :

$$\mathcal{S}(G) := \{H \mid H \text{ is an } \textit{almost subgroupoid} \text{ of } G \text{ and } ker(f) \subseteq H\}, \qquad (2)$$

$$\mathcal{N}(G) := \{N \mid N \text{ is a } \textit{normal almost subgroupoid} \text{ of } G \text{ and } ker(f) \subseteq N\}. \quad (3)$$

(ii) Denote by $\bar{\mathcal{S}}(G)$ (resp. $\bar{\mathcal{N}}(G)$) the set of almost subgroupoids (resp., the set of normal almost sugroupoids) of G', that is :

$$\bar{\mathcal{S}}(G') := \{H' \mid H' \text{ is an } \textit{almost subgroupoid} \text{ of } G'\}, \quad (4)$$

$$\bar{\mathcal{N}}(G') := \{N' \mid N' \text{ is a } \textit{normal almost subgroupoid} \text{ of } G'\}. \quad (5)$$

Theorem 4.11. (The correspondence theorem for almost subgroupoids) *For any surjective strong almost groupoid morphism* $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ *there exists a bijection (one-to-one correspondence) between the set* $\mathcal{S}(G)$ *of almost subgroupoids of* G *and the set* $\bar{\mathcal{S}}(G')$ *of almost sugroupoids of* G'.

Proof. We define $\varphi : \mathcal{S}(G) \to \bar{\mathcal{S}}(G')$ and $\psi : \bar{\mathcal{S}}(G') \to \mathcal{S}(G)$, given by:

$$\varphi(H) := f(H), \ (\forall) H \in \mathcal{S}(G) \quad \text{and} \quad \psi(H') := f^{-1}(H'), \ (\forall)\ H' \in \bar{\mathcal{S}}(G'). \quad (6)$$

By Proposition 4.10(i), it follows that $f(H)$ is an almost subgroupoid of G' for all $H \in \mathcal{S}(G)$. Hence, φ is well-defined. Also, by Proposition 4.6(i), it follows that $f^{-1}(H')$ is an almost subgroupoid of G, for all $H' \in \bar{\mathcal{S}}(G')$. Hence, ψ is well-defined. The maps φ and ψ given by (6) have the following properties:

$$\psi \circ \varphi = Id_{\mathcal{S}(G)} \quad \text{and} \quad \varphi \circ \psi = Id_{\bar{\mathcal{S}}(G')}. \quad (7)$$

The equalities (6) are equivalently with:

$$f^{-1}(f(H)) = H, \ (\forall)\ H \in \mathcal{S}(G) \quad \text{and} \quad f(f^{-1}(H')) = H', \ (\forall)\ H' \in \bar{\mathcal{S}}(G'). \quad (8)$$

• Let $H \in \mathcal{S}(G)$.
(1). For any $x \in H$ we have $f(x) \in f(H)$ and $x \in f^{-1}(f(H))$. Hence, $H \subseteq f^{-1}(f(H))$.
(2). If $x \in f^{-1}(f(H))$, then $f(x) \in f(H)$ and $(\exists) h \in H$ such that $f(x) = f(h)$. From $f(x) = f(h)$ follows $f(x \cdot h^{-1}) = \theta'(f(h))$ and $x \cdot h^{-1} \in Ker(f) \subseteq H$. Therefore, $f^{-1}(f(H)) \subseteq H$. From (1) and (2) it follows the first equality of (8).
• Let $H' \in \bar{\mathcal{S}}(G')$.
(3). Let $y \in f(f^{-1}(H'))$. Then $y = f(x)$ with $x \in f^{-1}(H')$ and follows $f(x) \in H'$. Hence $y \in H'$. Therefore, $f(f^{-1}(H')) \subseteq H'$.
(4). For any $x' \in H'$, $(\exists) x \in G$ such that $f(x) = x'$, since f is surjective. Then $x \in f^{-1}(H')$, since $f(x) \in H'$. Therefore $x' \in f(f^{-1}(H'))$. Hence $H' \subseteq f(f^{-1}(H'))$. From (3) and (4) it follows the second equality of (8). Finally, from (7), it follows that φ is a bijection. □

Similarly, applying Propositions 4.10(ii) and 4.6(ii), we can prove the following theorem.

Theorem 4.12. (The correspondence theorem for normal almost subgroupoids) *For any surjective strong almost groupoid morphism* $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ *there exists a bijection between the set* $\mathcal{N}(G)$ *of normal almost subgroupoids of* G *and the set* $\bar{\mathcal{N}}(G')$ *of normal almost sugroupoids of* G'.

Remark 4.13. The Teorems 4.11 and 4.12 generalize the correspondence theorems for subgroups and normal subgroups by a surjective morphism of groups. Using the concept of strong groupoid morphism has been proved the isomorphism theorems for Brandt groupoids ([1]).

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