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On the simple Suzuki groups $Sz(8)$ by the average orders

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Abstract. We prove that simple suzuki group $Sz(8)$, can be uniquely determined by its average order.

1. Introduction

Throughout this paper G is a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. Next, the sum of element orders in a group G denoted by $\psi(G) = \sum_{g\in G} o(g)$ where $o(g)$ denotes the order of $g \in G$ and also we define average order of G as $o(G) = \frac{\psi(G)}{|G|}$. In the way, the function $\psi(G)$ was introduced by Amiri, Jafarian and Isaacs [2]. They proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(Z_n)$. In fact Z_n is characterized by $\psi(Z_n)$ and $|Z_n|$. In this paper we goal discuss about average order of group. We show that the group G is characterized by $o(G)$, when ever there exist the group H, so that if $o(G) = o(H)$, then $G \cong H$. In the way, for example the authors in [1, 3, 5, 6, 8, 12, 15] proved that $PSL(2;5)$ and $PSL(2;7)$ are uniquely determined by their orders and the sum of the element orders. But only some of groups is characterized by average order. For example, in [17] and [21] is proved the alternating group A_5 and the symmetric group S_4 can be determined by average order.

In this paper, we prove that the simple Suzuki group $Sz(8)$, can be uniquely determined by its average order. Namely, we prove

Main Theorem. Let $SZ(8)$ be the simple Suzuki group and G be a droup such that $o(G) = o(Sz(8))$. Then $G \cong Sz(8)$.

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Notation and preliminaries

Let $m_i(G)$ be the number of elements of order i and $\pi_e(G)$ denotes the set of element orders of G. We define $\psi(G) = \sum_{i\in\pi_e(G)} im_i(G)$ and $|G| = \sum_{i\in\pi_e(G)} m_i(G)$ and $o(G) = \frac{\psi(G)}{|G|}$

Lemma 2.1. [9] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3. [4] *Let G be a 2-Frobenius group of even order. Then*

!. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*

2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. In particular, every 2-Frobenius group is soluble group.*

Lemma 2.4. [21] *Let G be a group and H a non-trivial normal subgroup of G.*

1. *For each $x \in G$ and $h \in H$, $o(xH) \mid o(xh)$,*
2. *$\frac{\psi(H)-|H|}{|G|} + o(\frac{G}{H}) \leqslant o(G)$. In particular $o(G/H) < o(G)$.*

Lemma 2.5. [20] *Let G be a non-solvable group. Then G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$*

Lemma 2.6. [2] *Let $P \in Syl_p(G)$, and assume that $P \unlhd G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 2.7. [11] *Let G_1, G_2 be finite groups, p be a prime number and let a, b be positive integers. Then the following hold:*

1. *$\psi(G_1 \times G_2) = \psi(G_1) \times \psi(G_2)$ if and only if $(|G_1|, |G_2|) = 1$ i.e. ψ is multiplicative;*
2. *$\psi(p^a) = \frac{p^{2a+1}+1}{p+1}$,*
3. *$\psi(p^a) \mid \psi(p^b)$ if and only if $2a + 1 \mid 2b + 1$,*
4. *$(\psi(p), \psi(p^2)) = (\psi(p), \psi(p^3)) = (\psi(p^2), \psi(p^3)) = 1$.*

Theorem 2.8. [17] *Let G be a finite group. If $o(G) < \frac{211}{60}$ then G is solvable.*

Proof of the Main Theorem

First, assume $|Sz(8)| = 2^6.5.7.13$ and $o(Sz(8)) = \frac{219311}{29120} = o(G) = 7.53$. Now, by Theorem 2.8 G is not soluble group. Since $o(G) > \frac{211}{60}$. Therefore G is not 2-Frobenius group. Now, we prove G is not a Frobenius group. On opposite, assume G be a Frobenius group. so $t(G) = 2$, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since 13 is an isolated vertex in $\Gamma(G)$. So case (i) $|H| = 13$ and $|K| = \frac{|G|}{13}$ and case (ii) $|K| = 13$ and $|H| = \frac{|G|}{13}$. We can see easily case(ii) not be accurs. Hence, we consider case (i). Since that $|H|$ divided $|K| - 1$ so $13| \frac{29120}{13} - 1$, where this is impossible. Hence G is not a Frobenius group. Now, by Lemma 2.5 and Lemma 2.4 then $o(G/H) < o(G)$. On the other hand, we know, suzuki group $Sz(q)$ is only group where 3 not divided order of it. So $G/H \cong Sz(q')$, where $q' = 2^{2m+1}$ and $o(G/H) < 7.53$ in special case $q' = 8$. In other words, $G/H \cong Sz(8)$, on the other hand G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ as $H = 1$, so $G \cong Sz(8)$, the proof be completed.

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