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Some properties of gr-n-submodules

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Abstract. We introduce the concept of graded G.n-submodule, which is a generalization of the notion of graded n-submodule. We find some characterizations of graded G.n-submodules and we examine the way the aforementioned notions are related to each other.

1. Introduction

Throughout this article, we assume that $\mathcal{A}$ is a commutative G-graded ring with identity and $\mathcal{D}$ is a unitary graded $\mathcal{A}$-module.

In graded module theory, graded prime submodules are defined similar to graded prime ideals in graded ring theory and play an important role. Graded prime submodules of graded modules over graded commutative rings, have been introduced and studied by many authors, (see for example [3, 6, 16]). Al-Zoubi, Al-Turman and Celikel in [4] introduced and studied the concepts of graded n-ideals. Recently, M. Al-Azaizeh and K. Al-Zoubi in [1] introduced and studied the concept of graded n-submodules. Here, we investigate some properties of graded n-submodules. Furthermore, we introduce the concept of graded G.n-submodule, which generalizes the concept of graded n-submodule. Several results are discussed.

2. Preliminaries

In this section we will give the definitions and results which are required in the next section.

Definition 2.1. (a) Let G be a group with identity e and $\mathcal{A}$ be a commutative ring with identity $1_{\mathcal{A}}$. Then $\mathcal{A}$ is G-graded ring if there exist

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additive subgroups $\mathcal{A}_g$ of $\mathcal{A}$ indexed by the elements $g \in G$ such that $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. The elements of $\mathcal{A}_g$ are called homogeneous of degree g. The set of all homogeneous elements of $\mathcal{A}$ is denoted by $h(\mathcal{A})$, i.e. $h(\mathcal{A}) = \bigcup_{g\in G} \mathcal{A}_g$, see [15].

(b) Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be G-graded ring. An ideal P of $\mathcal{A}$ is called a graded ideal if $P = \sum_{h\in G} P \cap \mathcal{A}_h = \sum_{h\in G} P_h$. By $P \leq_G^{id} \mathcal{A}$, we mean that P is a G-graded ideal of $\mathcal{A}$. Also, by $P <_G^{id} \mathcal{A}$, we mean that P is a proper G-graded ideal of $\mathcal{A}$, see [15].

(c) A left $\mathcal{A}$-module $\mathcal{D}$ is said to be a G-graded $\mathcal{A}$-module if $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ with $\mathcal{A}_g\mathcal{D}_h \subseteq \mathcal{D}_{gh}$ for all $g, h \in G$, where $\mathcal{D}_g$ is an additive subgroup of $\mathcal{D}$ for all $g \in G$. The elements of $\mathcal{D}_g$ are called homogeneous of degree g. The set of all homogeneous elements of $\mathcal{D}$ is denoted by $h(\mathcal{D})$, i.e, $h(\mathcal{D}) = \bigcup_{g\in G} \mathcal{D}_g$. Note that $\mathcal{D}_h$ is an $\mathcal{A}_e$-module for every $h \in G$, see [15].

(d) A submodule K of $\mathcal{D}$ is called a graded submodule of $\mathcal{D}$ if $K = \oplus_{h\in G}(K \cap \mathcal{D}_h) := \oplus_{h\in G} K_h$. By $K \leq_G^{sub} \mathcal{D}$, we mean that K is a G-graded submodule of $\mathcal{D}$. Also, by $K <_G^{sub} \mathcal{D}$, we mean that K is a proper G-graded submodule of $\mathcal{D}$, see [15].

(e) If K is graded submodule of $\mathcal{D}$, then $(K :_{\mathcal{A}} \mathcal{D}) = \{a \in \mathcal{A} \mid a\mathcal{D} \subseteq K\}$ is graded ideal of $\mathcal{A}$, (see [8]). Furthermore, the annihilator of K in $\mathcal{A}$ is denoted and defined by $Ann_{\mathcal{A}}(K) = \{a \in \mathcal{A} \mid aK = \{0\}\}$, see [8].

(f) The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $t = \sum_{g\in G} t_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $t_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$, see [17].

(g) A proper graded submodule P of $\mathcal{D}$ is called a graded prime submodule if whenever $a \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ with $am \in P$, then either $a \in (P :_{\mathcal{A}} \mathcal{D})$ or $m \in P$, see [8].

(h) The graded radical of a graded submodule U of $\mathcal{D}$, denoted by $Gr_{\mathcal{D}}(U)$, is defined to be the intersection of all graded prime submodules of $\mathcal{D}$ containing U. If U is not contained in any graded prime submodule of $\mathcal{D}$, then $Gr_{\mathcal{D}}(U) = \mathcal{D}$, see [11].

(i) A proper graded $\mathcal{A}$-submodule U of a graded module $\mathcal{D}$ is called graded primary (gr-primary) if $rm \in U$, then either $m \in U$ or $r \in Gr((U :_{\mathcal{A}} \mathcal{D}))$, where $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$, see [16].

Definition 2.2. (a) A graded $\mathcal{A}$-module $\mathcal{D}$ is called a graded torsion-free (briefly, gr-torsion-free) module if whenever $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ with $rm = 0$, then either $r = 0$ or $m = 0$. Let $T(\mathcal{D}) = \{x \in \mathcal{D} : Ann_{\mathcal{A}}(x) \neq \{0\}\}$, see [8].

(b) A graded $\mathcal{A}$-module $\mathcal{D}$ is said to be a graded multiplication (briefly, gr-multiplication) module if for every graded submodule U, there exists a graded ideal I of a graded ring $\mathcal{A}$ such that $U = I\mathcal{D}$. Also, every graded submodule U of a graded multiplication module, $U = (U :_{\mathcal{A}} \mathcal{D})\mathcal{D}$, see [9].

(c) A graded $\mathcal{A}$-module $\mathcal{D}$ is called graded comultiplication (briefly, gr-comultiplication) module if for every graded submodule U of $\mathcal{D}$, there exists a graded ideal P of $\mathcal{A}$ such that $U = (0 :_{\mathcal{D}} P)$, equivalently, for each graded submodule U of $\mathcal{D}$, we have $U = (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U))$, see [5].

(d) A proper graded $\mathcal{A}$-submodule K of a graded module $\mathcal{D}$ is called a graded r-submodule (briefly, gr-r-submodule) of $\mathcal{D}$ if whenever $a \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ such that $am \in K$ with $Ann_{\mathcal{D}}(a) = 0$, then $m \in K$. For more properties, see [2].

(e) A non-zero graded $\mathcal{A}$-module $\mathcal{D}$ is called a graded secondary if for every $r \in h(\mathcal{A})$, the endomorphism of $\mathcal{D}$ given by multiplication by r is either surjective or nilpotent, see [11].

(f) A graded $\mathcal{A}$-module $\mathcal{D}$ is a called a graded simple (briefly, gr-simple) if (0) and $\mathcal{D}$ are its only graded submodules, [7].

(g) A proper graded submodule U of a graded $\mathcal{A}$-module $\mathcal{D}$ is said to be a graded n-submodule (briefly, gr-n-submodule) if whenever $r \in h(\mathcal{A})$, $m \in h(\mathcal{D})$ with $rm \in U$ and $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, then $m \in U$, see [1].

3. Results

Theorem 3.1. *Let $f : \mathcal{D} \to \mathcal{D}'$ be a graded homomorphism of graded $\mathcal{A}-$modules.*

(i) Assume that f is a graded monomorphism. If U' is a gr-n-submodule of $\mathcal{D}'$, then $f^{-1}(U') = \mathcal{D}$ or $f^{-1}(U')$ is a gr-n-submodule of $\mathcal{D}$.

(*ii*) *Assume that f is a graded epimorphism and $Ker(f) \subseteq U$. If U is a gr-n-submodule of $\mathcal{D}$, then $f(U)$ is a gr-n-submodule of $\mathcal{D}'$.*

Proof. (*i*). Assume that U' is a gr-n-submodule of $\mathcal{D}'$ and $f^{-1}(U') \neq \mathcal{D}$. Let $rm \in f^{-1}(U')$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Then $f(rm) = rf(m) \in U'$. As f is a graded monomorphism and $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, we get $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}'))$. Since U' is a gr-n-submodule of $\mathcal{D}'$, then $f(m) \in U'$ and so $m \in f^{-1}(U')$. Hence, $f^{-1}(U')$ is a gr-n-submodule of $\mathcal{D}$.
(*ii*). Assume that U is a gr-n-submodule of $\mathcal{D}$. Let $rx \in f(U)$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}'))$ and $x \in h(\mathcal{D}')$. As f is a graded epimorphism, then there exists $m \in h(\mathcal{D})$ such that $x = f(m)$, so $rx = rf(m) = f(rm) \in f(U)$. Since $Ker(f) \subseteq U$, we have $rm \in U$ and $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, and since U is a gr-n-submodule of $\mathcal{D}$, we get $m \in U$ and so $x = f(m) \in f(U)$. Therefore, $f(U)$ is a gr-n-submodule of $\mathcal{D}'$. □

Corollary 3.2. *Let $U \subseteq K$ be two graded submodules of $\mathcal{D}$. Then the followings hold:*

(*a*) *If K is a gr-n-submodule of $\mathcal{D}$, then K/U is a gr-n-submodule of $\mathcal{D}/U$.*

(*b*) *If K/U is a gr-n-submodule of $\mathcal{D}/U$ and $(U :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, then K is a gr-n-submodule of $\mathcal{D}$.*

(*c*) *If K/U is a gr-n-submodule of $\mathcal{D}/U$ and U is a gr-n-submodule of $\mathcal{D}$, then K is a gr-n-submodule of $\mathcal{D}$.*

Proof. (*a*). Assume that K is a gr-n-submodule of $\mathcal{D}$. Let $\Psi : \mathcal{D} \to \mathcal{D}/U$ be a graded epimorphism defined by $\Psi(m) = m + U$. Then $Ker(\Psi) = U \subseteq K$, so by Theorem 3.1 (*ii*), K/U is a gr-n-submodule of $\mathcal{D}/U$.
(*b*). Assume that K/U is a gr-n-submodule of $\mathcal{D}/U$ and $(U :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. Let $rm \in K$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Then we have $(r + I)(m + U) = rm + U \in K/U$ and $(r + I) \notin Gr(Ann_{\mathcal{A}/I}(\mathcal{D}/I))$ where $I = (U :_{\mathcal{A}} \mathcal{D})$. Since K/U is a gr-n-submodule of $\mathcal{D}/U$, then $(m + U) \in K/U$ and so $m \in K$. Hence, K is a gr-n-submodule of $\mathcal{D}$.
(*c*). It follows from [1, Theorem 3.3(i)] and part (*b*) . □

Corollary 3.3. *Let $U \leq_G^{sub} \mathcal{D}$. If K is a gr-n-submodule of $\mathcal{D}$ such that $U \nsubseteq K$, then $K \cap U$ is a gr-n-submodule of U.*

Proof. Assume that K is a gr-n-submodule of $\mathcal{D}$ such that $U \nsubseteq K$. Consider the graded monomorphism $\Phi : U \to \mathcal{D}$, defined by $\Phi(m) = m$ Then $\Phi^{-1}(K) = K \cap U$, so by Theorem 3.1 (i), we have $K \cap U$ is a gr-n-submodule of U. □

The following result studies the behavior of a gr-n-submodules under localization.

Theorem 3.4. *Let $S \subseteq h(\mathcal{A})$ be a multiplication closed subset of $\mathcal{A}$. Then;*

(i) *If U is a gr-n-submodule of $\mathcal{D}$, then $S^{-1}U = S^{-1}\mathcal{D}$ or $S^{-1}U$ is a gr-n-submodule of $S^{-1}\mathcal{D}$.*

(ii) *If $S^{-1}U$ is a gr-n-submodule of $S^{-1}\mathcal{D}$, $\mathcal{D}$ is a finitely generated module and $S \cap (Ann_{\mathcal{A}}(\mathcal{D}) :_{\mathcal{A}} r) = \phi$ for every $r \notin Ann_{\mathcal{A}}(\mathcal{D})$, then either $U = \mathcal{D}$ or U is a gr-n-submodule of $\mathcal{D}$.*

Proof. (i). Assume that U is a gr-n-submodule of $\mathcal{D}$ and $S^{-1}U \neq S^{-1}\mathcal{D}$. Let $\frac{r}{s_1}\frac{m}{s_2} \in S^{-1}U$ where $\frac{r}{s_1} \notin Gr(Ann_{S^{-1}\mathcal{A}}(S^{-1}\mathcal{D}))$. Then $trm \in U$ for some $t \in S$. Since $\frac{r}{s_1} \notin Gr(Ann_{S^{-1}\mathcal{A}}(S^{-1}\mathcal{D}))$, then $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, and since U is a gr-n-submodule, then $tm \in U$ and so $\frac{m}{s_2} = \frac{tm}{ts_2} \in S^{-1}U$. Hence, $S^{-1}U$ is a gr-n-submodule of $S^{-1}\mathcal{D}$.
(ii). Assume that $S^{-1}U$ is a gr-n-submodule of $S^{-1}\mathcal{D}$, $\mathcal{D}$ is a finitely generated module and $S \cap (Ann_{\mathcal{A}}(\mathcal{D}) :_{\mathcal{A}} r) = \phi$ for every $r \notin Ann_{\mathcal{A}}(\mathcal{D})$. Assume that $U \neq \mathcal{D}$. Let $rm \in U$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Then $\frac{r}{1}\frac{m}{1} \in S^{-1}U$. Now, we want to show that $\frac{r}{1} \notin Gr(Ann_{S^{-1}\mathcal{A}}(S^{-1}\mathcal{D}))$. Assume that $\frac{r}{1} \in Gr(Ann_{S^{-1}\mathcal{A}}(S^{-1}\mathcal{D}))$, then there exists $k \in \mathbb{N}$ such that $(\frac{r}{1})^k S^{-1}\mathcal{D} = 0$, and so $ur^k\mathcal{D} = 0$ for some $u \in S$, as $\mathcal{D}$ is a finitely generated module. Now, since $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, then $r^k\mathcal{D} \neq 0$ and so $u \in S \cap (Ann_{\mathcal{A}}(\mathcal{D}) :_{\mathcal{A}} r^k) = \phi$, a contradiction. Therefore, $\frac{r}{1} \notin Gr(Ann_{S^{-1}\mathcal{A}}(S^{-1}\mathcal{D}))$. Since $S^{-1}U$ is a gr-n-submodule of $S^{-1}\mathcal{D}$, then $\frac{m}{1} \in S^{-1}U$ and so $m \in U$. Hence, U is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.5. *Let $\mathcal{D}$ be a finitely generated graded $\mathcal{A}$-module such that for every multiplicative closed set $S \subseteq h(\mathcal{A})$, the kernel of $\varphi : \mathcal{D} \to S^{-1}\mathcal{D}$ is either (0) or $\mathcal{D}$. Then (0) is a gr-n-submodule of $\mathcal{D}$.*

Proof. Let $rm = 0$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. So $r^k \neq 0$ for every $k \in \mathbb{N}$. Put $S = \{r^k : k \in \mathbb{N} \cup \{0\}\}$. Then S is a multiplicative closed set in $h(\mathcal{A})$. If $Ker(\varphi) = 0$, then $\varphi(m) = \frac{m}{1} = \frac{rm}{r} = 0$ and so $r = 0$. Now, if $Ker(\varphi) = \mathcal{D}$. As $\mathcal{D}$ is a finitely generated graded $\mathcal{A}$-module, then

$\mathcal{D} = \mathcal{A}m_1 + \mathcal{A}m_2 + \ldots + \mathcal{A}m_l$ for some $m_1, m_2, \ldots, m_l \in h(\mathcal{D})$. Therefore, $\varphi(m_i) = \frac{m_i}{1} = 0$ for every $1 \leq i \leq l$. This implies that, for every i, there exists $t_i \in \mathbb{N}$ such that $r^{t_i} m_i = 0$. Put $j = \max\{t_1, t_2, \ldots, t_l\}$. Then $r^j \mathcal{D} = 0$ and so $r \in Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, which is a contradiction. Therefore, $m = 0$ and so (0) is a gr-n-submodule of $\mathcal{D}$. □

A nonempty subset S of a G-graded ring $\mathcal{A}$ with $h(\mathcal{A}) - Gr(0) \subseteq S \subseteq h(\mathcal{A})$ is called a gr-n-multiplicatively closed subset of $\mathcal{A}$ if whenever $r \in h(\mathcal{A}) - Gr(0)$ and $s \in S$, then $rs \in S$, see [4].

Theorem 3.6. *Let $\mathcal{D}$ be a finitely generated graded $\mathcal{A}$-module and $U <_G^{sub} \mathcal{D}$. If $(U :_{\mathcal{A}} \mathcal{D}) \cap S = \phi$, where S is a gr-n-multiplicatively closed subset of $\mathcal{A}$, then there exists a gr-n-submodule K of $\mathcal{D}$ containing U such that $(K :_{\mathcal{A}} \mathcal{D}) \cap S = \phi$.*

Proof. Assume that $(U :_{\mathcal{A}} \mathcal{D}) \cap S = \phi$, where S is a gr-n-multiplicatively closed subset of $\mathcal{A}$. Consider the set $\Omega = \{K : K \leq_G^{sub} \mathcal{D}; (K :_{\mathcal{A}} \mathcal{D}) \cap S = \phi\}$. Since $U \in \Omega$, then $\Omega \neq \phi$. Since $\mathcal{D}$ is a finitely generated, then by Zorn's lemma, we have a maximal element $K \in \Omega$. Now, we must show that K is a gr-n-submodule of $\mathcal{D}$. Let $rm \in K$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D}) - K$. This implies that $m \in (K :_{\mathcal{D}} r)$ and $K \subseteq (K :_{\mathcal{D}} r)$. By maximality of K, we have $((K :_{\mathcal{D}} r) :_{\mathcal{A}} \mathcal{D}) \cap S \neq \phi$ and so there exists $s \in S$ such that $s\mathcal{D} \subseteq (K :_{\mathcal{D}} r)$. Also, $rs \in S$ since $r \in h(\mathcal{A}) - Gr(0)$, $s \in S$ and S is a gr-n-multiplicatively closed subset of $\mathcal{A}$. Thus $(K :_{\mathcal{A}} \mathcal{D}) \cap S \neq \phi$, a contradiction. Hence, K is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.7. *Let $U \leq_G^{sub} \mathcal{D}$ such that $I = Gr(Ann_{\mathcal{A}}(\mathcal{D})) \subseteq (U :_{\mathcal{A}} \mathcal{D})$. Then U is a gr-n-submodule of $\mathcal{D}$ if and only if $\mathcal{D}/U$ is a gr-torsion-free $\mathcal{A}/I$-module.*

Proof. Assume that U is a gr-n-submodule of $\mathcal{D}$. Let $(r+I)(m+U) = 0_{\mathcal{D}/U}$ where $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$. Then $rm \in U$. Now, if $r \in I$, then $r + I = 0_{\mathcal{A}/I}$ and if $r \notin I$, then $m \in U$ since U is a gr-n-submodule of $\mathcal{D}$ and so $(m + U) = 0_{\mathcal{D}/U}$. Hence $\mathcal{D}/U$ is a gr-torsion-free $\mathcal{A}/I$-module. Conversely, assume that $\mathcal{D}/U$ is a gr-torsion-free $\mathcal{A}/I$-module. Let $rm \in U$ where $r \in h(\mathcal{A}) - I$ and $m \in h(\mathcal{D})$. Therefore $(r + I)(m + U) = rm + U = U = 0_{\mathcal{D}/U}$. Since $\mathcal{D}/U$ is a gr-torsion-free $\mathcal{A}/I$-module and $r \notin I$, then $m \in U$. Thus U is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.8. *Let U be a gr-n-submodule of $\mathcal{D}$. Then either*

$$U = (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U)) \quad or \quad Gr(Ann_{\mathcal{A}}(U)) = Gr(Ann_{\mathcal{A}}(\mathcal{D})).$$

Proof. Assume that $Gr(Ann_{\mathcal{A}}(U)) \neq Gr(\ Ann_{\mathcal{A}}(\mathcal{D}))$. It is clear that $U \subseteq (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U))$. Since $Gr(Ann_{\mathcal{A}}(\mathcal{D})) \subsetneq Gr(Ann_{\mathcal{A}}(U))$, then there exists

$$r \in h(\mathcal{A}) \bigcap Gr(Ann_{\mathcal{A}}(U)) - Gr(Ann_{\mathcal{A}}(\mathcal{D})),$$

so there exists $k \in \mathbb{N}$ such that $r^k \in Ann_{\mathcal{A}}(U)$. Now, we want to show that $(0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U)) \subseteq U$. Let $m \in (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U)) \bigcap h(\mathcal{D})$. Then $r^k m = 0 \in U$ and so $m \in U$, since U is a gr-n-submodule of $\mathcal{D}$ and $r \notin Gr(\ Ann_{\mathcal{A}}(\mathcal{D}))$. Hence $U = (0 :_{\mathcal{D}} Ann_{\mathcal{A}}(U))$. □

Theorem 3.9. *Let $\mathcal{D}$ be a gr-torsion-free $\mathcal{A}$-module and $U <^{sub}_{G} \mathcal{D}$. If $rU = U$ for every $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, then U is a gr-n-submodule of $\mathcal{D}$.*

Proof. Assume that $rU = U$ for every $r \in h(\mathcal{A}) - Gr(\ Ann_{\mathcal{A}}(\mathcal{D}))$. Let $rm \in U$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Since $rU = U$, then $rm \in rU$ and so $m \in U$ because $\mathcal{D}$ is a gr-torsion-free $\mathcal{A}$-module. Thus U is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.10. *Let $S \subseteq h(\mathcal{A})$ be a multiplicative closed subset of $\mathcal{A}$. If $< 0 >$ is a gr-n-submodule of $\mathcal{D}$, then the kernel of $\varphi : \mathcal{D} \to S^{-1}\mathcal{D}$ is either $< 0 >$ or $\mathcal{D}$.*

Proof. Assume that $< 0 >$ is a gr-n-submodule of $\mathcal{D}$ and $Ker(\varphi) \neq < 0 >$. So there exists $0 \neq x \in Ker(\varphi) \bigcap h(\mathcal{D})$ and so there exists $s \in S$ such that $sx = 0$. As $< 0 >$ is a gr-n-submodule of $\mathcal{D}$ and $x \neq 0$, we have $s \in Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. Hence, $Ker(\varphi) = \mathcal{D}$. □

Theorem 3.11. *Let $\mathcal{A}$ be a G-graded ring, $\mathcal{D}_1$ and $\mathcal{D}_2$ be a graded $\mathcal{A}$-module. Let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ and $Gr(Ann_{\mathcal{A}}(\mathcal{D}_1)) = Gr(Ann_{\mathcal{A}}(\mathcal{D}_2))$.*

- (i) *If U_1 is a gr-n-submodule of $\mathcal{D}_1$, then $U_1 \times \mathcal{D}_2$ is a gr-n-submodule of $\mathcal{D}$.*
- (ii) *If U_1 is a gr-n-submodule of $\mathcal{D}_1$ and U_2 is a gr-n-submodule of $\mathcal{D}_2$, then $U_1 \times U_2$ is a gr-n-submodule of $\mathcal{D}$.*

Proof. (i). Assume that U_1 is a gr-n-submodule of $\mathcal{D}_1$. Let $r(x_g, m_g) \in U_1 \times \mathcal{D}_2$ where $r \in h(\mathcal{A}) - Ann_{\mathcal{A}}(\mathcal{D})$, $(x_g, m_g) \in h(\mathcal{D})$ and $g \in G$.

Since $Gr(Ann_{\mathcal{A}}(\mathcal{D}_1)) = Gr(Ann_{\mathcal{A}}(\mathcal{D}_2)) = Gr(Ann_{\mathcal{A}}(\mathcal{D}_1) \cap Ann_{\mathcal{A}}(\mathcal{D}_2)) = Gr(Ann_{\mathcal{A}}(\mathcal{D}_1 \times \mathcal{D}_2))$, then $r \in h(\mathcal{A}) - Ann_{\mathcal{A}}(\mathcal{D}_1)$ and $rx_g \in U_1$ and so $x_g \in U_1$ as U_1 is a gr-n-submodule of $\mathcal{D}_1$. Which implies that, $(x_g, m_g) \in U_1 \times \mathcal{D}_2$ and so $U_1 \times \mathcal{D}_2$ is a gr-n-submodule of $\mathcal{D}$.

(ii). The proof is similar to that of part (i). □

Theorem 3.12. *If every graded proper submodule of $\mathcal{D}$ is a gr-n-submodule, then $\mathcal{D}$ is a graded secondary $\mathcal{A}$-module.*

Proof. Assume that every graded proper submodule of $\mathcal{D}$ is a gr-n-submodule. Let $r \in h(\mathcal{A})$ and $\varphi_r : \mathcal{D} \to \mathcal{D}$ is defined by $\varphi_r(m) = rm$. If φ_r is not surjective, then $Im(\varphi_r) \neq \mathcal{D}$ and so there exists $m \in h(\mathcal{D}) - Im(\varphi_r)$. Therefore, $\varphi_r(m) = rm \in \varphi_r(\mathcal{D})$, since $Im(\varphi_r)$ is a proper graded submodule, then by our assumption, $\varphi_r(\mathcal{D})$ is a gr-n-submodue and so $r \in Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. Hence, $\varphi_r(\mathcal{D})$ is nilpotent and so $\mathcal{D}$ is a graded secondary $\mathcal{A}$-module. □

Theorem 3.13. *Let $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ be a finitely generated graded ideal of $\mathcal{A}$. If every proper graded submodule of $\mathcal{D}$ is a gr-n-submodule, then every ascending chain of its cyclic graded submodules stops.*

Proof. Let $\mathcal{A}m_1 \subset \mathcal{A}m_2 \subset \cdots \subset \mathcal{A}m_k \subset \cdots$ be an ascending chain of cyclic graded submodules of $\mathcal{D}$. Then $m_1 = r_2m_2 = r_2r_3m_3 = \cdots = r_2r_3\cdots r_km_k = \cdots$ for some $r_i \in h(\mathcal{A})$. As $\mathcal{A}m_i$ is a gr-n-submodule, $r_i \in Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. Now, since $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is a finitely generated graded ideal, there exists $t \in \mathbb{N}$ such that $(Gr(Ann_{\mathcal{A}}(\mathcal{D})))^t \subseteq Ann_{\mathcal{A}}(\mathcal{D})$, so $m_1 = r_2r_3\cdots r_tr_{t+1}m_{t+1} = 0$ and so $m_i = 0$ for each $i \in \mathbb{N}$, which is a contradiction. Hence every ascending chain of cyclic graded submodules of $\mathcal{D}$ stops. □

Theorem 3.14. *Let $\mathcal{A}$ be a G-graded ring and $\mathcal{D}$ be a graded $\mathcal{A}$-module. Then, the following statements are equivalent.*

(*i*) *Every proper graded submodule of $\mathcal{D}$ is a gr-n-submodule.*

(*ii*) *Every proper graded cyclic submodule of $\mathcal{D}$ is a gr-n-submodule.*

Proof. $(i) \Rightarrow (ii)$. This is clear.
$(ii) \Rightarrow (i)$. Assume that every proper graded cyclic submodule of $\mathcal{D}$ is a gr-n-submodule. Let U be a proper graded submodule of $\mathcal{D}$ and $rm \in U$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Then there exists $x \in h(U)$ such that $rm \in < \mathcal{A}x >$ and since $< \mathcal{A}x >$ is gr-n-submodule, then $m \in < \mathcal{A}x > \subseteq U$. Hence, U is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.15. *Let $\mathcal{D}$ be a graded finitely generated $\mathcal{A}$-module. Then, the following statements are equivalent.*

(i) *Every proper graded submodule of* $\mathcal{D}$ *is a gr-n-submodule.*

(ii) $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ *is a graded maximal ideal of* $\mathcal{A}$.

Proof. $(i) \Rightarrow (ii)$' Assume that every proper graded submodule of $\mathcal{D}$ is a gr-n-submodule. As $\mathcal{D}$ is graded finitely generated $\mathcal{A}$-module, by [10, Lemma 2.7(ii)] $\mathcal{D}$ has a graded maximal submodule U. Then $(U :_{\mathcal{A}} \mathcal{D})$ is a graded maximal ideal of $\mathcal{A}$. But by [1, Theorem 3.3(i)], $(U :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and so $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is a graded maximal ideal of $\mathcal{A}$.
$(ii) \Rightarrow (i)$. Assume that $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is a graded maximal ideal of $\mathcal{A}$. Let U be a proper graded submodule of $\mathcal{D}$ and $am \in U$ where $a \in h(\mathcal{A}) - Ann_{\mathcal{A}}(\mathcal{D})$ and $m \in h(\mathcal{D})$. As $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is a graded maximal ideal, then $Ann_{\mathcal{A}}(\mathcal{D})+ < a\mathcal{A} >= \mathcal{A}$. Therefore, there exist $t \in Ann_{\mathcal{A}}(\mathcal{D})$ and $r \in \mathcal{A}$ such that $t + ar = 1$, so $m = tm + arm \in U$. Hence, U is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.16. *Let* $\mathcal{D}$ *be a gr-comultiplication* $\mathcal{A}$*-module and* $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ $= Ann_{\mathcal{A}}(\mathcal{D})$. *If* $\mathcal{D}$ *has a gr-n-submodule, then the following are hold.*

(1) *Every gr-n-submodule is maximal.*

(2) $< 0 >$ *is a gr-n-submodule of* $\mathcal{D}$.

(3) $\mathcal{D}$ *is a gr-simple.*

Proof. Assume that $\mathcal{D}$ has a gr-n-submodule say U.
(1). Let $L <_G^{sub} \mathcal{D}$ such that $U \subseteq L$. Then $Gr(Ann_{\mathcal{A}}(\mathcal{D})) = Ann_{\mathcal{A}}(\mathcal{D}) \subset Ann_{\mathcal{A}}(L)$, so there exists $r \in Ann_{\mathcal{A}}(L) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. Suppose that $t \in L$, then $rt = 0 \in U$. Since U is gr-n-submodule and $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, we get $t \in U$. Therefore, U is a gr-maximal submodule of $\mathcal{D}$.
(2). Using (1), U is a graded maximal submodule of $\mathcal{D}$, so $(U :_{\mathcal{A}} \mathcal{D})$ is a graded maximal ideal of $\mathcal{A}$. Therefore, $Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ is a graded maximal ideal of $\mathcal{A}$. Hence, $< 0 >$ is a gr-n-submodule of $\mathcal{D}$.
(3). Using (1) and (2), we have $< 0 >$ is a gr-maximal submodule of $\mathcal{D}$ and so $\mathcal{D}$ is a gr-simple. □

Theorem 3.17. *Let* $U_1, U_2, \ldots, U_k$ *be gr-primary submodules of* $\mathcal{D}$ *such that* $Gr((U_i :_{\mathcal{A}} \mathcal{D}))$ *are not comparable. Then,* $\bigcap_{i=1}^{k} U_i$ *is a gr-n-submodule of* $\mathcal{D}$ *if and only if* U_i *is a gr-n-submodule for each* $i \in \{1, 2, \ldots, k\}$.

Proof. (⇒) Assume that $\bigcap_{i=1}^{k} U_i$ is a gr-n-submodule of $\mathcal{D}$. Let $rm \in U_j$ for some $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, $m \in h(\mathcal{D})$ and $1 \leq j \leq k$. Since $Gr((U_i :_{\mathcal{A}} \mathcal{D}))$ are not comparable, there exists $x \in \bigcap_{i=1,\, i\neq j}^{k} Gr((U_i :_{\mathcal{A}} \mathcal{D})) \bigcap h(\mathcal{A}) - Gr((U_j :_{\mathcal{A}} \mathcal{D}))$. Therefore, there exists $h \in \mathbb{N}$ such that $x^h rm \in \bigcap_{i=1}^{k} U_i$, so $x^h m \in \bigcap_{i=1}^{k} U_i$ since $\bigcap_{i=1}^{k} U_i$ is a gr-n-submodule of $\mathcal{D}$ and so $x^h m \in U_j$. Which implies that $m \in U_j$ as U_j gr-primary submodule. Hence, U_j is a gr-n-submodule.
(⇐) It is clear by [1, Theorem 3.3(ii)]). □

Theorem 3.18. *Let $\{U_\alpha\}_{\alpha\in I}$ be a family of gr-prime submodules of $\mathcal{D}$. If $\bigcap_{\alpha\in I} U_\alpha$ is a gr-n-submodule of $\mathcal{D}$, then $\bigcap_{\alpha\in I} U_\alpha$ is a gr-prime submodule.*

Proof. Assume that $\bigcap_{\alpha\in I} U_\alpha$ is a gr-n-submodule of $\mathcal{D}$. Let $rm \in \bigcap_{\alpha\in I} U_\alpha$ where $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$. If $r \notin (\bigcap_{\alpha\in I} U_\alpha : \mathcal{D})$, then $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and since $\bigcap_{\alpha\in I} U_\alpha$ is a gr-n-submodule, then $m \in \bigcap_{\alpha\in I} U_\alpha$ and so $\bigcap_{\alpha\in I} U_\alpha$ is a gr-prime submodule. □

Theorem 3.19. *Let $\mathcal{D}$ be a finitely generated graded $\mathcal{A}$-module and U be a gr-n-submodule of $\mathcal{D}$. Then $Gr_{\mathcal{D}}(U)$ is a gr-n-submodule if and only if $Gr_{\mathcal{D}}(U)$ is a gr-prime submodule.*

Proof. (⇒) Using Theorem 3.18, if $Gr_{\mathcal{D}}(U)$ is a gr-n-submodule, then $Gr_{\mathcal{D}}(U)$ is a gr-prime submodule.
(⇐) Assume that $Gr_{\mathcal{D}}(U)$ is a graded prime submodule and let $rm \in Gr_{\mathcal{D}}(U)$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. Since U is a gr-n-submodule, then by [1, Theorem 3.3(i)], $(U :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(\mathcal{D}))$. This implies that $Gr((U :_{\mathcal{A}} \mathcal{D})) = Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and since $(Gr_{\mathcal{D}}(U) :_{\mathcal{A}} \mathcal{D}) = Gr((U :_{\mathcal{A}} \mathcal{D}))$, then $(Gr_{\mathcal{D}}(U) :_{\mathcal{A}} \mathcal{D}) = Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and so $m \in Gr_{\mathcal{D}}(U)$. Hence, $Gr_{\mathcal{D}}(U)$ is a gr-n-submodule of $\mathcal{D}$. □

Theorem 3.20. *Let $\mathcal{D}$ and W be graded $\mathcal{A}$-modules such that $\mathcal{D} \subseteq W$ and $Gr(Ann_{\mathcal{A}}(\mathcal{D})) = Gr(Ann_{\mathcal{A}}(W))$. If U is a gr-n-submodule of $\mathcal{D}$, then there exists a gr-n-submodule K of W such that $U = K \cap \mathcal{D}$.*

Proof. Let $\Omega = \{L : L \leq_G^{sub} W \text{ and } L \cap \mathcal{D} = U\}$. Since $U \in \Omega$, then $\Omega \neq \phi$. Using Zorn's lemma, we find a maximal element K of Ω. To show that K is a *graded submodule* of W. Suppose $k_1, k_2 \in K$, where k_1 and k_2 are *homogeneous elements*. The sum $k_1 + k_2$ must belong to K, or else we could enlarge K while still maintaining the intersection condition, contradicting maximality. Therefore, K is closed under addition. Suppose $r \in h(A)$ (a homogeneous element of the graded ring A) and $w \in h(W)$ such that $rw \in K$. Assume, for contradiction, that $w \notin K$. Consider the *graded submodule* $K + \langle w \rangle$, the smallest graded submodule containing both K and w. By the maximality of K, we must have

$$(K + \langle w \rangle) \cap D \neq U,$$

meaning there exists some $m \in D \setminus U$ such that $m \in (K + \langle w \rangle)$. However, this contradicts the assumption that U is a *graded n-submodule* of D, leading to $m \in U$, a contradiction. Therefore, $w \in K$. Let $k \in K$ be an arbitrary element. Since W is a graded module, we can write k as a sum of its homogeneous components:

$$k = \sum_i k_i, \quad \text{where } k_i \in W_i.$$

If $k \in K$, then each k_i must also be in K, because if $k_i \notin K$, we would have $K' = K + \langle k_i \rangle$, and since K is maximal in Ω, this would contradict maximality unless $k_i \in K$. Therefore, K contains all its homogeneous components. Thus K is a graded submodule of W.

Now, we must show that K is a gr-n-submodule of W. Let $rw \in K$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(W))$ and $w \in h(W)$. Suppose that $w \notin K$. As K is a maximal element of Ω, then $(K + < w >) \cap \mathcal{D} \nsubseteq U$ and so there exist $k \in h(W) \cap K$, $m \in h(\mathcal{D}) - U$ and $t \in h(\mathcal{A})$ such that $k + tw = m$, and so $rk + rtw = rm \in K \cap \mathcal{D} = U$. But U is a gr-n-submodule and $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$, which implies that $m \in U$, a contradiction. Hence K is a gr-n-submodule of W and $K \cap \mathcal{D} = U$. □

Theorem 3.21. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ and $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_k$ where $\mathcal{D}_i$ is a non-zero graded $\mathcal{A}_i$-module for every $1 \leq i \leq k$ and $k \geq 2$. Then $\mathcal{D}$ has no gr-n-submodule.*

Proof. Suppose that U is a gr-n-submdule of $\mathcal{D}$. Then $U \neq \mathcal{D}$ and so there exists j, $1 \leq j \leq k$ such that $(0, 0, \cdots, 0, \underbrace{m}_{jth}, 0, \cdots, 0) \in h(\mathcal{D}) - U$. Then

$(1,1,\cdots,1,\underbrace{0}_{jth},1,\cdots,1)(0,0,\cdots,0,\underbrace{m}_{jth},0,\cdots,0)\in U.$

Since $(1,1,\cdots,1,\underbrace{0}_{jth},1,\cdots,1)\in h(\mathcal{A})-Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and U is a gr-n-submdule, then $(0,0,\cdots,0,\underbrace{m}_{jth},0,\cdots,0)\in U$, a contradiction. Therefore, $\mathcal{D}$ has no gr-n-submodule. □

4. G.gr-n-submodule

Definition 4.1. Let $U <_G^{sub} \mathcal{D}$. Then U is called a generalization of gr-n-submodule (G.gr-n-submodule) if for each $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $m \in h(\mathcal{D})$ with $rm \in U$, then $m \in U$.

Its clear that every gr-n-submodule of $\mathcal{D}$ is a G.gr-n-submodule but the following example shows that the converse is not true in general.

Example 4.2. Let $G = \mathbb{Z}_2$ and $\mathcal{A} = \mathbb{Z}$. Then $\mathcal{A}$ is a G-graded ring with $\mathcal{A}_0 = \mathbb{Z}$ and $\mathcal{A}_1 = \{0\}$. Let $\mathcal{D} = \mathbb{Z}_4 \oplus \mathbb{Z}$. Then $\mathcal{D}$ is a graded $\mathcal{A}-$module with $\mathcal{D}_0 = \mathbb{Z}_4 \oplus \mathbb{Z}$ and $\mathcal{D}_1 = \{0\}$. Then $U =< 2 > \oplus < 0 >$ is a G.gr-n-submodule of $\mathcal{D}$ which is not gr-n-submodule.

Lemma 4.3. *If $\mathcal{D}$ is a gr-torsion-free $\mathcal{A}$-module the notion of gr-n-submodule and G.gr-n-submodule coincide.*

Proof. It is clear □

Theorem 4.4. *If $< 0 >$ is a gr-n-submodule of $\mathcal{D}$, then notion of gr-n-submodule and G.gr-n-submodule coincide.*

Proof. Let U be a G.gr-n-submodule of $\mathcal{D}$ and $rm \in U$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ and $m \in h(\mathcal{D})$. If $r \notin Gr(Ann_{\mathcal{A}}(U))$, then $m \in U$ since U is a G.gr-n-submodule. If $r \in Gr(Ann_{\mathcal{A}}(U))$, then there exists $k \in \mathbb{N}$ such that $r^k(rm) = 0$, so $r^{k+1}m = 0$. But $< 0 >$ is a gr-n-submodule and $r \notin Gr(Ann_{\mathcal{A}}(\mathcal{D}))$ so $m = 0$ and so $m \in U$. Therefore, U is a gr-n-submodule. □

Theorem 4.5. *Let $U <_G^{sub} \mathcal{D}$. Then the following are equivalent.*

(1) *U is a G.gr-n-submodule of $\mathcal{D}$.*

(2) *$U = (U :_{\mathcal{D}} r)$ for every $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$.*

(3) *For every graded ideal J of $\mathcal{A}$ and every graded submodule L of $\mathcal{D}$, $JL \subseteq U$ with $J \nsubseteq Gr(Ann_{\mathcal{A}}(U))$, then $L \subseteq U$.*

Proof. (1) $\Rightarrow$ (2). Assume that U is a G.gr-n-submodule of $\mathcal{D}$. The inclusion $U \subseteq (U :_{\mathcal{D}} r)$ always holds for every $r \in \mathcal{A}$. Now, let $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $m \in (U :_{\mathcal{D}} r) \bigcap h(\mathcal{D})$, then $rm \in U$. As U is a G.gr-n-submodule, we get $m \in U$ and hence $U = (U :_{\mathcal{D}} \mathcal{A})$.
(2) $\Rightarrow$ (3). Assume that $U = (U :_{\mathcal{D}} r)$ for every $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $JL \subseteq U$ where $J \leq_G^{id} \mathcal{A}$ and $L \leq_G^{sub} \mathcal{D}$ with $J \nsubseteq Gr(Ann_{\mathcal{A}}(U))$. Since $J \nsubseteq Gr(Ann_{\mathcal{A}}(U))$, then there exists $j \in h(\mathcal{A}) \cap J - Gr(Ann_{\mathcal{A}}(U))$. Therefore, $jL \subseteq U$ and so $L \subseteq (U :_{\mathcal{D}} j) = U$ by our assumption.
(3) $\Rightarrow$ (1). Assume that for every graded ideal J of $\mathcal{A}$ and every graded submodule L of $\mathcal{D}$, $JL \subseteq U$ with $J \nsubseteq Gr(Ann_{\mathcal{A}}(U))$, then $L \subseteq U$. Let $rm \in U$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $m \in h(\mathcal{D})$. Let $I = r\mathcal{A}$ and $L = m\mathcal{A}$, so $JL \subseteq U$ with $J \nsubseteq Gr(Ann_{\mathcal{A}}(U))$, and so by our assumption, $L \subseteq U$. Which implies that $m \in U$. Hence, U is a G.gr-n-submodule of $\mathcal{D}$. □

Theorem 4.6. *Let $\mathcal{D}$ be a graded $\mathcal{A}$-module. Then*

(1) *If K is a G.gr-n-submodule of $\mathcal{D}$, then $(K :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(K))$.*

(2) *If K is a G.gr-n-submodule of $\mathcal{D}$, then $(K :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(\mathcal{D}))$.*

(3) *Let $\{K_i\}_{i\in\Omega}$ be a nonempty set of G.gr-n-submodule of $\mathcal{D}$. Then $\bigcap_{i\in\Omega} K_i$ is a G.gr-n-submodule.*

(4) *Let $\{K_i\}_{i\in\Omega}$ be a finite chain of G.gr-n-submodule of a finitely generated graded $\mathcal{A}$-module $\mathcal{D}$. Then, $\bigcup_{i\in I} K_i$ is a G.gr-n-submodule of $\mathcal{D}$.*

Proof. (1). Assume that K is a G.gr-n-submodule of $\mathcal{D}$. If $(K :_{\mathcal{A}} \mathcal{D}) \nsubseteq Gr(Ann_{\mathcal{A}}(K))$, then there exists $r \in h(\mathcal{A}) \cap (K :_{\mathcal{A}} \mathcal{D}) - Gr(Ann_{\mathcal{A}}(K))$ and so $r\mathcal{D} \subseteq K$ and since K is a G.gr-n-submodule of $\mathcal{D}$, we get $K = \mathcal{D}$, a contradiction. Hence, $(K :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(K))$.
(2). Assume that K is a G.gr-n-submodule of $\mathcal{D}$. Let $r \in h(\mathcal{A}) \cap (K :_{\mathcal{A}} \mathcal{D})$. Then, by part (1), $r \in Gr(Ann_{\mathcal{A}}(K))$ and so there exists $m \in \mathbb{N}$ such that $r^m \in Ann_{\mathcal{A}}(K)$ and since $r \in (K :_{\mathcal{A}} \mathcal{D})$, we have $r^m r\mathcal{D} = r^{m+1}\mathcal{D} = 0$ and so $r \in Gr(Ann_{\mathcal{A}}(\mathcal{D}))$.
(3). Let $rm \in \bigcap_{i\in\Omega} K_i$ where $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D}) - \bigcap_{i\in\Omega} K_i$. Then

$m \notin K_j$ for some $j \in \Omega$. As K_j is a G.gr-n-submodule of $\mathcal{D}$, we obtain $r \in Gr(Ann_{\mathcal{A}}(K_j)) \subseteq Gr(Ann_{\mathcal{A}}(\bigcap_{i\in\Omega} K_i))$. Therefore, $\bigcap_{i\in\Omega} K_i$ is a G.gr-n-submodule.
(4). Let $rm \in \bigcup_{i\in I} K_i$ where $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D}) - \bigcup_{i\in I} K_i$. Then $m \notin K_i$ for all $i \in I$. Since K_i is a G.gr-n-submodule of $\mathcal{D}$, we get $r \in Gr(Ann_{\mathcal{A}}(K_i))$ for all $i \in I$ and as I is a finite set, we have $r \in Gr(Ann_{\mathcal{A}}(\bigcup_{i\in I} K_i))$. Thus, $\bigcup_{i\in I} K_i$ is a G.gr-n-submodule. □

Theorem 4.7. *Let U and K be proper graded submodules of $\mathcal{D}$ and I be a graded ideals of $\mathcal{A}$ such that $I \nsubseteq Gr(Ann_{\mathcal{A}}(U)) \bigcup Gr(Ann_{\mathcal{A}}(K))$. Then, the following statements are holds.*

(1) *If U and K are G.gr-n-submodules of $\mathcal{D}$ with $IU = IK$, then $U = K$.*

(2) *If IU is a G.gr-n-submodule of $\mathcal{D}$, then $IU = U$.*

Proof. (1). Assume that U and K are G.gr-n-submodules of $\mathcal{D}$ with $IU = IK$. Using Theorem 4.5 (3), we get $K \subseteq U$ and $U \subseteq K$. Hence, $U = K$.
(2). Assume that IU is a G.gr-n-submodule of $\mathcal{D}$. Since $IU \subseteq IU$, then by Theorem 4.5 (3), $U \subseteq IU$ and so $IU = U$. □

Theorem 4.8. *Let $\mathcal{A}$ be a G-graded ring and $\mathcal{D}$ be a graded torsion-free $\mathcal{A}$-module. Then the zero submodules is a G.gr-n-submodule of $\mathcal{D}$.*

Proof. Using Lemma 4.3 and [1, Lemma 3.5], we get the result. □

Lemma 4.9. *Let $\mathcal{A}$ be a G-graded ring and $\mathcal{D}$ be a gr-multiplication torsion-free $\mathcal{A}$-module. Then the zero submodule is the only G.gr-n-submodule of $\mathcal{D}$.*

Proof. Using Lemma 4.3 and [1, Lemma 3.7], we get the result. □

Theorem 4.10. *Let $K <_G^{sub} \mathcal{D}$. Then, K is a G.gr-n-submodule if and only if for every $m \in h(\mathcal{D})$, $(K :_{\mathcal{A}} m) = \mathcal{A}$ or $(K :_{\mathcal{A}} m) \subseteq Gr(Ann_{\mathcal{A}}(K))$.*

Proof. ($\Rightarrow$) Assume that K is a G.gr-n-submodule and $(K :_{\mathcal{A}} m) \nsubseteq Gr(Ann_{\mathcal{A}}(K))$ where $m \in h(\mathcal{D})$. Then there exists $r \in h(\mathcal{A}) \bigcap (K :_{\mathcal{A}} m) - Gr(Ann_{\mathcal{A}}(K))$, so $rm \in K$ and $r \notin Gr(Ann_{\mathcal{A}}(K))$. Since K is a G.gr-n-submodule, then $m \in K$ and so $(K :_{\mathcal{A}} m) = \mathcal{A}$.
($\Leftarrow$) Assume that for every $m \in h(\mathcal{D})$, $(K :_{\mathcal{A}} m) = \mathcal{A}$ or $(K :_{\mathcal{A}} m) \subseteq$

$Gr(Ann_{\mathcal{A}}(K))$. Let $rm \in K$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(K))$ and $m \in h(\mathcal{D})$. Then $r \in (K :_{\mathcal{A}} m) - Gr(Ann_{\mathcal{A}}(K))$. By our assumption, $(K :_{\mathcal{A}} m) = \mathcal{A}$ and so $m \in K$. Hence, K is a G.gr-n-submodule. □

Theorem 4.11. *Let $K <^{sub}_{G} \mathcal{D}$. Then K is a G.gr-n-submodule of $\mathcal{D}$ if and only if every graded zero-divisor of a graded $\mathcal{A}$-module $\frac{\mathcal{D}}{K}$ is in $Gr(Ann_{\mathcal{A}}(K))$.*

Proof. Let K be a G.gr-n-submodule of $\mathcal{D}$ and $r \in h(\mathcal{A})$ be a zero-divisor in $\frac{\mathcal{D}}{K}$. Then there exists $m \in h(\mathcal{D}) - K$ such that $rm \in K$. As K is a G.gr-n-submodule of $\mathcal{D}$, then $r \in Gr(Ann_{\mathcal{A}}(K))$, as we needed. Conversely, assume that every graded zero-divisor of a graded $\mathcal{A}$-module $\frac{\mathcal{D}}{K}$ is in $Gr(Ann_{\mathcal{A}}(K))$. Let $rm \in K$ for some $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D}) - K$. Then r is a zero-divisor in $\frac{\mathcal{D}}{K}$ and so $r \in Gr(Ann_{\mathcal{A}}(K))$. Hence, K is a G.gr-n-submodule of $\mathcal{D}$. □

Theorem 4.12. *Let K be a gr-prime submodule of $\mathcal{D}$. Then K is a G.gr-n-submodule of $\mathcal{D}$ if and only if $(K :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(K))$.*

Proof. (⇒) Using Theorem 4.6 (1), we get the result.
(⇐) Assume that $(K :_{\mathcal{A}} \mathcal{D}) \subseteq Gr(Ann_{\mathcal{A}}(K))$. Let $rm \in K$ where $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(K))$ and $m \in h(\mathcal{D})$. Since K is a gr-prime submodule, then either $r \in (K :_{\mathcal{A}} \mathcal{D})$ or $m \in K$ and by our assumption, we get $m \in K$. Hence, K is a G.gr-n-submodule. □

Theorem 4.13. *Every G.gr-n-submodule of $\mathcal{D}$ is a gr-r-submodule of $\mathcal{D}$.*

Proof. Assume that U is a G.gr-n-submodule of $\mathcal{D}$. Let $rm \in U$ for some $r \in h(\mathcal{A})$ and $m \in h(\mathcal{D})$ such that $Ann_{\mathcal{D}}(r) = 0$. Since U is a G.gr-n-submodule of $\mathcal{D}$, then either $r \in Gr(Ann_{\mathcal{A}}(U))$ or $m \in U$. If $r \in Gr(Ann_{\mathcal{A}}(U))$, then there exists a smallest positive integer k such that $r^k U = 0$ and $r^{k-1}U \neq 0$. As $r(r^{k-1}U) = r^k U = 0$, then $r^{k-1}U \subseteq Ann_{\mathcal{D}}(r) = 0$ and so $r^{k-1}U = 0$, a contradiction. Therefore, $m \in U$ and hence U is a graded r-submodule of $\mathcal{D}$. □

Theorem 4.14. *Let $\mathcal{D}$ and K be graded $\mathcal{A}$-modules such that $K \subseteq \mathcal{D}$. If U is a G.gr-n-submodule of K and K is a G.gr-n-submodule of $\mathcal{D}$, then U is a G.gr-n-submodule of $\mathcal{D}$.*

Proof. Assume that U is a G.gr-n-submodule of K and K is a G.gr-n-submodule of $\mathcal{D}$. Let $rm \in U$ for some $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $m \in h(\mathcal{D})$. Since $Gr(Ann_{\mathcal{A}}(K)) \subseteq Gr(Ann_{\mathcal{A}}(U))$, then $r \notin Gr(Ann_{\mathcal{A}}(K))$

and since K is a G.gr-n-submodule of $\mathcal{D}$, then $m \in K$. Now, $rm \in U$, $r \in h(\mathcal{A}) - Gr(Ann_{\mathcal{A}}(U))$ and $m \in h(K)$, then $m \in U$ as U is a G.gr-n-submodule of K. Hence U is a G.gr-n-submodule of $\mathcal{D}$. □

Theorem 4.15. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ be a G-graded ring and $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_k$, where $\mathcal{D}_i$ is a non-zero graded $\mathcal{A}_i$−module for $1 \leq i \leq k$. If U is a G.gr-n-submodule of $\mathcal{D}$, then there exists j, $1 \leq j \leq k$, such that $U = U_1 \times U_2 \times \cdots \times U_k$, U_j is a G.gr-n-submodule of $\mathcal{D}_j$ and for any $i \neq j$, $U_i = 0$.*

Proof. Assume that U is a G.gr-n-submodule of $\mathcal{D}$. Then $U \neq \mathcal{D}$ and so there exists j, $1 \leq j \leq k$, such that $(0, \dots, 0, \underbrace{m}_{jth}, 0, \dots, 0) \in h(\mathcal{D}) - U$.

Then,

$$(1, \dots, 1, \underbrace{0}_{jth}, 1, \dots, 1)(0, \dots, 0, \underbrace{m}_{jth}, 0, \dots, 0) \in U.$$

Since U is a G.gr-n-submodule, then $(1, \dots, 1, \underbrace{0}_{jth}, 1, \dots, 1) \in Gr(Ann_{\mathcal{A}}(U))$.

Which implies that $U = 0 \times \cdots \times 0 \times \underbrace{U_j}_{jth} \times 0 \times \cdots \times 0$, where U_j is a graded submodule of $\mathcal{D}_j$. Therefore, $Ann_{\mathcal{A}}(U) = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{j-1} \times Ann_{\mathcal{A}_j}(U_j) \times \mathcal{A}_{j+1} \times \cdots \times \mathcal{A}_k$. Now, let $rx \in U_j$ for some $r \in h(\mathcal{A}_j) - Gr(Ann_{\mathcal{A}_j}(U_j))$ and $x \in h(\mathcal{D}_j)$. As U is a G.gr-n-submodule and

$$(0, \dots, 0, \underbrace{r}_{jth}, 0, \dots, 0)(0, \dots, 0, \underbrace{x}_{jth}, 0, \dots, 0) \in U,$$

then $x \in U_j$. Hence, U_j is a G.gr-n-submodule of $\mathcal{D}_j$. □

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