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Construction of gyrogroups of order 2^n by cyclic 2-groups

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Abstract. Gyrogroups abound in group theory. They possess a rich group-like structure that forms a natural generalization of groups. A prominent example is provided by Einstein's addition law of relativistically admissible velocities. Being nonassociative, it turns out that Einstein addition is a nongroup gyrogroup binary operation. A gyrogroup is a rich structure constituting a non-empty set with a binary operation that obeys an associative-like law called the gyroassociative law. The aim of this article is to present a method of constructing novel gyrogroups of order 2^n, $n \geqslant 3$, by a cyclic 2-group that is $\mathbb{Z}_{2^n}$.

1. Definitions and history

Seemingly structureless, Einstein addition of relativistically admissible velocities is neither commutative nor associative. However, more than 80 years after the appearance of the theory of special relativity in one of the anni mirabiles (1905), A.A. Ungar discovered in 1988 that Einstein addition is both *gyrocommutative* and *gyroassociative* [17, 18], a discovery that signaled the birth of *gyrogroup theory* [20] along lines parallel to group theory. Gyrogroups turn out to be generalized groups in which associativity (commutativity) is generalized to gyroassociativity (gyrocommutativity). The formal gyrogroup definition follows.

A pair $(G, \oplus)$ consisting of a non-empty set G and a binary operation $\oplus$ in G is called a groupoid. Let $(G, \oplus)$ be a groupoid. A bijection from G to itself is called an automorphism of G if $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$ for all $a, b \in G$. The set of all automorphisms of G, denoted by $\mathrm{Aut}\,(G, \oplus)$, forms

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a group under function composition. A groupoid $(G, \oplus)$ is a *gyrogroup* if the following axioms hold:

(G1) (left identity) there exists an element $0 \in G$ such that for all $x \in G$, $0 \oplus x = x$;

(G2) (left inverse) for each $a \in G$, there exists $b \in G$ such that $b \oplus a = 0$;

(G3) (left gyroassociative law) there exists a function $\text{gyr} : G \times G \longrightarrow \text{Aut}\,(G, \oplus)$ such that for every $a, b, c \in G$, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c$, where $\text{gyr}[a, b]c = \text{gyr}(a, b)(c)$;

(G4) (left reduction property, or left loop property) for each $a, b \in G$, $\text{gyr}[a, b] = \text{gyr}[a \oplus b, b]$.

Note that the gyrogroup axioms (G1) – (G4) imply their right counterparts [23]. The axiom *loop property* in the gyrogroup definition is also known as the *reduction property* since it triggers a remarkable reduction in complexity [21].

For every $a, b \in G$, the mapping $\text{gyr}[a, b]$ is called the gyroautomorphism generated by a and b. Gyroautomorphisms are also known as *gyrations*. The gyrogroup rich structure is demonstrated, for instance, in the study of Lorentz groups in [21, 17]. Finally, a gyrogroup $(G, \oplus)$ is said to be gyrocommutative if for all $a, b \in G$,

$$a \oplus b = \text{gyr}[a, b](b \oplus a)\,.$$

Any group is a gyrogroup of which the gyroautomorphisms are trivial, that is, they are the identity automorphism.

Throughout this paper, our notation is standard, taken mainly from [6, 23]. We refer interested readers to consult the survey [20] for a complete account of the history of the gyrogroup concept. We also refer to [14, 15] for the study of subgyrogroups, gyrogroup homomorphisms, and quotient gyrogroup.

The concept of a gyrogroup is a natural extension of the traditional notion of a group, as demonstrated by Chen and Ungar in their work [2]. This generalization further extends to the concept of a bi-gyrogroup, as explored in the work of Ungar [19]. Subsequent studies by Foguel and Ungar [4, 5], as well as Feder [3], reveal the close interrelation between gyrogroups and left gyrogroups with concepts derived from group theory.

As Suksumran showed in his paper [12], there is a correspondence between the class of gyrogroups and triples components being groups and twisted subgroups. He used these triples, arising from the theory of gyrogroups, to study the generalized Heisenberg group [10]. This underscores the integral role of gyrogroup theory in the broader context of group theory.

The classification of gyrogroups of small orders, of orders at most 31 up to isomorphism, except for 24, 27, and 30, is found in [1]. By calculations and methods rooted in the theory of quasigroups and loops, we know that there are 1995 gyrogroups of order 16, 179 of which are gyrocommutative [1]. It is worthwhile to mention that there is a history of constructing gyrogroups of order 16 in the literature that begins with Zhang in 1996, [24] where Zhang proposed the dual of a gyrocommutative gyrogroup of order 16 (duality of a loop in the sense of [11]). This history of constructing finite gyrogroups continues by Foguel and Ungar works in 2000 and 2001 where they created and developed a method for constructing gyrogroups by twisted subgroups of a group [4, 5]. This leads our story to a work by Ungar in 2000 [22] where he proposed a nongyrocommutative gyrogroup of order 16 constructed by the mentioned method. In 2021 Mahdavi et al. proposed a construction of nongyrocommutative gyrogroups of order 2^n [8]. Then, in 2022, Maungchang and Suksumran proposed a new way of constructing gyrogroups of finite orders and constructed a gyrogroup of order 16 in [9]. Finally, in 2022 Suksumran proposed a method for constructing gyrogroups and constructed three gyrogroups of order 16 [13].

In this article we present a new method of constructing gyrocommutative gyrogroups of orders 2^n, $n \geqslant 3$, which isomorphically yields different gyrogroups. Specifically, we present a way of constructing gyrogroups of order 2^n by the cyclic 2-group $\mathbb{Z}_{2^n}$ for $n \geqslant 3$ that is denoted by $G_2(n)$. In contrast, Mahdavi et al. constructed in [8] a gyrogroup $G(n)$ of order 2^n that we denote in this paper by $G_1(n)$.

2. Preliminaries

In this section we introduce the notation and concepts that we use later for the construction. Our calculations are done with the aid of GAP [16].

Notation 2.1. For $n \geqslant 3$, let $P(n) = \{0, 1, 2, \ldots, 2^{n-1} - 1\}$, $H(n) = \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$ and $G_2(n) = P(n) \cup H(n)$. It is clear that $P(n) \cong \mathbb{Z}_{2^{n-1}}$ and $H(n) = P(n) + m$ where $m = 2^{n-1}$. Hence $G_2(n) = P(n) \cup$

$(P(n)+m)$. Let $O_P = \{i \in P(n) \,|\, i \text{ is odd}\}$, $E_P = \{i \in P(n) \,|\, i \text{ is even}\}$, $O_H = \{i \in H(n) \,|\, i \text{ is odd}\}$ and $E_H = \{i \in H(n) \,|\, i \text{ is even}\}$.

Define the binary operation $\oplus$ on $G_2(n)$ as follows:

$$i \oplus j = \begin{cases} t & : (i,j) \in (P(n) \times P(n)) \cup \Big[(H(n) \times H(n)) - (E_H \times O_H)\Big] \\ t+m & : (i,j) \in (P(n) \times H(n)) \cup \Big[(H(n) \times P(n)) - (E_H \times O_P)\Big] \\ s & : (i,j) \in E_H \times O_H \\ s+m & : (i,j) \in E_H \times O_P \end{cases}$$

where $t, s \in P(n)$ and

$$\begin{cases} t \equiv i+j (\operatorname{mod} m) \\ s \equiv i+j+\dfrac{m}{2} (\operatorname{mod} m) \end{cases}$$

Clearly, the operation $\oplus$ is well defined.

Denoting the greatest common divisor of positive integers r and s by (r,s), we have the following simple lemma.

Lemma 2.2 ([8])**.** *Assume Notation* 2.1.

(*I*) $(\frac{m}{2}-1, m) = (\frac{m}{2}+1, m) = 1$.

(*II*) *Suppose* $x \equiv y (\operatorname{mod} m)$. *If* $x, y \in P(n)$ *or* $x, y \in H(n)$, *then* $x = y$.

Lemma 2.3. *Assuming Notation* 2.1, *let* $A : G_2(n) \to G_2(n)$ *be a map on* $G_2(n)$ *given by*

$$A(i) = \begin{cases} r & : \ i \in O_P, \\ r+m & : \ i \in O_H, \\ i & : \ \textit{otherwise}, \end{cases}$$

where $r \in P(n)$ *and* $r \equiv i + \dfrac{m}{2} (\operatorname{mod} m)$. *Then* $A \in \operatorname{Aut}(G_2(n), \oplus)$.

Proof. Clearly A is a well defined bijective map on $G_2(n)$. It is enough to show that A is a homomorphism on $(G_2(n), \oplus)$. To do this, assume $i, j \in G_2(n)$ be two arbitrary elements. We consider four distinct cases:

1. $i, j \in P(n)$. This condition is the first case we consider in the four subcases:

(*a*). $i, j \in O_P$. By the definition of $\oplus$, $i \oplus j = t_1 \in E_P$ such that $t_1 \equiv i + j (\bmod\, m)$. By assumption

$$A(i \oplus j) = A(t_1) = t_1 \equiv i + j (\bmod\, m). \tag{2.1}$$

Also, $A(i) = r_1$, $A(j) = r_2$, so that

$$r_1 \equiv i + \frac{m}{2} (\bmod\, m) \qquad \text{and} \qquad r_2 \equiv j + \frac{m}{2} (\bmod\, m).$$

Since $r_1, r_2 \in O_P$, it follows from the last equations and the definition of $\oplus$ that

$$A(i) \oplus A(j) = r_1 \oplus r_2 \equiv r_1 + r_2 \equiv i + j (\bmod\, m). \tag{2.2}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, it follows from equations 2.1, 2.2 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(*b*). $i, j \in E_P$. By the definition of $\oplus$, $i \oplus j = t_1 \in E_P$, $t_1 \equiv i + j (\bmod\, m)$. By assumption

$$A(i \oplus j) = A(t_1) = t_1 \equiv i + j (\bmod\, m). \tag{2.3}$$

We also have $A(i) = i$, $A(j) = j$. Hence, by the definition of $\oplus$,

$$A(i) \oplus A(j) = i \oplus j \equiv i + j (\bmod\, m). \tag{2.4}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, it follows from equations 2.3, 2.4 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(*c*). $i \in O_P$ and $j \in E_P$. By the definition of $\oplus$, $i \oplus j = t_1 \in O_P$ such that $t_1 \equiv i + j (\bmod\, m)$. By assumption

$$A(i \oplus j) = A(t_1) = r_1 \equiv t_1 + \frac{m}{2} \equiv i + j + \frac{m}{2} (\bmod\, m). \tag{2.5}$$

Also $A(j) = j$ and $A(i) = r_1$ such that $r_1 \equiv i + \frac{m}{2} (\bmod\, m)$. By the definition of $\oplus$,

$$A(i) \oplus A(j) = r_1 \oplus j \equiv r_1 + j \equiv i + \frac{m}{2} + j (\bmod\, m). \tag{2.6}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, it follows from equations 2.5, 2.6 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(d). $i \in E_P$ and $j \in O_P$. The proof is similar to (c) and so it is omitted.

2. $i, j \in H(n)$. This condition is the second case we consider in the four subcases:

(a) . $i, j \in O_H$. By the definition of $\oplus$, $i \oplus j = t_1 \in E_P$ such that $t_1 \equiv i + j (\bmod m)$. By assumption

$$A(i \oplus j) = A(t_1) = t_1 \equiv i + j (\bmod m). \tag{2.7}$$

Also, $A(i) = r_1 + m$, $A(j) = r_2 + m$ such that

$$r_1 \equiv i + \frac{m}{2} (\bmod m) \qquad \text{and} \qquad r_2 \equiv j + \frac{m}{2} (\bmod m).$$

Since $r_1, r_2 \in O_P$, by the last equations and the definition of $\oplus$, we get

$$A(i) \oplus A(j) = (r_1 + m) \oplus (r_2 + m) \equiv r_1 + r_2 \equiv i + j (\bmod m). \tag{2.8}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, it follows from equations 2.7, 2.8 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(b). $i, j \in E_H$. The proof is similar to 1 (b) and so it is omitted.

(c). $i \in O_H$ and $j \in E_H$. By the definition of $\oplus$, $i \oplus j = t_1 \in O_P$ such that $t_1 \equiv i + j (\bmod m)$. By assumption

$$A(i \oplus j) = A(t_1) = r_1 \equiv t_1 + \frac{m}{2} \equiv i + j + \frac{m}{2} (\bmod m). \tag{2.9}$$

Also $A(j) = j$ and $A(i) = r_1 + m \in O_H$ such that $r_1 \equiv i + \frac{m}{2} (\bmod m)$. By the definition of $\oplus$,

$$A(i) \oplus A(j) = (r_1 + m) \oplus j \equiv (r_1 + m) + j \equiv i + \frac{m}{2} + j (\bmod m). \tag{2.10}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, it follows from equations 2.9, 2.10 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(d). $i \in E_H$ and $j \in O_H$. By the definition of $\oplus$, $i \oplus j = s \in O_P$ such that $s \equiv i + j + \frac{m}{2} (\bmod m)$. By assumption

$$A(i \oplus j) = A(s) = r \equiv s + \frac{m}{2} \equiv i + j (\bmod m). \tag{2.11}$$

Also $A(i) = i$ and $A(j) = r_1 + m \in O_H$ such that $r_1 \equiv j + \frac{m}{2} (\mathrm{mod}\, m)$. By the definition of $\oplus$,

$$A(i) \oplus A(j) = i \oplus (r_1 + m) \equiv i + (r_1 + m) + \frac{m}{2} \equiv i + j (\mathrm{mod}\, m). \quad (2.12)$$

Since $A(i) \oplus A(j), A(i \oplus j) \in P(n)$, if follows from equations 2.11, 2.12 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

3. $i \in P(n)$ and $j \in H(n)$. This condition is the third case we consider in the four cases:

(a). $i \in O_P$ and $j \in O_H$. By the definition of $\oplus$, $i \oplus j = t + m \in E_H$ such that $t \in P(n)$ and $t \equiv i + j (\mathrm{mod}\, m)$. By assumption

$$A(i \oplus j) = A(t + m) = t + m \equiv i + j (\mathrm{mod}\, m). \quad (2.13)$$

Also $A(i) = r_1 \in O_P$ and $A(j) = r_2 + m \in O_H$ such that

$$r_1 \equiv i + \frac{m}{2} (\mathrm{mod}\, m) \qquad \text{and} \qquad r_2 \equiv j + \frac{m}{2} (\mathrm{mod}\, m).$$

By the definition of $\oplus$,

$$A(i) \oplus A(j) = r_1 \oplus (r_2 + m) \equiv r_1 + (r_2 + m) \equiv i + j (\mathrm{mod}\, m). \quad (2.14)$$

Since $A(i) \oplus A(j), A(i \oplus j) \in H(n)$, it follows from equations 2.13, 2.14 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(b). $i \in E_P$ and $j \in E_H$. Clearly, $A(i) = i$ and $A(j) = j$. By the definition of $\oplus$, $i \oplus j = t + m \in E_H$ such that $t \in P(n)$ and $t \equiv i + j (\mathrm{mod}\, m)$. Therefore,

$$A(i \oplus j) = A(t + m) = t + m = i \oplus j = A(i) \oplus A(j).$$

(c). $i \in O_P$ and $j \in E_H$. By the definition of $\oplus$, $i \oplus j = t + m \in O_H$ such that $t \in P(n)$ and $t \equiv i + j (\mathrm{mod}\, m)$. By assumption

$$A(i \oplus j) = A(t + m) \equiv (t + m) + \frac{m}{2} \equiv i + j + \frac{m}{2} (\mathrm{mod}\, m). \quad (2.15)$$

Also $A(j) = j$ and $A(i) = r \in O_P$ such that $r \equiv i + \frac{m}{2} (\mathrm{mod}\, m)$. By the definition of $\oplus$,

$$A(i) \oplus A(j) = r \oplus j \equiv r + j \equiv i + \frac{m}{2} + j (\mathrm{mod}\, m). \quad (2.16)$$

Since $A(i) \oplus A(j), A(i \oplus j) \in H(n)$, it follows from equations 2.15, 2.16 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

(d). $i \in E_P$ and $j \in O_H$. The proof is similar to the proof above case (c).

4. $i \in H(n)$ and $j \in P(n)$. This condition is the fourth case we consider in the four cases as follows:

(a). $i \in O_H$ and $j \in O_P$. The proof is similar to 3 (a) and so it is omitted.

(b). $i \in E_H$ and $j \in E_P$. The proof is similar to 3 (b) and so it is omitted.

(c). $i \in O_H$ and $j \in E_P$. The proof is similar to 3 (c) and so it is omitted.

(d). $i \in E_H$ and $j \in O_P$. By the definition of $\oplus$, $i \oplus j = s + m \in O_H$ such that $s \in P(n)$ and $s \equiv i + j + \frac{m}{2} (\operatorname{mod} m)$. By assumption

$$A(i \oplus j) = A(s+m) \equiv (s+m) + \frac{m}{2} \equiv i + j (\operatorname{mod} m). \tag{2.17}$$

Also $A(i) = i$ and $A(j) = r \in O_P$ such that $r \equiv j + \frac{m}{2} (\operatorname{mod} m)$. By the definition of $\oplus$,

$$A(i) \oplus A(j) = i \oplus r \equiv i + r + \frac{m}{2} \equiv i + j (\operatorname{mod} m). \tag{2.18}$$

Since $A(i) \oplus A(j), A(i \oplus j) \in H(n)$, it follows from equations 2.17, 2.18 and Lemma 2.2 that $A(i \oplus j) = A(i) \oplus A(j)$.

This completes the proof. □

Notation 2.4. Assume Notation 2.1. Set $M = [O_P \times (O_H \cup E_H)] \bigcup [O_H \times (O_P \cup E_H)] \bigcup [E_H \times (O_P \cup O_H)]$. Define $\operatorname{gyr} : G_2(n) \times G_2(n) \to \operatorname{Aut}(G_2(n), \oplus)$ as follows:

$$\operatorname{gyr}(a,b) = \operatorname{gyr}[a,b] = \begin{cases} A & : (a,b) \in M, \\ I & : \text{otherwise}, \end{cases}$$

where $I, A \in \operatorname{Aut}(G_2(n), \oplus)$, I being the identity automorphism and A being the automorphism defined in Lemma 2.3. Obviously, the map gyr is well defined.

3. Main result

The aim of this section is to characterize gyrogroups of order 2^n constructed by the cyclic 2-group $\mathbb{Z}_{2^n}$, for $n \geqslant 3$.

Theorem 3.1. *Assuming Notations* 2.1 *and* 2.4*, the pair* $(G_2(n), \oplus)$ *is a gyrogroup.*

Proof. By Lemma 2.3, $\mathrm{gyr}[a,b] \in \mathrm{Aut}\,(G, \oplus)$. By the definition of $\oplus$, $0 \oplus i = i$ for all $i \in G_2(n)$ and so 0 is the left identity element of $G_2(n)$. The inverse, $\ominus x$, of any element $x \in G_2(n)$ is given by

$$\ominus x = \begin{cases} -x & : x \in P(n), \\ -t+m & : x = t+m \in H(n), \end{cases}$$

where $-x$ and $-t$ are, respectively, the inverse of x and t in $P(n)$.

We now prove the left loop property. Let (a,b) be an arbitrary element of $G_2(n) \times G_2(n)$. We have four cases:

1. $(a,b) \in P(n) \times P(n)$. Then $a \oplus b \in P(n)$. Clearly $(a \oplus b, b), (a,b) \notin M$. By the definition of gyr,

$$\mathrm{gyr}[a \oplus b, b] = I = \mathrm{gyr}[a,b].$$

2. $(a,b) \in H(n) \times H(n)$. In this case, we have two subcases:

(a). In the case $(a,b) \in (O_H \times O_H) \cup (E_H \times E_H)$ we have $a \oplus b \in E_P$ and $(a,b), (a \oplus b, b) \notin M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = I = \mathrm{gyr}[a,b]$.

(b). If $(a,b) \in (O_H \times E_H) \cup (E_H \times O_H)$, then $a \oplus b \in O_P$ and $(a,b), (a \oplus b, b) \in M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = A = \mathrm{gyr}[a,b]$.

3. $(a,b) \in P(n) \times H(n)$. In this case, $a \oplus b \in H$ and $a \oplus b \equiv a + b (\mathrm{mod}\, m)$. Thus we have two subcases:

(a). If $(a,b) \in (E_P \times O_H) \cup (E_P \times E_H)$, then $(a,b), (a \oplus b, b) \notin M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = I = \mathrm{gyr}[a,b]$.

(b). If $(a,b) \in (O_P \times O_H) \cup (O_P \times E_H)$, then $(a,b), (a \oplus b, b) \in M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = A = \mathrm{gyr}[a,b]$.

4. $(a,b) \in H(n) \times P(n)$. In this case, $a \oplus b \in H$ and $a \oplus b \equiv a + b (\mathrm{mod}\, m)$. Thus we have two subcases:

(a). If $(a,b) \in (O_H \times E_P) \cup (E_H \times E_P)$, then $(a,b), (a \oplus b, b) \notin M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = I = \mathrm{gyr}[a,b]$.

(*b*). If $(a, b) \in (O_H \times O_P) \cup (E_H \times O_P)$, then $(a, b), (a \oplus b, b) \in M$. Therefore, $\mathrm{gyr}[a \oplus b, b] = A = \mathrm{gyr}[a, b]$.

Therefore, the left loop property is valid.

Finally, we investigate the left gyroassociative law. We have four cases:

1. $(a, b) \in P(n) \times P(n)$. Then $a \oplus b \in P(n)$ and $\mathrm{gyr}[a, b] = I$.
We have two subcases:

(*a*) $c \in P(n)$. Thus $b \oplus c \in P(n)$ and by the definition of $\oplus$:

$$\begin{aligned} (a \oplus b) \oplus \mathrm{gyr}[a, b]c &= (a \oplus b) \oplus c (\mathrm{mod}\, m) \\ &\equiv (a \oplus b) + c (\mathrm{mod}\, m) \\ &\equiv a + b + c (\mathrm{mod}\, m) \\ &\equiv a + (b \oplus c) (\mathrm{mod}\, m) \\ &\equiv a \oplus (b \oplus c) (\mathrm{mod}\, m). \end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$.

(*b*) $c \in H(n)$. Thus $b \oplus c \in H(n)$ and the proof of this case is similar to the previous case.

2. $(a, b) \in H(n) \times H(n)$. By Notation 2.4,

$$\mathrm{gyr}[a, b] = \begin{cases} A & : \ (a, b) \in N, \\ I & : \ \text{otherwise}, \end{cases}$$

in which $N = (O_H \times E_H) \cup (E_H \times O_H) \subseteq M$. We consider two subcases:

(*a*). $(a, b) \notin N$. In this subcase, $\mathrm{gyr}[a, b] = I$ and ($a, b \in O_H$ or $a, b \in E_H$). If $a, b \in O_H$, then $a \oplus b \in E_P$. So we have the following subcases:

(*i*). $c \in P(n)$. Thus $b \oplus c \in H(n)$ and the proof of this case is similar to 1(a).

(*ii*). $c \in H(n)$. Thus $b \oplus c \in P(n)$ and the proof of this case is similar to 1(a).

If $a, b \in E_H$, then $a \oplus b \in E_P$. So we have the following cases:

(iii). $c \in O_P$. Thus $b \oplus c \in O_H$ and by the definition of $\oplus$:

$$\begin{aligned}(a \oplus b) \oplus \mathrm{gyr}[a, b]c &= (a \oplus b) \oplus c \\ &\equiv (a \oplus b) + c (\mathrm{mod}\, m) \\ &\equiv a + b + c (\mathrm{mod}\, m) \\ &\equiv a + (b + c + \frac{m}{2}) + \frac{m}{2} (\mathrm{mod}\, m) \\ &\equiv a + (b \oplus c) + \frac{m}{2} (\mathrm{mod}\, m) \\ &\equiv a \oplus (b \oplus c) (\mathrm{mod}\, m).\end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$.

(iv). If $c \in E_P$, then $b \oplus c \in E_H$. The proof of this case is similar to 1(a).

(v). If $c \in O_H$, then $b \oplus c \in O_P$. The proof of this case is similar to 2(a)(iii).

(vi). If $c \in E_H$, then $b \oplus c \in E_P$. So, the proof is similar to 1(a).

(b). $(a, b) \in N$. In this subcase $\mathrm{gyr}[a, b] = A$, also ($a \in O_H$ & $b \in E_H$) or ($a \in E_H$ & $b \in O_H$). If ($a \in O_H$ & $b \in E_H$), then $a \oplus b \in O_P$. So we have the following cases:

(i). $c \in O_P$. Thus $b \oplus c \in O_H$ and $\mathrm{gyr}[a, b](c) \in O_P$. By Lemma 2.3 and the definition of $\oplus$ we have

$$\begin{aligned}(a \oplus b) \oplus \mathrm{gyr}[a, b](c) &\equiv (a \oplus b) + \mathrm{gyr}[a, b](c) (\mathrm{mod}\, m) \\ &\equiv a + b + c + \frac{m}{2} (\mathrm{mod}\, m) \\ &\equiv a + (b \oplus c) (\mathrm{mod}\, m) \\ &\equiv a \oplus (b \oplus c) (\mathrm{mod}\, m).\end{aligned}$$

By lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b](c)$.

(ii). $c \in E_P$. Thus $b \oplus c \in E_H$ and $\mathrm{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

(iii). $c \in O_H$. Then $b \oplus c \in O_P$ and $\mathrm{gyr}[a, b](c) \in O_H$. The proof of this case is similar to 2(b)(i).

(iv). $c \in E_H$. Then $b \oplus c \in E_P$ and $\mathrm{gyr}[a, b]c = c$. The proof of this case is similar to 1(a).

If ($a \in E_H$ & $b \in O_H$) then $a \oplus b \in O_P$. So we have the following cases:

(v). $c \in O_P$. Thus $b \oplus c \in E_H$ and $\text{gyr}[a,b](c) \in O_P$. By Lemma 2.3 and the definition of $\oplus$ we have

$$\begin{aligned}(a \oplus b) \oplus \text{gyr}[a,b](c) &\equiv (a \oplus b) + \text{gyr}[a,b](c)(\text{mod}\, m)\\ &\equiv a + b + \frac{m}{2} + c + \frac{m}{2}(\text{mod}\, m)\\ &\equiv a + b + c(\text{mod}\, m)\\ &\equiv a + (b \oplus c)(\text{mod}\, m)\\ &\equiv a \oplus (b \oplus c)(\text{mod}\, m).\end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b](c)$.

(vi). $c \in E_P$. Thus $b \oplus c \in O_H$ and $\text{gyr}[a,b](c) = c$. By the definition of $\oplus$ we have

$$\begin{aligned}(a \oplus b) \oplus \text{gyr}[a,b](c) &= (a \oplus b) \oplus c(\text{mod}\, m)\\ &\equiv (a \oplus b) + c(\text{mod}\, m)\\ &\equiv a + b + \frac{m}{2} + c(\text{mod}\, m)\\ &\equiv a + (b \oplus c) + \frac{m}{2}(\text{mod}\, m)\\ &\equiv a \oplus (b \oplus c)(\text{mod}\, m).\end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b](c)$.

(vii). $c \in O_H$. Then $b \oplus c \in E_P$ and $\text{gyr}[a,b](c) \in O_H$. The proof of this case is similar to 2(b)v).

($viii$). $c \in E_H$. Then $b \oplus c \in O_P$ and $\text{gyr}[a,b](c) = c$. The proof of this case is similar to 2(b)(vi).

3. $(a,b) \in P(n) \times H(n)$. By Notation 2.4, clearly

$$\text{gyr}[a,b] = \begin{cases} A & : \ a \in O_P,\\ I & : \ a \notin O_P.\end{cases}$$

We consider two subcases:

(a). $a \notin O_P$. In this case $\text{gyr}[a,b] = I$. If $b \in O_H$, then $a \oplus b \in O_H$. So we have the following cases:

(i). $c \in P(n)$. Thus $b \oplus c \in H(n)$. So, the proof is similar to 1(a).

(ii). $c \in H(n)$. Thus $b \oplus c \in P(n)$. So, the proof is similar to 1(a).

If $b \in E_H$, then $a \oplus b \in E_H$. So we have the following cases:

(iii). $c \in O_P$. Thus $b \oplus c \in O_H$ and by the definition of $\oplus$ we have

$$\begin{aligned}
(a \oplus b) \oplus \text{gyr}[a, b](c) &= (a \oplus b) \oplus c \\
&\equiv (a \oplus b) + c + \frac{m}{2} (\text{mod}\, m) \\
&\equiv a + b + c + \frac{m}{2} (\text{mod}\, m) \\
&\equiv a + (b \oplus c) (\text{mod}\, m) \\
&\equiv a \oplus (b \oplus c) (\text{mod}\, m).
\end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b](c)$.

(iv). $c \in E_P$. Thus $b \oplus c \in E_H$ and the proof is similar to 1(a).

(v). $c \in O_H$. Thus $b \oplus c \in O_P$ and the proof is similar to 3(a)(iii).

(vi). $c \in E_H$. Thus $b \oplus c \in E_P$ and the proof is similar to 1(a).

(b). $a \in O_P$. In this case $\text{gyr}[a, b] = A$. If $b \in O_H$, then $a \oplus b \in E_H$. So we have the following cases:

(i). $c \in O_P$. Thus $b \oplus c \in E_H$ and $\text{gyr}[a, b](c) \in O_P$. By Lemma 2.3 and the definition of $\oplus$ we have

$$\begin{aligned}
(a \oplus b) \oplus \text{gyr}[a, b](c) &\equiv (a \oplus b) + \text{gyr}[a, b](c) + \frac{m}{2} (\text{mod}\, m) \\
&\equiv a + b + c + \frac{m}{2} + \frac{m}{2} (\text{mod}\, m) \\
&\equiv a + b + c (\text{mod}\, m) \\
&\equiv a + (b \oplus c) (\text{mod}\, m) \\
&\equiv a \oplus (b \oplus c) (\text{mod}\, m).
\end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b](c)$.

(ii). $c \in E_P$. Thus $b \oplus c \in O_H$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

(iii). $c \in O_H$. Thus $b \oplus c \in E_P$ and $\text{gyr}[a, b](c) \in O_H$. The proof of this case is similar to 3(b)(i).

(iv). $c \in E_H$. Thus $b \oplus c \in O_P$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

If $b \in E_H$, then $a \oplus b \in O_H$. So we have the following cases:

(v). $c \in O_P$. Thus $b \oplus c \in O_H$ and $\text{gyr}[a, b](c) \in O_P$. The proof of this case is similar to 2(b)(i).

(vi). $c \in E_P$.Thus $b \oplus c \in E_H$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

(vii). $c \in O_H$. Thus $b \oplus c \in O_P$ and $\text{gyr}[a, b](c) \in O_H$. The proof of this case is similar to 2(b)(i).

($viii$). $c \in E_H$. Thus $b \oplus c \in E_P$ and $\text{gyr}[a, b](c) = c$. and the proof of this case is similar to 1(a).

4. $(a, b) \in H(n) \times P(n)$. By Notation 2.4, clearly

$$\text{gyr}[a, b] = \begin{cases} A & : \ b \in O_P, \\ I & : \ b \notin O_P. \end{cases}$$

We consider two subcases:

(a). $b \notin O_P$. In this case $\text{gyr}[a, b] = I$. If $a \in O_H$, then $a \oplus b \in O_H$. So we have the following subcases:

(i). $c \in P(n)$. Thus $b \oplus c \in P(n)$ and the proof of this case is similar to 1(a).

(ii). $c \in H(n)$. So $b \oplus c \in H(n)$ and the proof of this case is similar to 1(a).

If $a \in E_H$, then $a \oplus b \in E_H$. So we have the following cases:

(iii). $c \in O_P$. Thus $b \oplus c \in O_P$ and by the definition of $\oplus$ we have

$$\begin{aligned} (a \oplus b) \oplus \text{gyr}[a, b](c) &= (a \oplus b) \oplus c \\ &\equiv (a \oplus b) + c + \frac{m}{2} (\text{mod}\, m) \\ &\equiv a + b + c + \frac{m}{2} (\text{mod}\, m) \\ &\equiv a + (b \oplus c) + \frac{m}{2} (\text{mod}\, m) \\ &\equiv a \oplus (b \oplus c) (\text{mod}\, m). \end{aligned}$$

By Lemma 2.2, $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b](c)$.

(iv). $c \in E_P$.Thus $b \oplus c \in E_P$ and the proof of this case is similar to 1(a).

(v). $c \in O_H$. Thus $b \oplus c \in O_H$ and the proof of this case is similar to 4(a)(iii).

(vi). $c \in E_H$. Thus $b \oplus c \in E_H$ and the proof of this case is similar to 1(a).

(b). $b \in O_P$. In this case $\text{gyr}[a, b] = A$. If $a \in O_H$, then $a \oplus b \in E_H$. So we have the following cases:

(i). $c \in O_P$. Thus $b \oplus c \in E_P$ and $\text{gyr}[a, b](c) \in O_P$. The proof of this case is similar to 3(b)(i).

(ii). $c \in E_P$. Thus $b \oplus c \in O_P$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

(iii). $c \in O_H$. Thus $b \oplus c \in E_H$ and $\text{gyr}[a, b](c) \in O_H$. The proof of this case is similar to 3(b)(i).

(iv). $c \in E_H$. Thus $b \oplus c \in O_H$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 1(a).

If $a \in E_H$, then $a \oplus b \in O_H$. So we have the following cases:

(v). $c \in O_P$. Thus $b \oplus c \in E_P$ and $\text{gyr}[a, b](c) \in O_P$. The proof of this case is similar to 2(b)(v).

(vi). $c \in E_P$. Thus $b \oplus c \in O_P$ and $\text{gyr}[a, b](c) = c$. The proof of this case is similar to 2(b)(vii).

(vii). $c \in O_H$. Thus $b \oplus c \in E_H$ and $\text{gyr}[a, b](c) \in O_H$. The proof of this case is similar to 2(b)(v).

$(viii)$. $c \in E_H$. Thus $b \oplus c \in O_H$ and $\text{gyr}[a, b](c) = c$. and the proof of this case is similar to 2(b)(vi).

By the above mentions, $(G_2(n), \oplus)$ is a gyrogroup and this completes the proof. □

Mahdavi et al. in [8, Theorem 2] proved that the gyrogroup $G_1(n)$ is non-gyrocommutative. In contrast, in the following theorem we show that the gyrogroup $G_2(n)$ is gyrocommutative. Consequently, the gyrogroups $G_1(n)$ and $G_2(n)$ are not isomorphic.

Theorem 3.2. *The gyrogroup $(G_2(n), \oplus)$ is gyrocommutative.*

Proof. Suppose (a, b) is an arbitrary member of $G_2(n) \times G_2(n)$. We consider two cases:

1. $(a,b) \notin M$. In this case gyr$[a,b]$=I. By the definition of $\oplus$,

$$\mathrm{gyr}[a,b](b\oplus a) = b\oplus a \equiv a+b \equiv a\oplus b (\mathrm{mod}\, m).$$

   By Lemma 2.2, gyr$[a,b](b\oplus a) = a\oplus b$.

2. $(a,b) \in M$. In this case gyr$[a,b] = A$. If $b\oplus a$ is an even number, then $(a,b) \in S \cup S^{-1}$ in which $S = O_P \times O_H$. By Lemma 2.3 and the definition of $\oplus$ we have

$$\mathrm{gyr}[a,b](b\oplus a) = b\oplus a \equiv a+b \equiv a\oplus b (\mathrm{mod}\, m).$$

   By Lemma 2.2, gyr$[a,b](b\oplus a) = a\oplus b$. If $b\oplus a$ is an odd number, then $(a,b) \in T \cup T^{-1}$ in which $T = E_H \times (O_P \cup O_H)$. Now, we consider two subcases:

   (i). $(a,b) \in T$. So $(b,a) \in T^{-1}$. Thus, by Lemma 2.3 and the definition of $\oplus$,

$$\mathrm{gyr}[a,b](b\oplus a) \equiv b\oplus a + \frac{m}{2} \equiv a+b+\frac{m}{2} \equiv a\oplus b (\mathrm{mod}\, m).$$

   By Lemma 2.2, gyr$[a,b](b\oplus a) = a\oplus b$.

   (ii). $(a,b) \in T^{-1}$. So $(b,a) \in T$. Thus, by Lemma 2.3 and the definition of $\oplus$,

$$\mathrm{gyr}[a,b](b\oplus a) \equiv b\oplus a + \frac{m}{2} \equiv a+b+\frac{m}{2}+\frac{m}{2} \equiv a+b \equiv a\oplus b (\mathrm{mod}\, m).$$

   By Lemma 2.2, gyr$[a,b](b\oplus a) = a\oplus b$.

This completes the proof of gyrocommutativity. □

Example 3.3. We investigate the gyrogroup $G_2(3)$ of order 8 constructed by the cyclic group $\mathbb{Z}_8$. By definition, $G_2(3) = \{0,1,2,3,4,5,6,7\}$ and the binary operation $\oplus$ is defined as follows:

$$i\oplus j = \begin{cases} t & : (i,j) \in (P(3)\times P(3)) \cup \Big[(H(3)\times H(3)) - (E_H \times O_H)\Big], \\ t+4 & : (i,j) \in (P(3)\times H(3)) \cup \Big[(H(3)\times P(3)) - (E_H \times O_P)\Big], \\ s & : (i,j) \in E_H \times O_H, \\ s+4 & : (i,j) \in E_H \times O_P \end{cases}$$

in which $P(3) = \{0, 1, 2, 3\}$, $H(3) = \{4, 5, 6, 7\}$ and $t, s \in P(3)$. Also

$$\begin{cases} t \equiv i + j (\mathrm{mod}\, 4), \\ s \equiv i + j + 2 (\mathrm{mod}\, 4). \end{cases}$$

The resulting addition table and the gyration table for $G_2(3)$ are presented in Table 1 in which A is the unique non-identity gyroautomorphism of $G_2(3)$ given by $A = (1, 3)(5, 7)$.

Table 1:

(a) The Cayley table of $G_2(3)$

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	6	7	4	1	2	3	0
6	6	5	4	7	2	1	0	3
7	7	4	5	6	3	0	1	2

(b) The gyration table of $G_2(3)$

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	I	I	A	A	A	A
2	I	I	I	I	I	I	I	I
3	I	I	I	I	A	A	A	A
4	I	A	I	A	I	A	I	A
5	I	A	I	A	A	I	A	I
6	I	A	I	A	I	A	I	A
7	I	A	I	A	A	I	A	I

Example 3.4. In this example, we investigate the gyrogroup $G_2(4)$ of order 16 constructed by the cyclic group $\mathbb{Z}_{16}$. By definition,

$$G_2(4) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

and the binary operation $\oplus$ is defined by

$$i \oplus j = \begin{cases} t & : (i, j) \in (P(4) \times P(4)) \cup \Big[(H(4) \times H(4)) - (E_H \times O_H)\Big], \\ t + 8 & : (i, j) \in (P(4) \times H(4)) \cup \Big[(H(4) \times P(4)) - (E_H \times O_P)\Big], \\ s & : (i, j) \in E_H \times O_H, \\ s + 8 & : (i, j) \in E_H \times O_P, \end{cases}$$

where $P(4) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $H(4) = \{8, 9, 10, 11, 12, 13, 14, 15\}$ and $t, s \in P(4)$. Also

$$\begin{cases} t \equiv i + j (\mathrm{mod}\, 8), \\ s \equiv i + j + 4 (\mathrm{mod}\, 8). \end{cases}.$$

The resulting addition table and the gyration table for $G_2(4)$ are presented in Tables 2 and 3, respectively, in which A is the unique non-identity gyroautomorphism of $G_2(4)$ given by $A = (1,5)(3,7)(9,13)(11,15)$.

Table 2: The Cayley table of $G_2(4)$

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	0	9	10	11	12	13	14	15	8
2	2	3	4	5	6	7	0	1	10	11	12	13	14	15	8	9
3	3	4	5	6	7	0	1	2	11	12	13	14	15	8	9	10
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	6	7	0	1	2	3	4	13	14	15	8	9	10	11	12
6	6	7	0	1	2	3	4	5	14	15	8	9	10	11	12	13
7	7	0	1	2	3	4	5	6	15	8	9	10	11	12	13	14
8	8	13	10	15	12	9	14	11	0	5	2	7	4	1	6	3
9	9	10	11	12	13	14	15	8	1	2	3	4	5	6	7	0
10	10	15	12	9	14	11	8	13	2	7	4	1	6	3	0	5
11	11	12	13	14	15	8	9	10	3	4	5	6	7	0	1	2
12	12	9	14	11	8	13	10	15	4	1	6	3	0	5	2	7
13	13	14	15	8	9	10	11	12	5	6	7	0	1	2	3	4
14	14	11	8	13	10	15	12	9	6	3	0	5	2	7	4	1
15	15	8	9	10	11	12	13	14	7	0	1	2	3	4	5	6

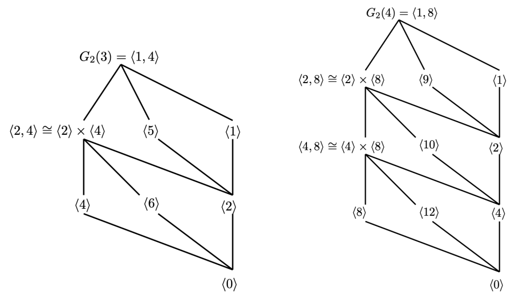
Table 3: The gyration table of $G_2(4)$

gyr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
1	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
2	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
3	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
4	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
5	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
6	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
7	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
8	I	A	I	A	I	A	I	A	I	A	I	A	I	A	I	A
9	I	A	I	A	I	A	I	A	A	I	A	I	A	I	A	I
10	I	A	I	A	I	A	I	A	I	A	I	A	I	A	I	A
11	I	A	I	A	I	A	I	A	A	I	A	I	A	I	A	I
12	I	A	I	A	I	A	I	A	I	A	I	A	I	A	I	A
13	I	A	I	A	I	A	I	A	A	I	A	I	A	I	A	I
14	I	A	I	A	I	A	I	A	I	A	I	A	I	A	I	A
15	I	A	I	A	I	A	I	A	A	I	A	I	A	I	A	I

Remark 3.5. By using the identity, $\mathrm{gyr}[a,b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$, one can simply calculate the group of all gyroautomorphisms of $G_2(n)$ con-

taining $\mathbb{Z}_2$. The group of gyroautomorphisms of $G_2(4)$ is isomorphic to $\mathbb{Z}_2$, by which we calculated the gyrosemidirect product of this gyrogroup with the following structure, $\mathbb{Z}_2 \times (\mathbb{Z}_8 \rtimes \mathbb{Z}_2)$.

FIGURE 1. The lattice of $G(3)$ and $G(4)$.



(A) The lattice of $G_2(3)$. (B) The lattice of $G_2(4)$.

In the next theorem, we will see that all proper subgyrogroups of $G_2(n)$ have trivial gyroautomorphisms and, hence, they are subgroups.

Theorem 3.6. *Let $G_2(n)$ be the 2-gyrogroup, by the means of the definition of $p-$gyrogroups in [7], of order 2^n. H is subgyrogroup of $G_2(n)$ if and only if it has one of the following forms:*

1. *$H = \langle 2^s \rangle$ is a subgroup of $P(n)$ such that $0 \leqslant s \leqslant n-1$.*
2. *$H = \langle 2^s, m \rangle$ such that $0 \leqslant s \leqslant n-1$ and $m = 2^{n-1}$. All H's are subgroups of $G_2(n)$ except $H = \langle 1, m \rangle = G_2(n)$*
3. *$H = \langle m + 2^s \rangle$ is a subgroup of $G_2(n)$ such that $0 \leqslant s \leqslant n-2$ and $m = 2^{n-1}$.*

Proof. We know that $G = P(n) \cup H(n)$ such that $P(n) \cap H(n) = \emptyset$. Since the restriction of $\oplus$ to $P(n)$ is the group addition, $P(n) \cong \mathbb{Z}_m$ where $m = 2^{n-1}$. Suppose H is a subgyrogroup of $G(n)$. If $H \subseteq P(n)$, then H will be a subgroup of $P(n)$ and $H = \langle 2^s \rangle$ such that $0 \leqslant s \leqslant n-1$, as desired. But if $H \nsubseteq P(n)$, then $H = H_1 \cup H_2$ in which $H_1 \subseteq P(n)$ and $H_2 \subseteq H(n)$. It is easy to see that H_1 is a subgroup of $P(n)$. We consider two cases as follows:

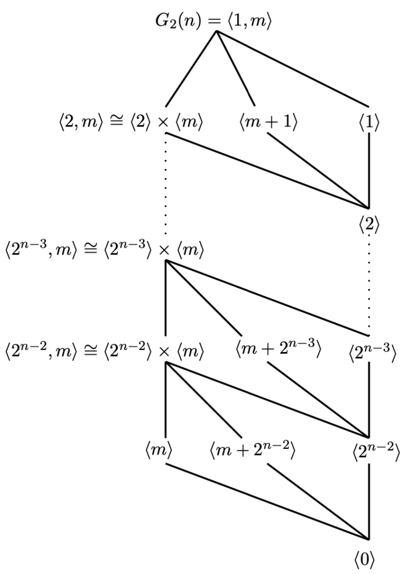
1. $m \in H_2$. In this case, the map $L_m \colon H_1 \to H_2$ defined by $L_m(x) = m \oplus x$ is bijective. Therefore, $H_2 = m \oplus H_1$. Since $H_1 \leqslant P(n) \cong \mathbb{Z}_m$, $H = H_1 \cup H_2 = H_1 \cup (m \oplus H_1) = \langle 2^s, m \rangle$ such that $0 \leqslant s \leqslant n-1$. For $s = 0$ and $s = n-1$, $\langle 1, m \rangle = G(n)$ and $\langle 0, m \rangle = \langle m \rangle$, respectively. If $s \neq 0$,

then by the definition of the gyroautomorphisms of $G_2(n)$, all the gyroautomorphisms of H are the identity automorphism. Therefore, H's are subgroups of $G_2(n)$ and $H \cong \langle 2^s \rangle \times \langle m \rangle$.

2. $m \notin H_2$. There exists an integer $i \in \mathbb{Z}_m$ such that $i \neq 0$ and i is the smallest number in which $m+i \in H_2$. Also, the map $L_{m+i} \colon H_1 \to H_2$ defined by $L_{m+i}(x) = (m+i) \oplus x$ is bijective. Therefore $H_2 = (m+i) \oplus H_1$. By the definition of $\oplus$, $(m+i) \oplus (m+i) = 2i \in H_1$. Since i is the smallest number that has been chosen and $H_1 \leqslant P(n) \cong \mathbb{Z}_m$, then $H_1 = \langle 2i \rangle$ and $H = H_1 \cup H_2 = H_1 \cup \Big((m+i) \oplus H_1\Big) = \langle 2i, m+i \rangle = \langle m+i \rangle$ such that $i = 2^s$ and $0 \leqslant s \leqslant n-2$.

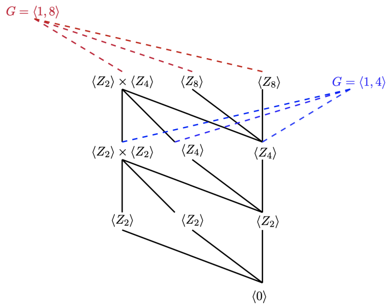
Clearly, $\langle 2^s \rangle \leqslant \langle 2^s, m \rangle$, $\langle m + 2^s \rangle \leqslant \langle 2^s, m \rangle$ and $\langle m \rangle \leqslant \langle 2^{n-2}, m \rangle \leqslant \langle 2^{n-3}, m \rangle \leqslant \cdots \leqslant \langle 2, m \rangle \leqslant \langle 1, m \rangle$. □

FIGURE 2. The lattice of $G_2(n)$.



Remark 3.7. As one can see from Figure 3 all proper subgyrogroups of $G_2(n)$ are included in subgyrogroups of $G_2(n+1)$.

FIGURE 3. The lattice of $G_2(3)$ and $G_2(4)$ together.



References

[1] **A.R. Ashrafi, K. Mavaddat Nezhaad, and M.A. Salahshour**, *Classification of gyrogroups of orders at most* 31, AUT J. Math. Computing **5** (2024), 11 – 18.

[2] **J. Chen and A.A. Ungar**, *From the group $SL(2, C)$ to gyrogroups and gyrovector spaces and hyperbolic geometry*, Foundations of Physics, **31** (11) (2001), 1611 – 1639.

[3] **T. Feder**, *Strong near subgroups and left gyrogroups*, J. Algebra, **259** (2003), 177 – 190.

[4] **T. Foguel and A.A. Ungar**, *Involutory decomposition of groups into twisted subgroups and subgroups*, J. Group Theory, **3** (2000), 27 – 46.

[5] **T. Foguel and A.A. Ungar**, *Gyrogroups and the decomposition of groups into twisted subgroups and subgroups*, Pacific J. Math., **197** (2001), 1 – 11.

[6] **A. Kerber**, *Applied Finite Group Actions*, Second edition, Algorithms and Combinatorics **19**, Springer-Verlag, Berlin, 1999.

[7] **S. Mahdavi, A.R. Ashrafi and M.A. Salahshour**, *Construction of new gyrogroups and the structure of their subgyrogroups*, Algebraic Structures and Their Applications, **8** (2) (2021), 17 – 30.

[8] **S. Mahdavi, A.R. Ashrafi, M.A. Salahshour and A.A. Ungar**, *Construction of* 2*-gyrogroups in which every proper subgyrogroup is either a cyclic or a dihedral group*, Symmetry, **13** (2021), 316.

[9] **R. Maungchang, T. Suksumran**, *On dihedralized gyrogroups and their Cayley graphs*, Mathematics, **10** (2022), 2276.

[10] **K. Sangkhanan and T. Suksumran**, *On generalized Heisenberg group: the symmetric case*, Results Math., **73** (2018), 91.

[11] **T. Suksumran**, *The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems*, In: Rassias, T., Pardalos, P. (eds), *Essays in Mathematics and its Applications*, Springer, Cham, 2016, 369 – 437.

[12] **T. Suksumran**, *Involutive groups, unique 2-divisibility, and related gyrogroup structures*, J. Algebra Appl., **16** (2017), 1750114.

[13] **T. Suksumran**, *On commutator-inversion invariant groups and gyrogroup construction*, J. Algebra Appl., (2022), 2450042.

[14] **T. Suksumran and K. Wiboonton**, *Lagranges theorem for gyrogroups and the Cauchy property*, Quasigroups and Related Systems, **22** (2014), 283 – 294.

[15] **T. Suksumran and K. Wiboonton**, *Isomorphism theorems for gyrogroups and L-subgyrogroups*, J. Geom. Symmetry Phys., **37** (2015), 67 – 83.

[16] **The Gap Team**, GAP–Groups, Algorithms and Programming, Lehrstuhl De für Mathematik, RWTH, Aachen, 1995.

[17] **A.A. Ungar**, *Thomas rotation and parametrization of the Lorentz transformation group*, Found. Phys. Lett., **1** (1988), 57 – 89.

[18] **A.A. Ungar**, *The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities*, Appl. Math. Lett., **1** (1988), 403 – 405.

[19] **A.A. Ungar**, *Parametric realization of the Lorentz transformation group in pseudo-euclidean spaces*, J. Geom. Symmetry Phys., **38** (2015), 39 – 108.

[20] **A.A. Ungar**, *The intrinsic beauty, harmony and interdisciplinarity in Einstein velocity addition law: Gyrogroups and gyrovector spaces*, Math. Interdisc. Res., **1** (2016), 5 – 51.

[21] **A.A. Ungar**, *Beyond pseudo-rotations in pseudo-euclidean spaces: An introduction to the theory of bi-gyrogroups and bi-gyrovector spaces*, Mathematical Analysis and its Applications, Elsevier/Academic Press, 2018.

[22] **A.A. Ungar**, *Hyperbolic trigonometry in the Einstein relativistic velocity model of hyperbolic geometry*, Computers Math. Applic., **40** (2000), 313 – 332.

[23] **A.A. Ungar**, *Beyond the Einstein addition law and its gyroscopic Thomas precession. The theory of gyrogroups and gyrovector spaces*, Fundamental Theories Physics, **117**, Kluwer Academic Publishers Group, Dordrecht, 2001.

[24] **J. Zhang**, *Constructing finite algebras with FALCON*, J. Automated Reasoning, **17** (1996), 1 – 22.

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