Quasigroups and Related Systems **32** (2024), 313 – 318

https://doi.org/10.56415/qrs.v32.22

# A correspondence induced by an involution centralizing an index-two subgroup

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**Abstract.** Given groups $(G, \circ)$ and $(H, \star)$ with disjoint support, we remark that the isomorphisms $(G, \circ) \to (H, \star)$ are in one-to-one correspondence with certain group operations on $G \cup H$ extending the operation of $G$ and interacting nicely with the structure on $H$. This correspondence follows from some general properties of groups admitting an index-two subgroup $N$ with an involution $x \notin N$ which centralizes $N$.

## 1. Introduction and main result

Let $(G, \circ)$ and $(H, \star)$ be groups with disjoint support. In this note, we highlight a one-to-one correspondence between the isomorphisms $(G, \circ) \to (H, \star)$ and certain groups $(G \cup H, \circledcirc)$ such that $\circledcirc$ extends $\circ$ and interacts nicely with the structure on $H$. The precise statement is as follows.

**Theorem 1.1.** *Let $(G, \circ)$ and $(H, \star)$ be groups such that $G \cap H = \varnothing$. Then the (possibly empty) set of all group isomorphisms $f : G \to H$ is in one-to-one correspondence with the set of all group operations $\circledcirc$ on $G \cup H$ satisfying the following properties:*

(*i*) *$\circledcirc$ extends $\circ$ .*

(*ii*) *The identity element $e_\star$ of $(H, \star)$ has order two in $(G \cup H, \circledcirc)$ and lies in the centralizer of $G$ in $(G \cup H, \circledcirc)$.*

(*iii*) *For all $h, h' \in H$, we have $h \star h' = h \circledcirc e_\star \circledcirc h'$.*

The proof of Theorem 1.1 is elementary, incorporating some constructions appearing in a previous work of the author, namely [4]. (Note: certain passages from this reference are recycled here verbatim, for the sake of self-containment.) As we shall see, the theorem follows from general properties of groups admitting an index-two subgroup $N$ and an involution $x \notin N$ which centralizes $N$. Our techniques resemble the well-known "group doubling" constructions from the theory of Moufang loops; see [1] and [2]. We conclude this note with some related observations concerning a peculiar nonassociative operation.

Before proceeding, let us specify our notation: In the group $(X, \circ)$, $e_\circ$ denotes the identity, and $x^\circ$ denotes the inverse of $x \in X$. We shall abuse notation slightly: If $G \subseteq X$

2010 Mathematics Subject Classification: 20A05

Keywords: group isomorphism, one-to-one correspondence, extension, translation, index-two subgroup.

is closed under the operation $\circ$, then we shall denote the algebraic system $(G, \circ|_{G\times G})$ simply by $(G, \circ)$.

We begin with a simple lemma, whose proof is an exercise.

**Lemma 1.2.** *Let $(G, \circ)$ be a group, and fix $z \in G$. Define a binary operation $\triangle$ on $G$ by*

$$x \triangle y := x \circ z^{\circ} \circ y$$

*for all $x, y \in G$. Then the right translation map $f : G \to G$ given by $f(x) = x \circ z$ is an isomorphism $(G, \circ) \to (G, \triangle)$.*

**Remark 1.3.** The operations $\circ$ and $\triangle$ in Lemma 1.2 are readily checked to *biassociate*, i.e.,

$$(a \circ b) \triangle c = a \circ (b \triangle c) \quad \text{and} \quad (a \triangle b) \circ c = a \triangle (b \circ c),$$

for all $a, b, c \in G$. Biassociative operations are isomorphic in fairly generic settings. For instance, let $(X, \circ)$, $(X, \triangle)$ be magmas such that $\circ$ and $\triangle$ biassociate, and suppose that $(X, \triangle)$ possesses an identity $e_\triangle$. The right translation map $f(x) = x \circ e_\triangle$ is then a homomorphism:

$$f(x) \triangle f(y) = (x \circ e_\triangle) \triangle (y \circ e_\triangle) = ((x \circ e_\triangle) \triangle y) \circ e_\triangle = (x \circ (e_\triangle \triangle y)) \circ e_\triangle = (x \circ y) \circ e_\triangle = f(x \circ y).$$

In particular, if $f$ is a bijection, then $(X, \circ) \cong (X, \triangle)$. In fact, if $\circ$ and $\triangle$ are biassociative group operations on $X$, then necessarily

$$x \triangle y = x \circ e_\triangle^{\circ} \circ y.$$

(See [3].)

The following proposition is important for Theorem 1.1.

**Proposition 1.4.** *Let $(G, \circ)$ and $(H, \star)$ be groups such that $G \cap H = \varnothing$. Suppose that $\circ$ extends to a group operation $\circledcirc$ on $G \cup H$ such that for all $h, h' \in H$, we have $h \star h' = h \circledcirc e_\star^{\circledcirc} \circledcirc h'$. Then $(G, \circ) \cong (H, \star)$.*

*Proof.* Define a binary operation $\circledast$ on $G \cup H$ by

$$x \circledast y := x \circledcirc e_\star^{\circledcirc} \circledcirc y.$$

By the lemma, $(G \cup H, \circledcirc) \cong (G \cup H, \circledast)$ via the translation map $f : G \cup H \to G \cup H$ given by $f(x) = x \circledcirc e_\star$. Since $\circledast$ extends $\star$, it suffices to show that $f(G) = H$. To see that $f(G) \subseteq H$, let $g \in G$, and assume that $g \circledcirc e_\star = y \in G$. One readily observes that $e_\circledcirc = e_\circ$, and it follows that $g^{\circledcirc} = g^{\circ} \in G$. Therefore, we obtain $e_\star = g^{\circ} \circledcirc y = g^{\circ} \circ y \in G \cap H$, a contradiction. Now, the inverse of $f$ is the map $f^{-1}(x) = x \circledcirc e_\star^{\circledcirc} = x \circledast e_\circ$, and a symmetric argument gives $f^{-1}(H) \subseteq G$, as needed. □

**Remark 1.5.** If $(X, \circ)$ is a group and $N$ is a proper subgroup, then we may fix $x \in X$ with $x \notin N$ and define a binary operation $\triangle$ on $X$ by $a \triangle b = a \circ x^{\circ} \circ b$. The right coset of $N$ containing $x$, namely $N \circ x$, is closed under $\triangle$. In fact,

$$(N, \circ) \cong (N \circ x, \triangle). \tag{1}$$

This follows, for instance, from Lemma 1.2; the right translation map $f : X \to X$ given by $a \mapsto a \circ x$ is an isomorphism $(X, \circ) \to (X, \triangle)$, but $f$ restricted to $N$ gives a bijection $N \to N \circ x$. Now, Proposition 1.4 may be interpreted as a consequence of (1). Indeed, after defining the operation $\circledast$, we get that $(G, \circledcirc) \cong (G \circledcirc e_\star, \circledast)$ via the translation map $f(a) = a \circledcirc e_\star$. It then suffices to argue that the coset $G \circledcirc e_\star$ equals $H$. (So, $G$ has index two as a subgroup of $(G \cup H, \circledcirc)$.)

The next proposition supplies the converse of Proposition 1.4.

**Proposition 1.6.** *Let $(G, \circ)$ and $(H, \star)$ be groups with disjoint support. Suppose that $f : (G, \circ) \to (H, \star)$ is an isomorphism. Then $\circ$ extends to a group operation $\circledcirc$ on $G \cup H$ such that for all $h, h' \in H$, we have $h \star h' = h \circledcirc e_\star^\circledcirc \circledcirc h'$.*

*Proof.* Extend $\circ$ to the binary operation $\diamond_f$ on $G \cup H$ given by

$$x \diamond_f y = \begin{cases} x \circ y & \text{if } x, y \in G \\ f^{-1}(x) \circ f^{-1}(y) & \text{if } x, y \in H \\ x \star f(y) & \text{if } x \in H, y \in G \\ f(x) \star y & \text{if } x \in G, y \in H. \end{cases}$$

We claim that $(G \cup H, \diamond_f)$ is a group. To see this, consider the direct product $(G, \circ) \times C_2$, where $C_2 = \{0, 1\}$ is a group of order 2. One can check that the mapping

$$\phi : G \times C_2 \to G \cup H$$

given by

$$\phi(x, i) = \begin{cases} x & \text{if } i = 0 \\ f(x) & \text{if } i = 1 \end{cases}$$

is a structure-preserving bijection between $G \times C_2$ and $(G \cup H, \diamond_f)$, and since $G \times C_2$ is a group, $(G \cup H, \diamond_f)$ must be as well. (The author initially proved that $(G \cup H, \diamond_f)$ is a group directly from the group axioms. Amitai Yuval provided this more insightful argument via [5]). Now, take $\circledcirc = \diamond_f$. Notice that the inverse of $e_\star$ in the group $(G \cup H, \diamond_f)$ is just $e_\star$. Thus, for $h, h' \in H$, we calculate

$$h \circledcirc e_\star^\circledcirc \circledcirc h' = h \diamond_f e_\star \diamond_f h' = (f^{-1}(h) \circ f^{-1}(e_\star)) \diamond_f h' = f^{-1}(h) \diamond_f h' = h \star h',$$

completing the proof. □

Hence, we have the following corollary, which refines the result of [4].

**Corollary 1.7.** *Let $(G, \circ)$ and $(H, \star)$ be groups with disjoint support. Then $(G, \circ) \cong (H, \star)$ if and only if $\circ$ extends to a group operation $\circledcirc$ on $G \cup H$ such that for all $h, h' \in H$, we have $h \star h' = h \circledcirc e_\star^\circledcirc \circledcirc h'$.*

**Remark 1.8.** Given that the algebraic system $(G \cup H, \diamond_f)$ from the proof of Proposition 1.6 is a group, it is worth noting how the isomorphism $(G \cup H, \diamond_f) \cong (G, \circ) \times C_2$ is a particular instance of a general group-theoretic principle. To see this, consider the following setup: $X$ is a group (whose operation we shall denote simply by juxtaposition). Let $N$ be a subgroup of $X$ of index two. Suppose there exists $x \in X$ of order two with

$x \notin N$. Assume further that $x$ lies in the centralizer of $N$ in $X$. We may then conclude that $X \cong N \times C_2$. Indeed, the mapping $\phi : N \times C_2 \to X$ defined by

$$\phi(n,i) = \begin{cases} n & \text{if } i = 0 \\ nx & \text{if } i = 1 \end{cases}$$

is bijective. It is also a homomorphism; for instance, if $n_1, n_2 \in N$, then we have

$$\phi[(n_1,1)(n_2,1)] = \phi(n_1 n_2, 0) = n_1 n_2 = n_1 n_2 x^2 = (n_1 x)(n_2 x) = \phi(n_1,1)\phi(n_2,1).$$

Now, $G$ is an index-two subgroup of $(G \cup H, \diamond_f)$ and $e_\star \notin G$, but this element commutes with each element of $G$ in the operation $\diamond_f$. Moreover, the order of $e_\star$ with respect to this operation is two. Taking $X = (G \cup H, \diamond_f)$, $N = G$, and $x = e_\star$ yields the desired isomorphism.

On a related note, we have the following proposition.

**Proposition 1.9.** *Let $(X, \circ)$ be a group and $N$ an index-two subgroup. Let $x \in X$ with $x \notin N$. Then there is a binary operation $\bullet_x$ on $X$ such that $(X, \bullet_x) \cong (N, \circ) \times C_2$.*

*Proof.* The bijection $\phi : N \times C_2 \to X$ given by

$$\phi(n,i) = \begin{cases} n & \text{if } i = 0 \\ n \circ x & \text{if } i = 1 \end{cases}$$

induces a group operation $\bullet_x$ on $X$. Explicitly, for $a, b \in X$, we have $a \bullet_x b =$

$$\begin{cases} n_1 \circ n_2 & \text{if } (a = n_1 \in N \text{ and } b = n_2 \in N) \text{ OR } (a = n_1 \circ x \in N \circ x \text{ and } b = n_2 \circ x \in N \circ x) \\ n_1 \circ n_2 \circ x & \text{if } (a = n_1 \in N \text{ and } b = n_2 \circ x \in N \circ x) \text{ OR } (a = n_1 \circ x \in N \circ x \text{ and } b = n_2 \in N), \end{cases}$$

and $\phi$ is an isomorphism $(N, \circ) \times C_2 \to (X, \bullet_x)$. □

The next theorem shows that when the element $x$ from Proposition 1.9 has order two in $(X, \circ)$ and centralizes $N$, we recover the original group operation from $\bullet_x$.

**Theorem 1.10.** *Let $(X, \circ)$ be a group with a subgroup $N$ of index two. Fix $x \in X$ with $x \notin N$. Suppose further that $x$ has order two in $(X, \circ)$ and belongs to the centralizer of $N$ in this group. Then $\bullet_x = \circ$.*

*Proof.* Let $a, b \in X$. If $a = n_1 \in N$ and $b = n_2 \in N$, then surely $a \bullet_x b = a \circ b$. Suppose that $a, b$ both belong to the coset $N \circ x$, say, $a = n_1 \circ x$ and $b = n_2 \circ x$. We have

$$a \bullet_x b = n_1 \circ n_2 = (n_1 \circ n_2) \circ e_\circ = (n_1 \circ n_2) \circ (x \circ x) = (n_1 \circ x) \circ (n_2 \circ x) = a \circ b,$$

as $x$ commutes $n_2$. Hence, $a \bullet_x b = a \circ b$ in this case as well. Continuing, assume that $a = n_1 \circ x \in N \circ x$ and $b = n_2 \in N$. Then

$$a \bullet_x b = n_1 \circ n_2 \circ x = (n_1 \circ x) \circ n_2 = a \circ b.$$

Finally, if $a = n_1 \in N$ and $b = n_2 \circ x \in N \circ x$, we have $a \bullet_x b = n_1 \circ (n_2 \circ x) = a \circ b$. It follows that $\bullet_x = \circ$, as needed. □

We are now ready to derive Theorem 1.1 as a consequence of Theorem 1.10.

*Proof.* Let

$$I(G,H) := \{f : G \to H | f \text{ is a group isomorphism}\}$$

and let $E(G,H)$ denote the collection of all group operations $\circledcirc$ on $G \cup H$ satisfying $(i)$, $(ii)$, and $(iii)$ in the statement of the theorem. If $f \in I(G,H)$, then the operation $\diamondsuit_f$ from Proposition 1.6 satisfies conditions $(i)$, $(ii)$, and $(iii)$. Therefore, we may define a map $\psi : I(G,H) \to E(G,H)$ by

$$\psi(f) = \diamondsuit_f.$$

We shall prove that $\psi$ is bijective. Injectivity is straightforward; assume that $f_1, f_2 \in I(G,H)$ with $\psi(f_1) = \psi(f_2)$. Then $\diamondsuit_{f_1} = \diamondsuit_{f_2}$. Hence, for any $g \in G$, we have $g \diamondsuit_{f_1} e_\star = g \diamondsuit_{f_2} e_\star$. In other words,

$$f_1(g) = f_1(g) \star e_\star = g \diamondsuit_{f_1} e_\star = g \diamondsuit_{f_2} e_\star = f_2(g) \star e_\star = f_2(g),$$

so $f_1 = f_2$. For surjectivity, let

$$\circledcirc \in E(G,H)$$

be arbitrary. We have seen (via Proposition 1.4) that the right translation map $f : G \to H$ given by $f(g) = g \circledcirc e_\star$ is an isomorphism. We claim that $\circledcirc = \diamondsuit_f$. Notice how we are in the setup of Theorem 1.10 with $(X, \circ) = (G \cup H, \circledcirc)$, $N = G$, and $x = e_\star$. Using $(i)$, $(ii)$, and $(iii)$, it is straightforward to verify that the group operation $\diamondsuit_f$ may be reformulated as $a \diamondsuit_f b =$

$$\begin{cases} g_1 \circledcirc g_2 & \text{if } (a = g_1 \in G \text{ and } b = g_2 \in G) \text{ or } (a = g_1 \circledcirc e_\star \in G \circledcirc e_\star \text{ and } b = g_2 \circledcirc e_\star \in G \circledcirc e_\star) \\ g_1 \circledcirc g_2 \circledcirc e_\star & \text{if } (a = g_1 \in G \text{ and } b = g_2 \circledcirc e_\star \in G \circledcirc e_\star) \text{ or } (a = g_1 \circledcirc e_\star \in G \circledcirc e_\star \text{ and } b = g_2 \in G), \end{cases}$$

which is precisely the operation $\bullet_{e_\star}$. For example, in the case when $a = g_1 \circledcirc e_\star \in G \circledcirc e_\star$ and $b = g_2 \in G$, we have

$$(g_1 \circledcirc e_\star) \diamondsuit_f g_2 = (g_1 \circledcirc e_\star) \star f(g_2) = (g_1 \circledcirc e_\star) \star (g_2 \circledcirc e_\star) = (g_1 \circledcirc e_\star) \circledcirc e_\star \circledcirc (g_2 \circledcirc e_\star) = g_1 \circledcirc g_2 \circledcirc e_\star.$$

By Theorem 1.10, we get that $\circledcirc = \diamondsuit_f$, so $\psi$ is surjective. Theorem 1.1 is obtained. □

**Remark 1.11.** Theorem 1.1 and Corollary 1.7 are easily adapted to the case when $G$ and $H$ are not necessarily disjoint sets. Simply view $\{0\}$ and $\{1\}$ as trivial groups, and apply the results to the disjoint union

$$G \sqcup H := (G \times \{0\}) \cup (H \times \{1\}).$$

# 2. Closing remarks

In closing, we report an interesting family of nonassociative (that is, not *necessarily* associative) operations, which is obtained by generalizing the operation $\diamondsuit_f$. The setup is as follows: Let $(G, \circ)$ and $(H, \star)$ be groups and suppose that $\alpha : G \to H$ and $\beta : H \to G$ are two group homomorphisms. For simplicity, assume that $G$ and $H$ are disjoint sets. (If not, we can form the disjoint union and proceed similarly.) Define a binary operation $\diamondsuit_\alpha^\beta$ on $G \cup H$ by

$$x \diamondsuit_\alpha^\beta y = \begin{cases} x \circ y & : x, y \in G \\ \beta(x) \circ \beta(y) & : x, y \in H \\ x \star \alpha(y) & : x \in H, y \in G \\ \alpha(x) \star y & : x \in G, y \in H. \end{cases}$$

Since $G$ and $H$ are groups, the system $(G \cup H, \diamond_\alpha^\beta)$ has an identity element (namely, $e_\circ$) and preserves inverses. However, it is not associative in general, since for example if $x \in G$ and $y, y' \in H$, then

$$(x \diamond_\alpha^\beta y) \diamond_\alpha^\beta y' = \beta(\alpha(x)) \circ \beta(y) \circ \beta(y'),$$

which need not equal

$$x \diamond_\alpha^\beta (y \diamond_\alpha^\beta y') = x \circ \beta(y) \circ \beta(y').$$

In fact, associativity holds if and only if $\beta = \alpha^{-1}$. Hence, $(G \cup H, \diamond_\alpha^\beta)$ is a group if and only if $\beta = \alpha^{-1}$, in which case we have seen that

$$(G \cup H, \diamond_\alpha^{\alpha^{-1}}) = (G \cup H, \diamond_\alpha) \cong (G, \circ) \times C_2.$$

**Acknowledgements.** I thank the anonymous referee for helpful feedback, which led to a richer exposition. I also thank Jeffery Liu for introducing me to the operation $\triangle$ from Lemma 1.2 and pointing out that the right coset $N \circ z$ forms a group under this operation.

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Received January 9, 2024

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