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The inclusion prime ideal graph of a semigroup and ring

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Abstract. We introduce the inclusion prime ideal graph $In_p(S)$ of nontrivial prime ideals of a commutative semigroup S. We characterize a semigroup S for which the graph $In_p(S)$ is null, complete or connected. Then we study various graph parameters, thickness, metric and partition dimension of the inclusion prime ideal graph of the multiplicative semigroup $\mathbb{Z}_n$ of integers of modulo n. Finally we characterize a ring R for which the graph $In_p(R)$ is null, complete or connected and also some graph theoretic and ring theoretic properties are studied.

1. Introduction

Nowdays in algebraic combinatorics, a major part of research is attached to the application of graph theory and combinatorics in abstract algebra. In particular, using graph theory to study a ring (resp. semigroup) draws so much attention. Since the structure of a ring (resp. semigroup) is closely tied with the behaviour of its ideals, it is interesting and worthy to consider a graph with vertex set as ideals of a ring (resp. semigroup). Also as the ideal structure reflects ring (semigroup) properties, several graphs that are based on the ideals were defined (see [1], [2], [3]). Akbari et al. [3] introduced the concept of inclusion ideal graph of a ring R, denoted by $In(R)$, is a graph with vertices are nontrivial left ideals of R and two distinct vertices I_1, I_2 are adjacent if and only if $I_1 \subset I_2$ or $I_2 \subset I_1$. In [4], they studied some graph parameters of $In(R)$ like connectedness, diameter, girth and perfectness. Recently Baloda et al. [8] studied the inclusion ideal graph of a semigroup. Then Khanra et al.[16] studied various graph parameters of inclusion ideal graph of a semigroup, in particular multiplicative semigroup

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$\mathbb{Z}_n$ of integers modulo n. Also we know that prime ideals play a highly important role in studying the structure of a ring (resp. semigroup). These observations prompted us to consider the inclusion prime ideal graph of a commutative semigroup S (resp. ring R), denoted by $In_p(S)$(resp. $In_p(R)$), is a graph with vertices are nontrivial prime ideals of S(resp. R) and two distinct vertices P_1, P_2 are adjacent if and only if $P_1 \subset P_2$ or $P_2 \subset P_1$.

Here by a graph $G = (V, E)$, we mean a simple undirected graph with vertex set $V(G)$ and edge set E. A graph G is said to be connected if there exists a path between any two distinct vertices of G. A graph G is said to be null if no two vertices of G are adjacent. If two distinct vertices u, $v \in V$ are adjacent, we write it by $u \sim v$, otherwise by $u \nsim v$. A graph in which any two distinct vertices are adjacent is called a complete graph. We use K_n to denote the complete graph with n vertices. The girth of G is the length of the shortest cycle in G and is denoted by $gr(G)$. An unicyclic graph is a connected graph containing exactly one cycle. A clique of a graph G is a complete subgraph of it and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. The chromatic number of G, denoted by $\chi(G)$, is the minimum number of colors required for a vertex coloring of G. A graph is said to be perfect if and only if $\omega(H) = \chi(H)$ for every induced subgraph H of G. Here by a surface, we mean a compact connected topological space such that each point has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$. We recall that a map $\phi : G \longrightarrow S$ is an embedding of a graph G into a surface S if ϕ represents a drawing of G on S without any crossings. The genus of a graph G, denoted as $g(G)$, is the minimal integer k such that the graph can be embedded in S_k, where S_k denote the sphere with k handles attached with it. The graphs of genus 0, 1, 2 are called planar, toroidal, bitoroidal respectively. An outerplanar graph is a planar graph that can be embedded in a plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. The thickness of a graph G, denoted as $\theta(G)$, is the minimum number of decomposition of G into it's planar subgraphs. We denote $[n] = \{1, 2, \ldots, n\}$ and for any set X we denote the cardinality of a set X by $|X|$. By $N[v]$, we denote the closed neighbourhood of a vertex v of G. For undefined graph terminology, we refer [15] and [20]; for semigroup theory, see [11, 17, 21].

This paper is arranged as follows : In Section 2, we characterize a commutative semigroup S for which $In_p(S)$ is null, complete or connected. Then in Section 2.1, we determine various graph parameters like degree of a vertex, chromatic and clique number, vertex cover number, metric dimension, strong metric dimension and partition dimension of the prime inclusion ideal graph $In_p(\mathbb{Z}_n)$ of $\mathbb{Z}_n$. Also, we a pose a conjecture regarding the Hamiltonian nature of $In_p(\mathbb{Z}_n)$. Finally in Section 3, we characterize a commutative ring R for which the graph $In_p(R)$ is null or complete and compute the clique and chromatic number. Also we studied some graph properties of the prime inclusion ideal graph of an arithmetical ring.

2. Prime inclusion ideal graph of semigroup

It is well known that a commutative semigroup S is archimedian if and only if it has no proper prime ideals ([21], Theorem 1). So we have the following immediate result.

Theorem 2.1. *Let S be a semigroup. Then the following statements are equivalent*
(1) *$In_p(S)$ is an empty graph.*
(2) *S is an archimedian semigroup.*

Example 2.2. It is easy to observe that nontrivial ideals of the monogenic semigroup $S_M = \{0, x, x^2, \dots, x^{m-1}\}$ with zero element is of the form $I_t = \{0, x^t, \dots, x^{m-1}\}$, where $2 \leqslant t \leqslant m-1$. Therefore S_M has no nontrivial prime ideals and consequently $In_p(S_M)$ is an empty graph.

Theorem 2.3. *Let S be a semigroup with zero element. Then $In_p(S)$ is a null graph if and only if each nontrivial prime ideal is minimal.*

Proof. Let S be a semigroup with zero such that $In_p(S)$ is a null graph. Now if S has exactly one nontrivial prime ideal then it is clear. So let P_1 and P_2 be two distinct nontrivial prime ideals of S. If P_1 is not minimal, then there exists a prime ideal P of S such that $0 \neq P \subset P_1$. It follows that $P \sim P_1$, which contradicts the fact that $In_p(S)$ is a null graph. Consequently, each nontrivial prime ideal of S is minimal. The proof of the converse part is obvious. □

Example 2.4. Let us consider the semigroup $S = \{0, a, b, ab : a^2 = a, b^2 = b, ab = ba\}$ with zero element. The only prime ideals of S are $\{0, a, ab\}$ and $\{0, b, ab\}$, both are minimal prime. Consequently $In_p(S)$ is a null graph.

We know that if a semigroup has unity but not a group, then it has a unique maximal ideal, which is prime also. So we have the following immediate result.

Corollary 2.5. *Let S be a semigroup with unity. Then the following statements are equivalent*
(1) $In_p(S)$ *is a null graph.*
(2) *Each nontrivial prime ideal of S is maximal.*
(3) *S has a unique nontrivial prime ideal.*

Since every ideal in a semigroup S with unity is contained in a unique maximal ideal of S, we have the following result:

Theorem 2.6. *The prime inclusion ideal graph $In_p(S)$ of a semigroup S with unity is connected. Moreover, diam $In_p(S) \leqslant 2$.*

Theorem 2.7. *Let S be a semigroup. Then the following statements are equivalent:*
(1) $In_p(S)$ *is a complete graph.*
(2) *S is a semiprimary semigroup.*
(3) *Prime ideals of S are linearly ordered.*
(4) *Semiprime ideals of S are linearly ordered.*
(5) *Semiprime ideals of S are prime.*
(6) *For each a, $b \in S$, there exists $k \in \mathbb{N}$ such that $a|b^k$ or $b|a^k$.*

Proof. (1) $\Leftrightarrow$ (2). Let I be a semiprimary ideal of S. Then $\sqrt{I} = \bigcap P_\alpha$, where P_α's are prime ideal of S containing I. Since the graph $In_p(S)$ is complete we have $\sqrt{I} = P$ for some prime ideals P of S and hence S is a semiprimary semigroup.

Conversely, let P_1, $P_2 \in V(In_p(S))$ but $P_1 \nsim P_2$. Then there exists x, $y \in S$ such that $x \in P_1 - P_2$ and $y \in P_2 - P_1$. Then $xy \in P_1 \cap P_2 = \sqrt{P_1P_2}$, a prime ideal of S as S is semiprimary. Hence $x \in P_1 \cap P_2$ or $y \in P_1 \cap P_2$, which is a contradiction. Consequently $In_p(S)$ is a complete graph.

(2) $\Leftrightarrow$ (3). It follows from Theorem 1 of [17].

(3) $\Leftrightarrow$ (4). Let I_1 and I_2 be two semiprime ideals of S. Clearly $I_1 \cap I_2$ is a semiprime ideal of S. Then $I_1 \cap I_2 = \sqrt{I_1 \cap I_2}$ is a prime ideal of S. Consequently semiprime ideals are linearly ordered. Converse is clear.

(4) $\Leftrightarrow$ (5). Let I be a semiprime ideal of S. Then $I = \sqrt{I}$ is a prime ideal of S. For converse part, let I_1 and I_2 are two distinct semiprime ideals of S. Clearly $I_1 \cap I_2$ is a prime ideal of S as semiprime ideals are prime. Now in a similar way as in the proof of above cases it is clear that semiprime ideals are linearly ordered.

(3) $\Leftrightarrow$ (6). Let a, $b \in S$ and prime ideals of S are linearly ordered. Then $\sqrt{(a)} \subseteq \sqrt{(b)}$ or $\sqrt{(b)} \subseteq \sqrt{(a)}$ which implies either $a^s \in (b)$ or $b^t \in (a)$ for some s, $t \in \mathbb{N}$. Let $k = max\{s, t\}$. Therefore either $a|b^k$ or $b|a^k$.

Conversely, let P_1 and P_2 be two distinct prime ideals of S and $ab \in P_1 \cap P_2$ for some a, $b \in S$. Then $ab \in P_1$ and $ab \in P_2$ which implies $a \in P_1$ or $b \in P_1$ and $a \in P_2$ or $b \in P_2$. Now if $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$ then it is clear that $P_1 \subset P_2$ or $P_2 \subset P_1$. Now we consider the remaining cases $a \in P_1$ and $b \in P_2$ or $a \in P_2$ and $b \in P_1$. Without loss of generality let $a \in P_1$ and $b \in P_2$. Then by assumption there exists $k \in \mathbb{N}$ such that $a|b^k$ or $b|a^k$. Now if $a|b^k$ then $b^k = as \in P_1$ for some $s \in S$, implies $b \in P_1$ and hence $b \in P_1 \cap P_2$. Similarly $b|a^k$ implies $a \in P_1 \cap P_2$. Consequently $P_1 \cap P_2$ is a prime ideal of S. Hence the result follows. □

2.1. The prime inclusion ideal graph of the semigroup $\mathbb{Z}_n$

Theorem 2.8. *Every nontrivial prime ideal P of $\mathbb{Z}_n$ is of the form $P = \cup\{p_i\mathbb{Z}_n : i \in [k]\}$, where p_1, $p_2, \ldots, p_k$ are k distinct prime divisor of n.*

Proof. It is well known that nonzero ideal of $\mathbb{Z}_n$ is of the form $\cup\{m_i\mathbb{Z}_n : i \in [k]\}$, where m_1, $m_2, \ldots, m_k$ are divisors of n such that $m_i \nmid m_j$ if $i \neq j$ ([22],Theorem 2). Let P be a nontrivial prime ideal of $\mathbb{Z}_n$. Then $P \in \cup\{m_i\mathbb{Z}_n : i \in [k]\}$, where m_1, $m_2, \ldots, m_k$ are divisors of n such that $m_i \nmid m_j$ if $i \neq j$. Now in the expression of P if one of m_i's is composite, say m_t, then $m_t = m_a m_b$ for some proper divisor m_a and m_b of m_t. Then $m_a m_b \in P$ but neither $m_a \in P$ nor $m_b \in P$, contradicts that P is a prime ideal of $\mathbb{Z}_n$. Since ideal generated by prime divisor of n is a prime ideal and union of any collection of prime ideals of a semigroup is a prime ideal, we have $P = \cup\{p_i\mathbb{Z}_n : i \in [k]\}$, where p_i's ($i \in [k]$) are distinct prime divisor of n. Hence the result follows. □

Proposition 2.9. *The number of nontrivial prime ideal of $\mathbb{Z}_n$ is equal to $2^k - 1$, where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$.*

Proof. The nontrivial prime ideal of $\mathbb{Z}_n$ is of the form $\cup\{p_i\mathbb{Z}_n : i \in [k]\}$ (cf. Theorem 2.8), where p_i's ($i \in [k]$) are distinct prime divisor of n. Therefore the number of nontrivial prime ideals of $\mathbb{Z}_n$ is ${}^kC_1 + {}^kC_2 + \cdots + {}^kC_k = 2^k - 1$. □

Corollary 2.10. *The order of the graph $In_p(Z_n)$ is equal to $2^k - 1$, where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$.*

Let $\Lambda = \{1, 2, \ldots, t\} \subseteq [k]$. We use the sign P_Λ or $P_{12\ldots t}$ to denote the prime ideal $(p_1) \cup (p_2) \cup \cdots \cup (p_t)$ of $\mathbb{Z}_n$. Clearly, for $k = 1$, we have $In_p(\mathbb{Z}_n) \cong K_1$. So throughout this paper we consider the graph $In_p(\mathbb{Z}_n)$, where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $k \geqslant 2$.

Since the multiplicative semigroup $\mathbb{Z}_n$ has identity, by Theorem 2.6 we have the following immediate result.

Corollary 2.11. *The graph $In_p(\mathbb{Z}_n)$ is a connected with diam$(In_p(\mathbb{Z}_n))=2$.*

Theorem 2.12. (1) *The girth of $In_p(\mathbb{Z}_n)$ is given by*

$$gr(In_p(\mathbb{Z}_n)) = \begin{cases} \infty\ , & \text{if } k = 2 \\ 3, & \text{if } k \geqslant 3 \end{cases}$$

(2) *The graph is triangulated if and only if $k \geqslant 3$.*

(3) *The graph is k-partiate if and only if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $k \geqslant 2$.*

Theorem 2.13. *The degree of a vertex P_Λ such that $|\Lambda| = t(t \in [k])$ is deg$(P_\Lambda) = 2^t + 2^{k-t} - 3$.*

Proof. Let P_Λ be any vertex of $In_p(\mathbb{Z}_n)$ such that $|\Lambda| = t(t \in [k])$. Then the number of nontrivial prime ideals properly contained in P_Λ is

$${}^tC_1 + {}^tC_2 + \cdots + {}^tC_{t-1} = 2^t - 2.$$

Also the number of nontrivial prime ideals properly containing P_Λ is

$${}^{k-t}C_1 + {}^{k-t}C_2 + \cdots + {}^{k-t}C_{k-t} = 2^{k-t} - 1.$$

Therefore the total number of vertices adjacent to P_Λ is $= 2^t - 2 + 2^{k-t} - 1 = 2^t + 2^{k-t} - 3$. Therefore $\deg(P_\Lambda) = 2^t + 2^{k-t} - 3$. □

It is well known that a simple connected graph G is Eulerian if and only if every vertex of G is of even degree. Since for $k \neq t$, $\deg(P_\Lambda) = 2^t + 2^{k-t} - 3$ is an odd number, where $|\Lambda| = t$, we have the following immediate result

Corollary 2.14. *The graph $In_p(\mathbb{Z}_n)$ is not Eulerian.*

Lemma 2.15. *Let P_{Λ_1} and P_{Λ_2} be any two vertex of $In_p(\mathbb{Z}_n)$. Then deg(P_{Λ_1}) =deg(P_{Λ_2}) if and only if $|\Lambda_1| = |\Lambda_2|$ or $|\Lambda_1| + |\Lambda_2| = k$.*

Proof. Let $|\Lambda_1| = s$ and $|\Lambda_2| = t$. Then deg(P_{Λ_1}) =deg(P_{Λ_2}). Consequently, $2^s + 2^{k-s} - 3 = 2^t + 2^{k-t} - 3$, which implies $2^s - 2^t = \frac{2^k(2^s-2^t)}{2^{s+t}}$. So either $2^s = 2^t$ or $2^s \neq 2^t$.

If $2^s = 2^t$, then $s = t \Rightarrow |\Lambda_1| = |\Lambda_2|$.

If $2^s \neq 2^t$, then $2^k = 2^{s+t} \Rightarrow k = s + t \Rightarrow |\Lambda_1| + |\Lambda_2| = k$. Conversely, if $|\Lambda_1| = |\Lambda_2|$, then clearly deg$(P_{\Lambda_1})$ = deg (P_{Λ_2}).

Now let $|\Lambda_1|+|\Lambda_2| = k \Rightarrow s+t = k$. Therefore deg$(P_{\Lambda_2}) = 2^t+2^{k-t}-3 = 2^{k-s} + 2^s - 3 =$ deg (P_{Λ_1}). □

Theorem 2.16. *The maximum and minimum degrees of* $In_p(\mathbb{Z}_n)$ *are* $\Delta(In_p(\mathbb{Z}_n)) = 2^k-2$ *and* $\delta(In_p(\mathbb{Z}_n)) = 2^{t+1}-3$ *(if* $k = 2t$*) and* $\delta(In_p(\mathbb{Z}_n)) = 3(2^t - 1)$ *(if* $k = 2t + 1$*). Moreover, the degree sequence* $DS(In_p(\mathbb{Z}_n))$ *is* $2^t+2^{k-t}-3$ $({}^kC_t$ *times*$), \ldots, 2+2^{k-1}-3$ $(2.{}^kC_1$ *times*$)$, 2^k-2 *when* $k = 2t$ *and* $2^t+2^{k-t}-3$ $(2.{}^kC_t$ *times*$), \ldots, 2+2^{k-1}-3$ $(2.{}^kC_1$ *times*$)$, 2^k-2 *when* $k = 2t + 1$.

Proof. Since the vertex $P_{12\ldots k}$ is adjacent to every other vertex in $In_p(\mathbb{Z}_n)$, we have $\Delta(In_p(\mathbb{Z}_n))$ =deg$(P_{12\ldots k}) = 2^k - 2$. To find the minimum degree vertices in $In_p(\mathbb{Z}_n)$, we consider the function $f : [1, k] \longrightarrow \mathbb{R}$ defined by

$$f(x) = 2^x + 2^{k-x} - 3 \tag{1}$$

Now $f'(x) = 0 \Rightarrow \frac{2^x}{log_e 2} - \frac{2^{k-x}}{log_e 2} = 0 \Rightarrow x = \frac{k}{2}$.

Also $f''(x) = \frac{2^x}{(log_e 2)^2} + \frac{2^{k-x}}{(log_e 2)^2}$. Hence $f''(\frac{k}{2}) = \frac{2^{\frac{k}{2}+1}}{(log_e 2)^2} > 0$. Therefore f has a minimimun value at $x = \frac{k}{2}$. Since we are interseted in integer solutions and also by combining Lemma 2.15, it is easy to observe that vertices of the form P_Λ such that $|\Lambda| = \frac{k}{2}$ (if k is even) and $|\Lambda| = \frac{k+1}{2}$ or $\frac{k-1}{2}$ (if k is odd) have minimum degrees. So we have $\delta(In^p(\mathbb{Z}_n)) = 2^{t+1} - 3$ (if $k = 2t$) and $\delta(In^p(\mathbb{Z}_n)) = 3(2^t - 1)$ (if $k = 2t + 1$).

It is clear to observe that for $m > n$ in $[1, t]$ we have $f(m) < f(n)$ and for $a > b$ in $[t, k]$ we have $f(a) > f(b)$ when $k = 2t$. Also for $a > b$ in $[p + 1, k]$ we have $f(a) > f(b)$ and $f(t) = f(t + 1)$ when $k = 2t + 1$.

Therefore by applying Theorem 2.13 and Lemma 2.15 we have the required degree sequence. □

Corollary 2.17. *The irregularity index of* $In_p(\mathbb{Z}_n)$ *is* $MWB(In_p(\mathbb{Z}_n)) = t + 1$, *where* $k = 2t$ *or* $2t + 1$.

Now we are interested in finding the number of edges of $In_p(\mathbb{Z}_n)$.

Theorem 2.18. *The number of edges of $In_p(\mathbb{Z}_n)$ is given by the equation $2|E_n| = \sum_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3|V_n|$, where $|V_n|$ and $|E_n|$ denotes the number of vertices and edges of $In_p(\mathbb{Z}_n)$ respectively.*

Proof. We know that sum of degrees of vertices of a graph is twice the number of edges, hence the total no of edges of $In_p(\mathbb{Z}_n)$ is

$$\begin{aligned} 2|E_n| &= {}^kC_1(2 + 2^{k-1} - 3) + {}^kC_2(2^2 + 2^{k-2} - 3) + \cdots + {}^kC_k(2^k + 2^0 - 3) \\ &= \sum_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3(2^k - 1) = \sum_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3|V_n|. \end{aligned}$$

Hence the result follows. □

Example 2.19. Let $n = \prod_{i=1}^{3} p_i^{\alpha_i}$. Then $k = 3$ and $|V_n| = 7$. Hence by Theorem 2.18 we have $2|E_n| = \sum_{i=1}^{3} {}^3C_i(2^i + 2^{3-i}) - 3.7 = 24 \Rightarrow |E_n| = 12$ (see Figure 1(a)).

Theorem 2.20. *The vertex connectivity of $In_p(\mathbb{Z}_n)$ is $\kappa(In_p(\mathbb{Z}_n)) = 2^t + 2^{k-t} - 3$, where $k = 2t$ or $2t + 1$, $t \in \mathbb{N}$.*

Proof. Let $k = 2t$ or $2t + 1$. Then the minimum degree of $In_p(\mathbb{Z}_n)$ is $2^t + 2^{k-t} - 3$, in fact, every vertex of the form $\{P_\Lambda : |\Lambda| = t\}$ is of minimum degree. Now consider the set $N(P_\Lambda)$ of neighborhoods of any vertex of the form $\{P_\Lambda : |\Lambda| = t\}$. Clearly $|N(P_\Lambda)| = 2^t + 2^{k-t} - 3$.

We claim that $N(P_\Lambda)$ is a minimal vertex cut of $In_p(\mathbb{Z}_n)$. It is easy to observe that $In_p(\mathbb{Z}_n)$ has two components C_1 and C_2 where $C_1 = \{P_\Lambda\}$ and C_2 is the induced subgraph of $In_p(\mathbb{Z}_n)$ with vertex set $V(In_p(\mathbb{Z}_n)) - \{N(P_\Lambda) \cup P_\Lambda\}$. Now if possible let $In_p(\mathbb{Z}_n) - S$ is a disconnected graph where $S \subset N(P_\Lambda)$ be a vertex cut of $In_p(\mathbb{Z}_n)$. Then there exists $P_{\Lambda_1} \in N(P_\Lambda) - S$ and hence $P_{\Lambda_1} \sim P_\Lambda$. Now it is easy to observe that $In_p(\mathbb{Z}_n) - S$ is a connected graph, a contradiction. Hence the result follows. □

Theorem 2.21. *The graphs $In_p(\mathbb{Z}_n)$ and $In_p(\mathbb{Z}_m)$ are isomorphic if and only if n and m have same number of prime factors.*

Proof. Let the two graphs $In_p(\mathbb{Z}_n)$ and $In_p(\mathbb{Z}_m)$ are isomorphic but n and m have different number of prime factors. So without loss of generality we assume $s > t$, where s and t are number of prime factors of n and m

respectively. Then $|V(In_p(\mathbb{Z}_n)| > |V(In_p(\mathbb{Z}_m)|$, a contradiction. Therefore n and m have same number of prime factors.

Conversely, Let n and m have same number of prime factors. Now we define a map $f : In_p(\mathbb{Z}_n) \longrightarrow In_p(\mathbb{Z}_m)$ by $f(P_\Lambda) = Q_\Lambda$, where Λ is an indexed subset of $[s] = [t]$ and Q_Λ is the prime ideal $\bigcup_{i=1}^{t}(q_i)$. One can easily see that f is a bijection and two vertices in $In_p(\mathbb{Z}_n)$ are adjacent if and only if their f-images are adjacent in $In_p(\mathbb{Z}_m)$. Hence the two graphs are isomorphic. □

Lemma 2.22. *Let Λ_1 and Λ_2 be two indexed subset of $[k]$. If $|\Lambda_1| = |\Lambda_2|$, then $P_{\Lambda_1} \nsim P_{\Lambda_2}$.*

Proof. For different Λ_1 and Λ_2 there exists $s \in \Lambda_1 - \Lambda_2$ and $t \in \Lambda_2 - \Lambda_1$. Hence $P_{\Lambda_1} \nsim P_{\Lambda_2}$. □

Theorem 2.23. *The clique number of the graph $In_p(\mathbb{Z}_n)$ is $\omega(In_p(\mathbb{Z}_n)) = k$.*

Proof. Clearly $L = \{P_1, P_{12}, \ldots, P_{12\ldots k}\}$ is a clique of size k in $In_p(\mathbb{Z}_n)$. If possible let $L \cup \{P\}$ be a clique containing L. Then $P = P_\Lambda$, where Λ is an indexed subset of $[k]$. Since P_Λ is different from elements of L, we have either $1 \notin \Lambda$ or there exists $s, t \in [k]$ with $s < t$ and $s \notin \Lambda$ but $t \in \Lambda$. Now if $1 \in \Lambda$, then $P_1 \nsim P_\Lambda$, contradicts that $L \cup \{P\}$ is a clique in $In_p(\mathbb{Z}_n)$. Again considering the second case we have $P_{12\ldots s} \nsim P_\Lambda$, a contradiction. Hence L is a maximal clique in $In_p(\mathbb{Z}_n)$.

Now if possible let L' be a clique of size greater than equals to $k + 1$. Then there exists i, $J \in L'$ such that $I = P_{\Lambda_1}$ and $J = P_{\Lambda_2}$ where $|\Lambda_1| = |\Lambda_2| = t$ for some $1 \leqslant t \leqslant k$. Hence from Lemma 2.22, we have $I \nsim J$, a contradiction. Therefore $\omega(In_p(\mathbb{Z}_n)) = k$. □

Theorem 2.24. *The chromatic number of $In_p(\mathbb{Z}_n))$ is $\chi(In_p(\mathbb{Z}_n)) = k$.*

Proof. Since the inclusion ideal graph is perfect ([16], Theorem 1), clearly $In_p(\mathbb{Z}_n)$ is perfect. Therefore we have $\chi(In_p(\mathbb{Z}_n)) = \omega(In_p(\mathbb{Z}_n)) = k$.

Also we know $\chi(In_p(\mathbb{Z}_n)) \geqslant \omega(In_p(\mathbb{Z}_n)) = k$. Here we demonstrate a proper k-coloring of $In_p(\mathbb{Z}_n)$. Let us define $A_i = \{P_\Lambda : \Lambda$ is an indexed subset of cardinality $i \in [k]\}$. Now put the colour i to the set of vertices in A_i, this is a proper k-coloring. Hence $\chi(In_p(\mathbb{Z}_n)) \leqslant k$. Consequently, $\chi(In_p(\mathbb{Z}_n)) = \omega(In_p(\mathbb{Z}_n)) = k$. □

A vertex cover of a graph G is a set of vertices that covers all the edges of G. The minimum cardinality of a vertex cover in G is called the vertex cover number of G, denoted by $\tau(G)$.

Theorem 2.25. *The vertex cover number of* $In_p(\mathbb{Z}_n)$ *is given by* $\tau(In_p(\mathbb{Z}_n)) = 2^k - 1 - {}^kC_r$, *where* $k = 2r$ *or* $2r+1$ *and* $r \in \mathbb{N}$.

Proof. First we need to define a cover set C for the graph $In_p(\mathbb{Z}_n)$. Now to cover highest number of edges with less number of vertices, we need to start adding vertices in C one by one that having highest number of neighbourhoods.

Since the vertex $P_{12\ldots k}$ is of highest degree it must be in C. Now we look into the vertices which is of second highest degree and these vertices is of the form $\{P_\Lambda : |\Lambda| = 1 \text{ or } k-1\}$ (see Theorem 2.16). Now to cover the rest of the edges incident with the vertices $\{P_\Lambda : |\Lambda| = 1\}$ and $\{P_\Lambda : |\Lambda| = k-1\}$, we have to add all these vertices. By continuing this way, the final step of adding the vertices in C is the set of vertices $\{P_\Lambda : |\Lambda| = r-1 \text{ and } r+1\}$ if k is even or $\{P_\Lambda : |\Lambda| = r \text{ or } r+1\}$ if k is odd. Hence the vertex cover of $In_p(\mathbb{Z}_n)$ is given by $C = V(In_p(\mathbb{Z}_n)) - \{P_\Lambda : |\Lambda| = r\}$. Consequently the vertex cover number of $In_p(\mathbb{Z}_n)$ is given by

$$\begin{aligned}\tau(In_p(\mathbb{Z}_n) &= {}^kC_1 + \cdots + {}^kC_{r-1} + {}^kC_{r+1} + \ldots {}^kC_k = \sum_{i=1}^{k} {}^kC_i -^k C_r \\ &= 2^k - 1 -^k C_r.\end{aligned}$$

□

We know a set I is independent if and only if it's complement is a vertex cover. Hence the number of vertices of a graph G is the sum of independence number and vertex cover number of G. So we have the following immediate result.

Corollary 2.26. *The independence number of* $In_p(\mathbb{Z}_n)$ *is* ${}^kC_r$, *where* $k = 2r$ *or* $2r+1$.

If $k = 2$ then $In_p(\mathbb{Z}_n) \cong P_3$, a path of length three and hence not a hamiltonian graph. For $k = 3$, $P_1 \sim P_{12} \sim P_2 \sim P_{23} \sim P_3 \sim P_{13} \sim P_{123} \sim P_1$ is a Hamiltonian cycle in $In_p(\mathbb{Z}_n)$ and if $k = 4$ then $P_1 \sim P_{12} \sim P_{123} \sim P_{13} \sim P_3 \sim P_{23} \sim P_{234} \sim P_{34} \sim P_4 \sim P_{24} \sim P_2 \sim P_{124} \sim P_{14} \sim P_{134} \sim P_{1234} \sim P_1$ is a Hamiltonian cycle in $In_p(\mathbb{Z}_n)$. Also for $k = 5$, $P_1 \sim P_{12} \sim P_{123} \sim P_{1234} \sim P_{124} \sim P_{24} \sim P_2 \sim P_{23} \sim P_{234} \sim P_{2345} \sim P_{235} \sim P_{35} \sim P_3 \sim P_{34} \sim P_{345} \sim P_{1345} \sim P_{134} \sim P_{14} \sim P_4 \sim P_{45} \sim P_{145} \sim P_{1245} \sim P_{245} \sim P_{25} \sim P_5 \sim P_{15} \sim P_{125} \sim P_{1235} \sim P_{135} \sim P_{13} \sim P_{12345} \sim P_1$ is a Hamiltinian cycle in $In_p(\mathbb{Z}_n)$. In a similar way we can find a Hamiltonian cycle $k = 6, 7$ and so on. Hence we are in a position to make the following conjecture

Conjecture 2.27. *The graph* $In_p(\mathbb{Z}_n)$ *is Hamiltonian if and only if* $k \geqslant 3$.

Theorem 2.28. *The following statements about $In_p(\mathbb{Z}_n)$ are equivalent:*
(1) *$In_p(\mathbb{Z}_n)$ is an outer-planar graph.*
(2) *$k = 2$.*
(3) *$In_p(\mathbb{Z}_n)$ is a ring graph.*

Proof. (1) $\Leftrightarrow$ (2). If $k = 2$, then $In_p(\mathbb{Z}_n) \cong P_3$, a path of order three and hence outer-planar. Now it is easy to observe from Figure 1(a) with $v_1 = P_1$, $v_2 = P_{13}$, $v_3 = P_3$, $v_4 = P_{23}$, $v_5 = P_{123}$, $v_6 = P_{12}$, $v_7 = P_2$ that if $k = 3$, then $In_p(\mathbb{Z}_n)$ contains a subdivision of the complete graph K_4 and hence not an outer-planar graph. Again if $k \geqslant 4$, then the induced subgraph formed by the set of vertices $\{P_1, P_{12}, P_{123}, P_{1234}\}$ is the complete graph K_4 and hence not outer-planar.

(2) $\Leftrightarrow$ (3). Since every outer-planar graph is a ring graph, it is clear that $In_p(\mathbb{Z}_n)$ is a ring graph for $k = 2$. Now if $k = 3$, then from Figure 1(a) it is clear that rank$(In_p(\mathbb{Z}_n)) = 6 \neq 7 =$frank$(In_p(\mathbb{Z}_n))$ (see Figure 1(a)) and hence not a ring graph ([14], Theorem 2.13). Also it is clear that for $k \geqslant 3$, $In_p(\mathbb{Z}_n)$ contains a subdivision of the complete graph K_4 and hence not a ring graph. □

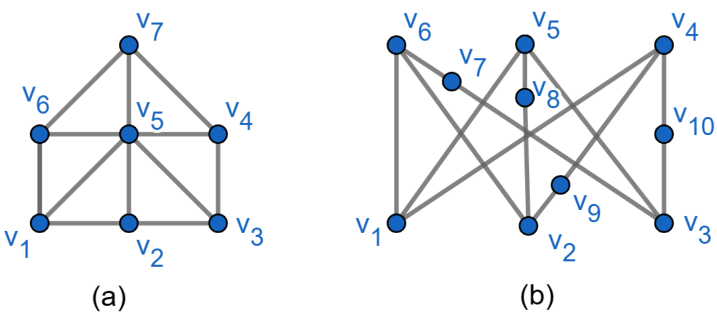

Figure 1: (a) $In_p(\mathbb{Z}_{\prod_{i=1}^{3} p_i^{\alpha_i}})$, (b) Subgraph of $In_p(\mathbb{Z}_{\prod_{i=1}^{4} p_i^{\alpha_i}})$ homeomorphic to $K_{3,3}$

Theorem 2.29. *The graph $In_p(\mathbb{Z}_n)$ is planar if and only if $2 \leqslant k \leqslant 3$.*

Proof. Since every ring graph is planar, it is clear that $In_p(\mathbb{Z}_n)$ is planar for $k = 2$. Since there is no crossing of edges in drawing the graph $In_p(\mathbb{Z}_n)$ for $k = 3$ in the plane (see Figure 1(a)), clearly it is planar. Now if $k = 4$, then

$In_p(\mathbb{Z}_n)$ contains a complete bipartiate graph $K_{3,3}$ as a minor (see Figure 1(b) with $v_1 = P_1$, $v_2 = P_2$, $v_3 = P_3$, $v_4 = P_{14}$, $v_5 = P_{13}$, $v_6 = P_{12}$, $v_7 = P_{123}$, $v_8 = P_{1234}$, $v_9 = P_{124}$, $v_{10} = P_{134}$) and hence not planar. Again if $k \geqslant 5$, then the subgraph forms by the set of vertices $\{P_1, P_{12}, P_{123}, P_{1234}, P_{12345}\}$ is the complete graph K_5 and hence not planar. □

Theorem 2.30. *The graph $In_p(\mathbb{Z}_n)$ has thickness one if and only if $2 \leqslant k \leqslant 3$ and has thickness two if and only if $k = 4$.*

Proof. We know that a graph has thickness one if and only if it is planar. Therefore $In_p(\mathbb{Z}_n)$ has thickness one if and only if $2 \leqslant k \leqslant 3$ (see Theorem 2.29).

Let us consider the case for $k = 4$. It is easy to calculate that for $k = 4$, $In_p(\mathbb{Z}_n)$ has 15 vertices and 50 edges (cf. Theorem 2.18). Therefore we have $\theta(In_p(\mathbb{Z}_n))_{k=4} \geqslant 2$ ([23], Lemma 2.3). The planar decomposition of $In_p(\mathbb{Z}_n))_{k=4}$ as shown in Figure 2, we have $\theta(In_p(\mathbb{Z}_n))_{k=4} = 2$.

Now if $k = 5$, then $In_p(\mathbb{Z}_n)$ has 31 vertices and 180 edges (cf. Theorem 2.18). Therefore $\theta(In_p(\mathbb{Z}_n))_{k=5} \geqslant 3$ ([23], Lemma 2.3). Also for $k \geqslant 5$, the graph $In_p(\mathbb{Z}_n)$ has a subgraph isomorphic to $In_p(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{5} p_i^{\alpha_i}$. Therefore for $k \geqslant 5$, $\theta(In_p(\mathbb{Z}_n)) \geqslant 3$ ([23], Lemma 2.1). Hence the result follows. □

Proposition 2.31. *The graph $In_p(\mathbb{Z}_n)$ is never toroidal and is not bitoroidal for $k > 4$.*

Proof. We complete the proof by considering the following cases:
Case (i). Let $k \leqslant 3$. Then we have $\gamma(In_p(\mathbb{Z}_n)) = 0$ (see Theorem 2.29).
Case (ii). Let $k = 4$. Then the graph $In_p(\mathbb{Z}_n)$ has $n = 15$ vertices and $e = 50$ edges (see Theorem 2.18). Therefore by applying Proposition 4.4.4 of [20], we have

$$g(In_p(\mathbb{Z}_n)) \geqslant \lceil \frac{50}{6} - \frac{15}{2} + 1 \rceil = 2 \tag{2}$$

Case (iii). Let $k = 5$. Then $In_p(\mathbb{Z}_n)$ has $n = 31$ vertices and $e = 180$ edges (see Theorem 2.18). Therefore $g(In_p(\mathbb{Z}_{\prod_{i=1}^{5} p_i^{\alpha_i}})) \geqslant 16$ ([20], Proposition 4.4.4). Now if $k \geqslant 6$, then $In_p(\mathbb{Z}_n)$ contains a subgraph isomorphic to $In_p(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{5} P_i^{\alpha_i}$ and hence $g(In_p(\mathbb{Z}_n)) \geqslant 16$. Combining all the above cases we have the desired result. □

Now we recall the following result which is important to determine the metric dimension of $In_p(\mathbb{Z}_n)$.

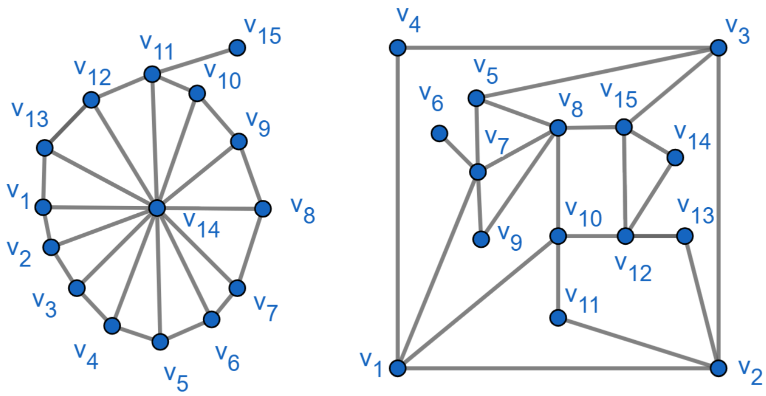

Figure 2: A planar decomposition of $In_p(\mathbb{Z}_{\prod_{i=1}^{4} p_i^{\alpha_i}})$

Theorem 2.32 ([9], Theorem 1)**.** *If G is a connected graph of order n and diameter d, then $f(n,d) \leqslant dim(G) \leqslant n-d$, where $f(n,d)$ is the least positive integer k such that $k + d^k \geqslant n$.*

Theorem 2.33. *The metric dimension of $In_p(\mathbb{Z}_n)$ is*

$$dim(In_p(\mathbb{Z}_n)) = \begin{cases} 1 & \text{if } k = 2 \\ k, & \text{if } k \geqslant 3 \end{cases}$$

Proof. Case (1). Let $k = 2$. Then $In_p(\mathbb{Z}_n) \cong P_3$. Hence $\dim(In_p(\mathbb{Z}_n)) = 1$ ([9], Theorem 2). Moreover any of the pendent vetrices P_1 or P_2 is a metric basis for $In_p(\mathbb{Z}_n)$.
Case (2). Let $k \geqslant 3$. Since $In_p(\mathbb{Z}_n)$ has $2^k - 1$ vertices and is of diameter 2, so by Theorem 2.32 we have $f(n,d) \leqslant \dim(In_p(\mathbb{Z}_n)) \leqslant 2^k - 3$, where $f(n,d)$ is the least positive integers l such that $l + 2^l \geqslant 2^k - 1$. Clearly k is the least positive integers such that $k + 2^k \geqslant 2^k - 1$. Therefore

$$k \leqslant \dim(In_p(\mathbb{Z}_n)) \leqslant 2^k - 3, \qquad \text{for } k \geqslant 3. \tag{3}$$

Now we prove that $W = \{P_1, P_2, \ldots, P_k\}$ is a resolving set for $In_p(\mathbb{Z}_n)$. On the contarary, if possible let there exists distinct vertices P_{Λ_1} and $P_{\Lambda_2} \in V(In_p(\mathbb{Z}_n)) - W$ such that $r(P_{\Lambda_1}, W) = (a_1, \ldots, a_k) = (b_1, \ldots, b_k) = r(P_{\Lambda_2}, W)$ where $a_i = b_i = 1$ or 2 for $i \in [k]$.

We claim that $|\Lambda_1| = |\Lambda_2|$. If not, without loss of generality, let $|\lambda_1| < |\Lambda_2|$. Then there exists $t \in \Lambda_2 - \Lambda_1$, $t \in [k]$.Then $(a_1, \ldots, a_t = 2, \ldots a_k) \neq (b_1, \ldots, b_t = 1, \ldots b_k)$, a contradiction. Therefore $|\Lambda_1| = |\Lambda_2|$. Now since P_{Λ_1} and P_{Λ_2} are distinct there exists $i_1 \in \Lambda_1 - \Lambda_2$ and $i_2 \in \Lambda_2 - \Lambda_1$, where i_1, $i_2 \in [k]$.

Then $(a_1, \ldots, a_{i_1} = 1, \ldots a_{i_2} = 2, \ldots, a_k) = (b_1, \ldots, b_{i_1} = 2, \ldots, b_{i_2} = 1, \ldots b_k)$, a contradiction. Hence $P_{\Lambda_1} = P_{\Lambda_2}$. Therefore distinct vertices of $In_p(\mathbb{Z}_n)$ has distinct representations with respect to W. So W is a resolving set of $In_p(\mathbb{Z}_n)$, which implies

$$dim(In_p(\mathbb{Z}_n)) \leqslant k \tag{4}$$

Now combining equation (3) and (4) we have $\dim(In_p(\mathbb{Z}_n)) = k$ for $k \geqslant 3$ with W as metric basis. □

To determine the strong metric dimension of $In_p(\mathbb{Z}_n)$ we recall the following result.

Theorem 2.34 ([19], Theorem 2.2). *For any graph G with diameter 2, $sdim(G) = |V(G)| - \omega(\mathbb{R}_G)$, where $\mathbb{R}_G$ is the reduced graph of G.*

Theorem 2.35. *The strong metric dimension is $sdim(In_p(\mathbb{Z}_n)) = 2^k - k - 1$ where $k \geqslant 2$.*

Proof. Let P_{Λ_1} and P_{Λ_2} be any two vertices of $In_p(\mathbb{Z}_n)$. Now if $N[P_{\Lambda_1}] = N[P_{\Lambda_2}]$ then we must have $\deg(P_{\Lambda_1}) = \deg(P_{\Lambda_2})$, which is possible only if $|\Lambda_1| = |\Lambda_2|$ or $|\Lambda_1| + |\Lambda_2| = k$ (see Lemma 2.15).
Case 1. Let $|\Lambda_1| = |\Lambda_2|$. Then $P_{\Lambda_1} \in N[P_{\Lambda_1}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_1}]$. Also $P_{\Lambda_2} \in N[P_{\Lambda_2}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_2}]$. Therefore $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$.
Case 2. Let $|\Lambda_1| + |\Lambda_2| = k$. Without loss of generality, let $|\Lambda_1| < |\Lambda_2|$. Then there exists $a \in [k]$ such that $a \in \Lambda_2 - \Lambda_1$. Then $P_a \in N[P_{\Lambda_2}]$ but $P_a \notin N[P_{\Lambda_1}]$ and hence $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$.

Thus in any cases $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$ and since P_{Λ_1} and P_{Λ_2} are arbitrary, we have $\mathbb{R}_{In_p(\mathbb{Z}_n)} = In_p(\mathbb{Z}_n)$. We know that $In_p(\mathbb{Z}_n)$ is a graph with $2^k - 1$ vertices and $\omega(In_p(\mathbb{Z}_n)) = k$. Therefore by Theorem 2.34 we have $\text{sdim}(In_p(\mathbb{Z}_n)) = 2^k - k - 1$. □

The above result can be proved in a different way, which is as follows: If $N[P_{\Lambda_1}] = N[P_{\Lambda_2}]$ then we must have $P_{\Lambda_1} \sim P_{\Lambda_2}$ otherwise $P_{\Lambda_1} \in N[P_{\Lambda_1}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_1}]$ and $P_{\Lambda_1} \notin N[P_{\Lambda_2}]$ but $P_{\Lambda_2} \in N[P_{\Lambda_2}]$. Now since $P_{\Lambda_1} \sim P_{\Lambda_2}$, either $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$. Without loss of generality , let $\Lambda_1 \subset \Lambda_2$.

So there exists $t \in \Lambda_2$ but $t \notin \Lambda_1$ and hence $P_t \sim P_{\Lambda_2}$ but $P_t \nsim P_{\Lambda_1}$ which implies $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$, a contradiction. Therefore $\mathbb{R}_{In_p(\mathbb{Z}_n)} = In_p(\mathbb{Z}_n)$. Hence the result follows.
Now we are interested to find the partition dimension of the graph $In_p(\mathbb{Z}_n)$.

Theorem 2.36 ([10], Theorem 3.1)**.** *If G is a graph of order $n \geqslant 3$ and diameter d, then $g(n,d) \leqslant pd(G) \leqslant n-d+1$, where $g(n,d)$ is the least positive integer k for which $(d+1)^k \geqslant n$ for integers n and d with $n \geqslant d \geqslant 2$.*

Since for $k = 2$, $In_p(\mathbb{Z}_n) \cong P_3$, we have $\mathrm{pd}(In_p(\mathbb{Z}_n)) = 2$ ([10], Proposition 2.1).

Theorem 2.37. *The partition dimension of the graph $In_p(\mathbb{Z}_n)$ satisfy the inequality $k-1 \leqslant pd(In_p(\mathbb{Z}_n)) \leqslant k$ for $k \geqslant 3$.*

Proof. Let $k \geqslant 3$. Since $In_p(\mathbb{Z}_n)$ is a graph of diameter 2, by applying Theorem 2.36 we have $g(n,2) \leqslant \mathrm{pd}(In_p(\mathbb{Z}_n))$, where $g(n,2)$ is the least positive integer l for which $(2+1)^l = 3^l \geqslant n = 2^k - 1$. Clearly $l = k-1$. Therefore we have the inequality

$$pd(In_p(\mathbb{Z}_n)) \geqslant k-1 \tag{5}$$

Now we present a k-resolving partition $\Pi = (S_1, S_2, \ldots, S_k)$ of $V(In_p(\mathbb{Z}_n))$ where
$S_1 = N[P_1]$,
$S_2 = \{P_2, \{P_{\Lambda_2} : |\Lambda_2| = 2 \text{ and } 2 \in \Lambda_2 \text{ but } 1 \notin \Lambda_2\}, \ldots, \{P_{\Lambda_{k-1}} : |\Lambda_{k-1}| = k-1 \text{ and } 2 \in \Lambda_{k-1} \text{ but } 1 \notin \Lambda_{k-1}\}\}$,
...
$S_{k-1} = \{P_{k-1}, P_{k-1k}\}$,
$S_k = \{P_k\}$.

Let $v_1 \in S_1$. Then $v_1 = P_\Lambda$ such that $1 \in \Lambda$ and $r(v_1, \Pi) = (0, a_2, \ldots, a_k)$, where $a_i = 1$ if $i \in \Lambda$ otherwise $a_i = 2$.

Let $v_2 \in S_2$. Then $v_2 = P_{\Lambda_2}$ such that $2 \in \Lambda_2$ but $1 \notin \Lambda_2$. Then $r(v_2, \Pi) = (1, 0, a_3, \ldots, a_k)$ where $a_i = 1$ if $i \in \Lambda_2$ otherwise $a_i = 2$.

...

Let $v_t \in S_t$, $t \in [k]$. Then $v_t = P_{\Lambda_t}$ such that $t \in \Lambda_t$ but $1, 2, \ldots, t-1 \notin \Lambda_t$. Then $r(v_t, \Pi) = (1, a_2, \ldots, a_t = 0, \ldots, a_k)$ where $a_i = 1$ if $i \in \Lambda_t$ otherwise $a_i = 2$.

...

Similarly as above for $v_k = P_k \in S_k$ we have $r(v_k, \Pi) = (0, 1, 1 \ldots, 1)$.

Since representations of all vertices with respect to the partition Π of $V(In_p(\mathbb{Z}_n))$ are distinct, clearly Π is a resolving partition of $In_p(\mathbb{Z}_n)$. Therefore

$$pd(In_p(\mathbb{Z}_n)) \leqslant k \tag{6}$$

Combining equation (5) and (6) we have the required result. □

Remark 2.38. Also note that for $k = 3$, no 2-partition can be a resolving partition of $In_p(\mathbb{Z}_n)_{k=3}$. Since for $k = 3$, $In_p(\mathbb{Z}_n)$ has seven vertices so one set of a 2-partitions of $V(In_p(\mathbb{Z}_n))$ must contain at least four vertex but we cannot have four distinct representations with respect to Π for this four vertices. So $\mathrm{pd}(In_p(\mathbb{Z}_n))_{k=3} = 3$.

In a similar way we can prove that $\mathrm{pd}(In_p(\mathbb{Z}_n))_{k=4} = 4$.

3. Inclusion prime ideal graph of a ring

Throughout this section, by a ring R, we mean a commutative ring with unity and domain is a commutative ring with unity having no zero divisors. Here we consider the inclusion prime ideal graph of a ring R, denoted by $In_p(R)$, is a graph with vertices are nontrivial prime ideals of R and two distinct vertices are adjacent if and only if one is contained in the other. We denote the Jacobson radical, set of all nonzero prime ideals and the set of all maximal ideals of a ring R by $J(R)$, $spec(R)$ and $\mathrm{Max}(R)$ respectively. For a prime ideal P of R, we define $M(P) = \{M_i \in Max(R) : P \subseteq M_i\}$. For undefined terminology in commutative ring theory, see[6] and [24].

Theorem 3.1. *Let R be a ring. Then the following statements are equivalent:*
(1) $In_p(R)$ is an empty graph.
(2) R is a field.
(3) $In(R)$ is an empty graph.

Proof. (1) ⇔ (2). Let $In_p(R)$ be an empty graph but R is not a field. Then $I = (0)$ is not a maximal ideal of R. Hence there exists a maximal ideal M containg (0) ([6], Corollary 1.4), which is prime also. Therefore $M \in V(In_p(R))$, which contradicts that $In_p(R)$ is empty. Consequently R is a field. The converse is obvious.
(2) ⇔ (3). The proof is clear as every simple commutative ring with unity is a field. □

Theorem 3.2. *Let R be a ring which is not an integral domain. Then the following statements are equivalent:*
(1) *$In_p(R)$ is a null graph.*
(2) *Prime ideals of R are maximal as well as minimal.*
(3) *R is a zero dimensional ring.*
(4) *Any quotient ring of R that is an integral domain is also a field.*

Proof. (1) $\Leftrightarrow$ (2). Let $In_p(R)$ be a null graph and P be a nontrivial prime ideals of R which is not maximal. Since R has identity, P must be contained in some maximal ideal M of R ([6], Corollary 1.4). Since M is prime also, we have $P \sim M$, which contradicts that $In_p(R)$ is a null graph. Hence every nontrivial prime ideals are maximal. Also if possible let there exists a prime ideal P_1 which is not minimal. Then there exists another non zero prime ideal P_2 of R such that $P_2 \subset P_1$. Then $P_1 \sim P_2$, which is a contradiction. Consequently every nontrivial prime ideals of R are minimal.

Conversely, let P_1, $P_2 \in V(In_p(R))$. Then by hypothesis P_1 and P_2 are maximal as well as minimal. Therefore $P_1 \nsim P_2$ and hence $In_p(R)$ is a null graph.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (1) It is clear.
(2) $\Leftrightarrow$ (4). The proof is clear by just recalling that an ideal I of a commutative ring R with unity is prime (resp. maximal) if and only if the quotient ring R/I is an integral domain (resp. field). □

Example 3.3. The graph $In_p(R)$ of a regular ring R is a null graph since every prime ideals of R is maximal ([18]). The graph $In_p(R)$ of an Artinian ring R is a null graph as R is zero dimensional ([6], Theorem 8.5).

Theorem 3.4. *Let R be an integral domain which is not a field. Then the following statements are equivalent:*
(1) *$In_p(R)$ is a null graph.*
(2) *Every non zero prime ideals of R are maximal.*
(3) *Semiprimary ideals of R are primary.*
(4) *Valuation ideals are primary.*

Proof. (1) $\Leftrightarrow$ (2). The proof is straightforward.
(2) $\Leftrightarrow$ (3). Let J be a semiprimary ideal of R. If $J = (0)$, then J is prime and hence semiprimary. Again, if $J \neq 0$, then $\sqrt{J}$ is a maximal ideal of R. Therefore J is primary ([24], p. 153) and hence semiprimary. The proof of the converse part follows from Corollary 3.2 of [12].
(2) $\Leftrightarrow$ (4). It is clear from Theorem 3.1 of [13]. □

Remark 3.5. Clearly if R is a principal ideal domain, then $In_p(R)$ is a null graph. Moreover, rings R in which nonzero prime ideals are maximal, the graph $In_p(R)$ is complete or connected if and only if R is local if and only if R has exactly one nonzero prime ideal. So we have the following immediate result.

Corollary 3.6. *Let R be an Artinian ring. Then the following statements are equivalent:*
(1) *$In_p(R)$ is a complete graph.*
(2) *$In_p(R)$ is a connected graph.*
(3) *$In_p(R) \cong K_1$.*
(4) *R is local.*

Theorem 3.7. *Let R be a commutative ring with unity. Then the following statements are equivalent:*
(1) *$In_p(R)$ is a complete graph.*
(2) *prime ideals of R are linearly ordered.*
(3) *radical ideals of R are linearly ordered.*
(4) *each proper radical ideal are prime.*
(5) *The radical ideals of principal ideals of R are linearly ordered.*
(6) *For each a, $b \in R$, there exists $k \in \mathbb{N}$ such that $a|b^k$ or $b|a^k$.*
(7) *Intersection of two prime ideals of R is a prime ideal.*
(8) *2-absorbing primary ideals of R are semiprimary.*
(9) *R is local with incomparable prime ideals are co-maximal.*

Proof. (1) $\Leftrightarrow$ (2). It is clear.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow \cdots \Leftrightarrow$ (6). It follows from Theorem 1 of [7].
(2) $\Leftrightarrow$ (7). It is clear.
(2) $\Leftrightarrow$ (8). Let I be a 2-absorbing primary ideal of R. Since prime ideals of R are linearly ordered, clearly $\sqrt{I}$ is a prime ideal and hence I is a semiprimary ideal of R.

Conversely, let P_1 and P_2 be two distinct prime ideals of R. Then $P_1 \cap P_2$ is a 2-absorbing primary ideal ([5], Theorem 1) and hence semiprimary by hypothesis. Since $\sqrt{P_1 \cap P_2} = P_1 \cap P_2$, clearly $P_1 \cap P_2$ is a prime ideal of R. Therefore prime ideals of R are linearly ordered.
(1) $\Leftrightarrow$ (9). Let $In_p(R)$ be a complete graph. Then clearly R is local and as there is no incomparable prime ideals in R, vacuosly they are comaximal. For converse part, let P_1 and P_2 be two distinct nontrivial prime ideals of R. Since R is a local, P_1 and P_2 must be contained in the unique maximal ideal M of R. Now if they are incomparable, then $P_1 + P_2 = R$, contradicts

that they both contained in M. Therefore $P_1 \sim P_2$ and hence $In_p(R)$ is a complete graph. □

Theorem 3.8. *Let R be a domain which is not a field. Then the following statements are equivalent:*
(1) *$In_p(R)$ is a complete graph.*
(2) *R is a local treed domain.*

Proof. (1) $\Leftrightarrow$ (2). It is clear by applying Theorem 3.7 and recalling that a treed domain is a domain in which incomparable prime ideals are comaximal. □

Theorem 3.9. *Let R be an arithmetical ring. Then*
(1) *a vertex $P \in V(In_p(R))$ is universal if and only if $P \subseteq J(R)$.*
(2) *$In_p(R)$ is a complete graph if and only if R is local.*
(3) *The independence number of $In_p(R)$ is $\alpha(In_p(R)) = |Max(R)|$.*
(4) *$gr(In_p(R))$*

$$= \begin{cases} 3\ , & \text{if } M(P_1) \cap M(P_2) \neq \phi \text{ for some } P_1, P_2 \in spec(R) - Max(R) \\ \infty, & \text{otherwise.} \end{cases}$$

(5) *$In_p(R)$ is a star graph if and only if $J(R)$ contains a prime ideal P of R such that $spec(R) - Max(R) = \{P\}$.*

Proof. (1). Let $P \in V(In_p(R))$ be a universal vertex. Then $P \subseteq M_i$ for every $M_i \in Max(R)$ and hence $P \subseteq J(R)$.

Conversely, let $P \subseteq J(R)$ and $Q \in V(In_p(R)) - \{P\}$. Clearly $M(P) \cap M(Q) \neq \phi$ and hence there exists $M_i \in Max(R)$ such that P, $Q \subseteq M_i$. Since incomparable prime ideals of an arithmetical ring are comaximal, clearly $P \sim Q$. Since Q is arbitrary, clearly P is a universal vertex in $In_p(R)$.

(2). Let $In_p(R)$ be a complete graph. Then clearly R is local.

Conversely, let R be a local ring with maximal ideal M and P_1, P_2 be two distinct vertices in $In_p(R)$. Clearly $P_1, P_2 \subseteq M$ and hence not comaximal. Since in an arithmetical ring incomparable prime ideals are comaximal, clearly $P_1 \sim P_2$. Therefore $In_p(R)$ is a complete graph.

(3). Since no two maximal ideals of R adjacent in $In_p(R)$, clearly $Max(R)$ is an independent set in $In_p(R)$. Now if there is an independent set I such that $Max(R) \subset I$, then there exists a prime ideal $P_1 \in I$ such that $P_1 \sim M$ for some maximal ideal $M \in Max(R)$, which is a contradiction. Now we prove

that there does not exists any independent set of cardinality strictly greater than $|Max(R)|$. If possible, let I_1 be such an independent set in $In_p(R)$. Then there must exists two prime ideals which are contained in some unique maximal ideal of R and hence adjacent, which is a contradiction. Therefore, $\alpha(In_p(R)) = |Max(R)|$.
(4). If there exists two prime ideals P_1, $P_2 \in spec(R) - Max(R)$ such that $M(P_1) \cap M(P_2) \neq \phi$, then P_1 and P_2 must be contained in some maximal ideal $M_i \in Max(R)$. Therefore $P_1 \sim P_2 \sim M_i \sim P_1$ is a 3-cycle in $In_p(R)$ and hence $\text{gr}(In_p(R)) = 3$, otherwise $\text{gr}(In_p(R)) = \infty$.
(5). Let $In_p(R)$ be a star graph with P as universal vertex. Then by (1), $P \subseteq J(R)$. Now if there exists another prime ideal $P_1 \in spec(R) - Max(R)$, then P_1 and P must be contained in some maximal ideal M of R. Therefore $P_1 \sim P \sim M \sim P_1$ is a cycle in $In_p(R)$, contradicts that $In_p(R)$ is a star graph. Hence $spec(R) - Max(R) = \{P\}$.

Conversely, let $J(R)$ contains a prime ideal P such that $spec(R) - Max(R) = \{P\}$. Then P is a universal vertex in $In_p(R)$ and no two prime ideal ideals in $spec(R) - \{P\}$ are adjacent. Consequently, $In_p(R)$ is a star graph. □

Theorem 3.10. *The graph $In_p(R)$ is a perfect graph. Moreover,* $\omega((In_p(R)) = \chi(In_p(R)) =$

$$\begin{cases} dim(R) \text{ or } dim(R)+1\ , & \text{if } R \text{ is a domain or not with } dim(R) < \infty \\ \infty, & \text{if } dim(R) = \infty \end{cases}$$

Proof. In a similar way as in Theorem 2.8 of [16], we can prove that $In_p(R)$ is a comparability graph and hence perfect. To compute the clique number of $In_p(R)$, we consider the following cases:
Case (1). $dim(R) = \infty$. Since any chain of nonzero prime ideals of length k is a clique with k vertices, clearly $\omega(In_p(R)) = \infty$.
Case (2). Let $dim(R) = k < \infty$. Then clearly $\omega(In_p(R)) \geqslant k$ or $k+1$ according as R is an domain or not. Now we assume that $W = \{P_i : 1 \leqslant i \leqslant \omega(In_p(R))\}$ is the set of vertex of a clique in $In_p(R)$. We now prove by method of induction, that the prime ideals $P_1, P_2, \ldots, P_t$ form a chain, where $t = \omega(In_p(R))$. The statement is clear for $t = 1$. Now let the prime ideals in $A_1 = \{P_1, P_2, \ldots, P_r\}$, where $1 < r < t$, form a chain. We now show that prime ideals in $A_2 = \{P_1, P_2, \ldots, P_{r+1}\}$ also form a chain. Let $P_{i_1} \subset P_{i_2} \subset \cdots \subset P_{i_r}$, where $i_j \in [r]$, be the chain of elements in A_1. If $P_{r+1} \subset P_{i_1}$, then there exists the chain $P_{r+1} \subset P_{i_1} \subset P_{i_2} \subset \cdots \subset P_{i_r}$ in

A_2. Otherwise, let P_{i_t} be the maximal element of A_1 which is a subset of P_{r+1}. Then easily $P_{r+1} \subset P_{i_{t+1}}$. Therefore we have the chain $P_{i_1} \subset P_{i_2} \subset P_{i_t} \subset P_{r+1} \subset P_{i_t+1} \subset \cdots \subset P_{i_r}$. Therefore $\omega(In_p(R)) \leqslant k$ or $k+1$ as R is a domain or not. Since $In_p(R)$ is perfect, we have the desired result. $\square$

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