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The structures of full terms preserving a partition under different operations

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Abstract. Full terms that preserve a partition in a finite set are extension of full terms, which can be applied to classify algebras of the same type into subclasses known as the full solid variety preserving a partition. In this paper, two associative binary operations induced by a superassociative superposition defined on the set of full terms preserving a partition are given. As a generalization, sets of such full terms and their binary operations are also discussed.

1. Introduction and preliminaries

One of the exceptional classes of terms is a full term introduced in [6]. To achieve this, let n be a fixed positive integer, I an index set and τ_n a type of operation symbols of arity n, that is $\tau_n = (n_i)_{i\in I}$ where $n_i = n$ for all $i \in I$. We recall from [6] that the set T_n of all mappings α on a finite set $\bar{n} := \{1, \dots, n\}$ and a binary composition of functions forms a semigroup known as a transformation semigroup. Thus, an n-ary full term of type τ_n is inductively defined by the following: (1) $f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ is an n-ary full term of type τ_n if $\alpha \in T_n$ and (2) if $t_1, \dots, t_n$ are n-ary full terms of type τ_n and f_i is an operation symbol of type τ_n, then $f_i(t_1, \dots, t_n)$ is an n-ary full term of type τ_n. Hence, the set of all n-ary full term of type τ_n, denoted by $W^F_{\tau_n}(X_n)$, is closed under the following superposition $S^n : (W^F_{\tau_n}(X_n))^{n+1} \to W^F_{\tau_n}(X_n)$ which is given by the following : $S^n(t, t_1, \dots, t_n) = f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)})$ where $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ for any mapping α in T_n and $S^n(t, t_1, \dots, t_n) =$

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$f_i\left(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n)\right)$ if $t=f_i(s_1,\ldots,s_n)$. As a consequence, $\left(W^F_{\tau_n}(X_n),S^n\right)$ has been formed. This algebra always plays a key role in the theory of full solid variety. For this importance, see [1, 3].

One of the generalizations of terms is a set of terms. In fact, sets of terms are called tree languages, see [1, 4, 12]. The symbol $P(W^F_{\tau_n}(X_n))$ denotes the set of all subsets or tree languages of all n-ary full terms of type τ_n. For instance, we see that $\{f(x_1,x_2,x_3)\}$ and $\{g(x_2,x_2,x_2),f(x_1,x_1,x_1)\}$ are examples of tree languages in the power set $P(W^F_{(3,3)}(X_3))$. On the other hand, a set $\{g(x_2,f(x_2,x_3,x_1),x_3)\}$ is not a tree language of ternary full terms of type $(3,3)$. To compute the result of tree languages of full terms in the theory of full hyperidentities, in [16], a non-deterministic superposition operation on the set $P(W^F_{\tau_n}(X_n))$ was defined. By the definition, a mapping $\widehat{S}^n : P(W^F_{\tau_n}(X_n))^{n+1}\to P(W^F_{\tau_n}(X_n))$ is defined as follows:

(1) $\widehat{S}^n\left(\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\},B_1,\ldots,B_n\right)=\{f_i(r_{\alpha(1)},\ldots,r_{\alpha(n)})\mid r_{\alpha(j)}\in B_{\alpha(j)}, j=1,\ldots,n\}$,

(2) $\widehat{S}^n\left(\{f_i(t_1,\ldots,t_n)\},B_1,\ldots,B_n\right)=\{f_i(r_1,\ldots,r_n)\mid r_j\in\widehat{S}^n\left(\{t_j\},B_1,\ldots,B_n\right), j=1,\ldots,n\}$,

(3) if $|A|>1$, then $\widehat{S}^n(A,B_1,\ldots,B_n)=\cup_{a\in A}\{\widehat{S}^n(\{a\},B_1,\ldots,B_n)\}$,

(4) $\widehat{S}^n(A,B_1,\ldots,B_n)=\emptyset$ if $A=\emptyset$ or $B_j=\emptyset$ for some j.

Recently, subclasses of full terms are given using different transformations. For example, in [18], a semigroup $S(\bar{n},Y)=\{\beta\in T_n\mid\beta(Y)\subseteq Y\}$ of transformations on a finite set $\bar{n}$ leaving $Y\subseteq\bar{n}$ invariant was applied to set a new term such that each pair of these terms was extended to be $S(\bar{n},Y)$-hyperidentity of a variety V. Binary operations, $+$ and $*$, defined on this set are presented, which lead to construct semigroups of full terms with an invariant subset. In [17], the theorem that establishes the freeness of an algebra consisting of the set of all terms generated by transformations with a restricted range and an $(n+1)$-ary operation satisfying certain equational laws has been mentioned.

Recall from [15] that terms defined by transformations preserving a partition are presented and their structures are established. Actually, let $\mathcal{P}$ be a partition on a finite set $\bar{n}$ and consider the set

$$T(\bar{n},\mathcal{P})=\{f\in T_n\mid\forall A_i\in\mathcal{P},\,\exists A_j\in\mathcal{P},\,f(A_i)\subseteq A_j\}.$$

It is known that $T(\bar{n},\mathcal{P})$ is a subsemigroup of T_n. Hence, an n-ary full term preserving a partition $\mathcal{P}$ on $\bar{n}$ of type τ_n is inductively defined in [15] as follows: (1) $f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ is an n-ary full term preserving a partition $\mathcal{P}$ on $\bar{n}$ of type τ_n if $\alpha \in T(\bar{n},\mathcal{P})$, and (2) if $t_1,\ldots,t_n$ are n-ary full terms preserving a partition $\mathcal{P}$ on $\bar{n}$ of type τ_n, then $f_i(t_1,\ldots,t_n)$ so is. Let $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ be the set of all n-ary full terms preserving a partition $\mathcal{P}$ on $\bar{n}$ of type τ_n. Moreover, the set $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ is closed under the operation S^n defined for the set of full terms.

For a superassociative algebra, also called a Menger algebra, we refer to a pair of a nonempty set with an operation $\circ$ of type $(n+1)$ satisfying the superassociative law, i.e.,

$$\circ\,(\circ(a,b_1,\ldots,b_n),c_1,\ldots,c_n) = \circ\,(a,\circ(b_1,c_1,\ldots,c_n),\ldots,\circ(b_n,c_1,\ldots,c_n))\,.$$

Recent developments in superassociative algebras can be found in [2, 7, 8, 9, 10, 11, 14]. It was proved that the sets $W_{\tau_n}^{F}(X_n)$ and $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ together with the superassociative operation S^n form superassociative algebras. Furthermore, the operation $\widehat{S}^n$ defined on $P(W_{\tau_n}^{F}(X_n))$ is also superassociative.

In this work, based on the study of $T(\bar{n},\mathcal{P})$-full terms described in the paper [15] and semigroups of full terms with an invariant subset given in [13], we continue to investigate algebraic properties of full terms preserving a partition in depth. Thus, the paper is organized as the following: Section 2 is devoted to the study of binary operations defined on the set of full terms preserving a partition, which detemined by the superassociative operation S^n, and their power sets. We further prove the embeddability of the semigroups of $T(\bar{n},\mathcal{P})$-full terms into the semigroups of tree languages of $T(\bar{n},\mathcal{P})$-full terms. In Section 3, mappings whose images are full terms preserving a partition and their sets are examined and connections between these mappings and a non-deterministic full hypersubstitution σ^{nd} which preserves a partition is a mapping that sends any n-ary operation symbol to a set of full term preserving a partition are described.

2. Binary operations defined on $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$

To enhance understanding, we begin the results in this section with some examples of full terms preserving a partition of some type.

Example 2.1. Let $\tau_4 = (4,4)$ be a type with two quaternary operation symbols f_1 and f_2. Let $\mathcal{P} = \{\{1,2\},\{3,4\}\}$ be a partition of $\bar{4}$. It can be seen that

$$f_1(x_1,x_1,x_3,x_3), f_1(x_2,x_2,x_3,x_3), f_2(x_4,x_3,x_2,x_1), f_2(x_1,x_1,x_1,x_1)$$

are elements in the set of $W_{\tau_4}^{T(\bar{4},\{\{1,2\},\{3,4\}\})}(X_4)$ because $\begin{pmatrix}1&2&3&4\\1&1&3&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}$ belong to $T(\bar{4},\{\{1,2\},\{3,4\}\})$.

In other ways,
$f_1(x_1,x_3,x_2,x_4), f_2(x_1,x_4,x_2,x_3), f_1(x_1,x_3,x_2,x_2) \notin W_{\tau_4}^{T(\bar{4},\{\{1,2\},\{3,4\}\})}(X_4)$ since $\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&3&2&2\end{pmatrix} \notin T(\bar{4},\{\{1,2\},\{3,4\}\})$.

We now define two binary operations on $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$.

Definition 2.2. The binary operation $+ : (W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))^2 \to W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ is defined by $s+t = S^n(s,\underbrace{t,\ldots,t}_{n-\text{times}})$ for all $s,t \in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$.

Definition 2.3. On the Cartesian product $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n$, the binary operation $*$ is defined by $(s_1,\ldots,s_n)*(t_1,\ldots,t_n) = (S^n(s_1,t_1,\ldots,t_n),\ldots, S^n(s_n,t_1,\ldots,t_n))$ for all $(s_1,\ldots,s_n),(t_1,\ldots,t_n) \in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n$.

As a consequence, the following theorem is proved.

Theorem 2.4. *The following statements are obtained:*

(1) $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n),+\right)$ *is a semigroup,*

(2) $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n,*\right)$ *is a semigroup,*

(3) *the semigroup* $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n),+\right)$ *can be embeddable into* $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n,*\right)$.

Proof. We first prove that the statement (1) holds. Let s,t,u be any elements in $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. We have $(s+t)+u = S^n(s,t,\ldots,t)+u = S^n(S^n(s,t,\ldots,t),u,\ldots,u) = S^n(s,S^n(t,u,\ldots,u),\ldots,S^n(t,u,\ldots,u)) = s+S^n(t,u,\ldots,u) = s+(t+u)$ since the operation S^n satisfies the superassociativity. As a result, $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n),+\right)$ is a semigroup. For the

statement (2), let $(s_1,\ldots,s_n),(t_1,\ldots,t_n),(u_1,\ldots,u_n)\in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n$. We have
$\Big((s_1,\ldots,s_n)*(t_1,\ldots,t_n)\Big)*(u_1,\ldots,u_n)=(s_1,\ldots,s_n)*\Big((t_1,\ldots,t_n)*(u_1,\ldots,u_n)\Big)$ because the operation S^n satisfies the superassociative law. Finally, to prove the statement (3), we define the mapping $\delta: W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\to W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n$ by $\delta(t)=(\underbrace{t,\ldots,t}_{n-\text{times}})$ for all $t\in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. To show that δ is a homomorphism, we let $s,t\in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. Then, by the definition, we have $\delta(s+t)=(\underbrace{s+t,\ldots,s+t}_{n-\text{times}})=\Big(S^n(s,t,\ldots,t),\ldots,S^n(s,t,\ldots,t)\Big)=(s,\ldots,s)*(t,\ldots,t)=\delta(s)*\delta(t)$. It is obvious that δ is an injection. As a consequence, we can say that $\Big(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n),+\Big)$ can be embedded into $\Big(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n,*\Big)$. □

We now focus on subsets of $W_{\tau_n}(X_n)$. Normally, the set of all subsets of full terms preserving a partition is denoted by $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ and its elements are called tree languages that preserve a partition.

Example 2.5. Let $\tau_4=(4,4)$ be a type with quaternary operation symbols f_1 and f_2. Let $\mathcal{P}=\{\{1,2\},\{3,4\}\}$ be a partition of $\bar{4}$. Since $\begin{pmatrix}1&2&3&4\\1&1&3&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}$ are in $T\,(\bar{4},\{\{1,2\},\{3,4\}\})$. Then we have that $\emptyset,\{f_1(x_1,x_1,x_3,x_3)\},\{f_2(x_2,x_2,x_3,x_3)\}$, $\{f_1(x_4,x_3,x_2,x_1)\},\{f_1(x_1,x_1,x_3,x_3),f_2(x_2,x_2,x_3,x_3)\}$, $\{f_1(x_4,x_3,x_2,x_1), f_2(x_1,x_1,x_1,x_1)\}$ and $\{f_1(x_1,x_1,x_3,x_3),f_2(x_1,x_1,x_1,x_1),f_1(x_4,x_3,x_2,x_1)\}$ are elements in the set $P(W_{\tau_4}^{T(\bar{4},\mathcal{P})}(X_4))$. On the other hand, we see that $\{f_1(x_1,x_3,x_2,x_4)\},\{f_2(x_1,x_4,x_2,x_3)\},\{f_1(x_1,x_3,x_2,x_2)\}\notin P(W_{\tau_4}^{T(\bar{4},\mathcal{P})}(X_4))$ because $\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$, $\begin{pmatrix}1&2&3&4\\1&3&2&2\end{pmatrix}\notin T\,(\bar{4},\{\{1,2\},\{3,4\}\})$.

Construction of the operation of type $(n+1)$ on the set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ can be naturally defined by the following.

Definition 2.6. Let n be a positive integer and $A,A_1,\ldots,A_n\in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. Then we define the operation $\dot{S}^n$ on $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ by

$$\dot{S}^n(A, A_1, \ldots, A_n) = \begin{cases} \{S^n(t, t_1, \ldots, t_n) \mid t \in A, t_j \in A_j, 1 \leq j \leq n\} \\ \quad \text{if all of the sets } A, A_1, \ldots, A_n \text{ are not empty,} \\ \\ \emptyset \quad \text{if at least one of the sets } A, A_1, \ldots, A_n \text{ is empty.} \end{cases}$$

Let us consider the following example.

Example 2.7. Let $\tau_3 = (3, 3)$ be a type with two ternary operation symbols f_1 and f_2. On the set $W_{(3,3)}^{T(\bar{3},\{\{1,2\},\{3\}\})}(X_3)$, we let $A = \{f_1(x_1, x_2, x_3)\}, A_1 = \{f_1(x_2, x_2, x_1)\}, A_2 = \{f_2(x_3, x_3, x_2)\}, A_3 = \{f_2(x_3, x_3, x_3), f_2(x_1, x_1, x_1)\}$. Then we have

$$\begin{aligned} \dot{S}^3(A, A_1, A_2, A_3) &= \dot{S}^3(\{f_1(x_1, x_2, x_3)\}, \{f_1(x_2, x_2, x_1)\}, \{f_2(x_3, x_3, x_2)\}, \\ &\quad \{f_2(x_3, x_3, x_3), f_2(x_1, x_1, x_1)\}) \\ &= \{S^3(f_1(x_1, x_2, x_3), f_1(x_2, x_2, x_1), f_2(x_3, x_3, x_2), f_2(x_3, x_3, x_3))\} \\ &\quad \cup \{S^3(f_1(x_1, x_2, x_3), f_1(x_2, x_2, x_1), f_2(x_3, x_3, x_2), f_2(x_1, x_1, x_1))\} \\ &= \{f_1(f_1(x_2, x_2, x_1), f_1(x_2, x_2, x_1), f_2(x_3, x_3, x_3))\} \\ &\quad \cup \{f_1(f_1(x_2, x_2, x_1), f_1(x_2, x_2, x_1), f_2(x_1, x_1, x_1))\} \\ &= \{f_1(f_1(x_2, x_2, x_1), f_1(x_2, x_2, x_1), f_2(x_3, x_3, x_3)), \\ &\quad f_1(f_1(x_2, x_2, x_1), f_1(x_2, x_2, x_1), f_2(x_1, x_1, x_1))\}. \end{aligned}$$

A relationship between the operations $\dot{S}^n$ and $\widehat{S}^n$ is now explained.

Proposition 2.8. *Let n be a positive integer and $A, A_1, \ldots, A_n \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. Then $\dot{S}^n(A, A_1, \ldots, A_n) \subseteq \widehat{S}^n(A, A_1, \ldots, A_n)$.*

Proof. We first show by induction on a characteristic of an n-ary full term t preserving a partition that $\widehat{S}^n(\{t\}, \{a_1\}, \ldots, \{a_n\}) \subseteq \widehat{S}^n(\{t\}, A_1, \ldots, A_n)$ for all $a_i \in A_j$, $i = 1, \ldots, n$. Suppose that α is a mapping in $T(\bar{n}, \mathcal{P})$. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then we obtain

$$\begin{aligned} & \widehat{S}^n(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, \{a_1\}, \ldots, \{a_n\}) \\ = \; & \{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \mid r_{\alpha(j)} \in \{a_{\alpha(j)}\}, j = 1, \ldots, n\} \\ = \; & \{f_i(a_{\alpha(1)}, \ldots, a_{\alpha(n)})\} \\ \subseteq \; & \{f_i(p_{\alpha(1)}, \ldots, p_{\alpha(n)}) \mid p_{\alpha(j)} \in A_{\alpha(j)}, \forall j = 1, \ldots, n\} \\ = \; & \widehat{S}^n(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, A_1, \ldots, A_n). \end{aligned}$$

Assume that $t = f_i(t_1, \ldots, t_n)$ and our claimed is satisfied for $t_1, \ldots, t_n$. Then we get

$$\begin{array}{rl} & \widehat{S}^n\left(\{f_i\left(t_1,\ldots,t_n\right)\},\{a_1\},\ldots,\{a_n\}\right) \\ = & \{f_i\left(r_1,\ldots,r_n\right) \mid r_j \in \widehat{S}(\{t_j\},\{a_1\},\ldots,\{a_n\}), \text{ for all } j=1,\ldots,n\} \\ \subseteq & \{f_i\left(r_1,\ldots,r_n\right) \mid r_j \in \widehat{S}(\{t_j\},A_1,\ldots,A_n), \text{ for all } j=1,\ldots,n\} \\ = & \widehat{S}^n\left(\{f_i\left(t_1,\ldots,t_n\right)\},A_1,\ldots,A_n\right). \end{array}$$

From these preparations, we conclude that

$$\begin{array}{rcl} \dot{S}^n\left(A,A_1,\ldots,A_n\right) & = & \left\{\widehat{S}^n\left(a,a_1,\ldots,a_n\right) | a \in A, a_j \in A_j, \forall j=1,\ldots,n\right\} \\ & = & \cup_{a\in A}\widehat{S}^n\left(\{a\},\{a_1\},\ldots,\{a_n\}\right) \\ & \subseteq & \cup_{a\in A}\widehat{S}^n\left(\{a\},A_1,\ldots,A_n\right) \\ & = & \widehat{S}^n\left(A,A_1,\ldots,A_n\right). \end{array}$$

This completes a proof. □

We remark that in general the operation $\dot{S}^n$ does not satifies the superassociative law as the following counterexample. Following Example 2.7, we now consider the sets $\dot{S}^3\left(A, \dot{S}^3(A_1,A_2,A_1,A_3), \dot{S}^3(A_2,A_2,A_1,A_3), \dot{S}^3(A_3,A_2,A_1,A_3)\right)$ and $\dot{S}^3\left(\dot{S}^3(A,A_1,A_2,A_3), A_2, A_1, A_3\right)$. It follows from a direct calculation by the definition of the operation $\dot{S}^n$ and S^n that these two sets are different.

Then we have:

Theorem 2.9. *The set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ is closed under the non-deterministic superposition $\widehat{S}^n$.*

Proof. For this, we let $A, B_1, \ldots, B_n \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. We must show that $\widehat{S}^n\left(A,B_1,\ldots,B_n\right) \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. We will consider the following 3 cases.

Case 1 : If at least one of $A, B_1, \ldots, B_n$ is an empty set, then $\widehat{S}^n\left(A,B_1,\ldots,B_n\right) = \emptyset$. Thus $\widehat{S}^n\left(A,B_1,\ldots,B_n\right) \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$.

Case 2 : Assume that A is a one element set.

Case 2.1 : If $A = \{f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)\}$, where $\alpha \in T(\bar{n},\mathcal{P})$, then $\widehat{S}^n\left(\{f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)\},B_1,\ldots,B_n\right) = \{f_i\left(r_{\alpha(1)},\ldots,r_{\alpha(n)}\right) \mid r_{\alpha(j)} \in B_{\alpha(j)}, j=1,\ldots,n\}$ belongs to the set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ because $B_{\alpha(j)}$ elements in $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ for all $j=1,\ldots,n$.

Case 2.2 : If $A = \{f_i\left(t_1,\ldots,t_n\right)\}$ for each $\widehat{S}^n\left(\{t_j\},B_1,\ldots,B_n\right)$ elements in $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. Thus $\widehat{S}^n\left(\{f_i\left(t_1,\ldots,t_n\right)\},B_1,\ldots,B_n\right) = \{f_i(r_1,\ldots,r_n) \mid r_j \in \widehat{S}^n\left(\{t_n\},B_1,\ldots,B_n\right), j=1,\ldots,n\}$ is in the set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$.

Case 3 : Let $a \in A$. Since for each $\widehat{S}^n(\{a\}, B_1, \ldots, B_n) \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$, thus $\widehat{S}^n(A, B_1, \ldots, B_n) = \bigcup_{a\in A}\left\{\widehat{S}^n(\{a\}, B_1, \ldots, B_n)\right\} \in P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. Therefore, we conclude that the power set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ is closed under the non-deterministic superposition $\widehat{S}^n$. □

On the set $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$, two binary operation are defined by the following:

Definition 2.10. Let A and B be two elements of $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$. Then we define the binary operation $\widehat{+}$ on $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ by

$$A\widehat{+}B = \widehat{S}^n(A, \underbrace{B, \ldots, B}_{n\text{-times}}).$$

Definition 2.11. Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be elements of the Cartesian product $P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n$. Then the binary operation $\widehat{*}$ on $P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n$ is defined by

$(A_1, \ldots, A_n)\widehat{*}(B_1, \ldots, B_n) = \left(\widehat{S}^n(A_1, B_1, \ldots, B_n), \ldots, \widehat{S}^n(A_n, B_1, \ldots, B_n)\right)$.

Then we prove:

Theorem 2.12. *The following statements are obtained:*

(1) $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right), \widehat{+}\right)$ *is a semigroup,*

(2) $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n, \widehat{*}\right)$ *is a semigroup,*

(3) $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right), \widehat{+}\right)$ *is embeddable into* $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n, \widehat{*}\right)$.

Proof. First, we will show that the statement (1) holds. Let A, B, C be elements in $P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)$. Then we have that $\left(A\widehat{+}B\right)\widehat{+}C = \widehat{S}^n(A, B, \ldots, B)\widehat{+}C = \widehat{S}^n\left(\widehat{S}^n(A, B, \ldots, B), C, \ldots, C\right) = \widehat{S}^n\left(A, \widehat{S}^n(B, C, \ldots, C), \ldots, \widehat{S}^n(B, C, \ldots, C)\right) = A\widehat{+}\left(\widehat{S}^n(B, C, \ldots, C)\right) = A\widehat{+}\left(B\widehat{+}C\right)$. Hence, $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right), \widehat{+}\right)$ is a semigroup. To prove that the statement (2) holds, we let $(A_1, \ldots, A_n), (B_1, \ldots, B_n), (C_1, \ldots, C_n) \in P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n$. Since a non-deterministic superposition operation is superassociative, we obtain that $\Big((A_1,\ldots,A_n)\,\widehat{*}\,(B_1,\ldots,B_n)\Big)\widehat{*}\,(C_1,\ldots,C_n)$ and $(A_1,\ldots,A_n)\,\widehat{*}\Big((B_1,\ldots,B_n)\,\widehat{*}\,(C_1,\ldots,C_n)\Big)$ are identical. Finally, we prove that (3) holds. We define the mapping $\gamma : P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right) \to P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n$ by $\gamma(A) = (\underbrace{A,\ldots,A}_{n\text{-times}})$ for all $A \in P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)$. Obviously, γ is an injective mapping. It easy to see that $\gamma\left(A\widehat{+}B\right) = \gamma(A)\widehat{*}\gamma(B)$. Thus γ is a homomorphism. Therefore, the semigroup $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right), \widehat{+}\right)$ is embeddable into $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n, \widehat{*}\right)$. □

The following theorem gives connection between structures of full terms preserving a partition under different operations and their extensions.

Theorem 2.13. *The following statements are valid:*

(1) $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n), S^n\right)$ *is embeddable into* $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n, \widehat{S}^n\right)$,

(2) $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n), +\right)$ *is embeddable into* $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right), \widehat{+}\right)$,

(3) $\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)^n, *\right)$ *is embeddable into* $\left(P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)^n, \widehat{*}\right)$.

Proof. In order to show that the statement (1) holds, we define the mapping $\mu : W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n) \to P\left(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)\right)$ by $\mu(t) = \{t\}$ for all $t \in W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. Clearly, μ is an injective mapping. Moreover, it has a homomorphism property, i.e., $\mu\left(S^n(t, s_1,\ldots,s_n)\right) = \widehat{S}^n\left(\mu(t),\mu(s_1),\ldots,\mu(s_n)\right)$. We give a proof by induction of the complexity of t. If $t = f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)$ and $\alpha \in T(\bar{n},\mathcal{P})$, we get that

$$
\begin{aligned}
& \mu\left(S^n(f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right), s_1,\ldots,s_n)\right)\\
= \; & \mu\left(f_i\left(s_{\alpha(1)},\ldots,s_{\alpha(n)}\right)\right)\\
= \; & \left\{f_i\left(s_{\alpha(1)},\ldots,s_{\alpha(n)}\right)\right\}\\
= \; & \left\{f_i\left(r_{\alpha(1)},\ldots,r_{\alpha(n)}\right) | r_{\alpha(j)} \in \{s_{\alpha(j)}\}, j = 1,\ldots,n\right\}\\
= \; & \widehat{S}^n\left(\left\{f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)\right\}, \{s_1\},\ldots,\{s_n\}\right)\\
= \; & \widehat{S}^n\left(\mu\left(f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right)\right), \mu(s_1),\ldots,\mu(s_n)\right).
\end{aligned}
$$

If $t = f_i\left(t_1,\ldots,t_n\right)$ and assume that for each t_j, $j = 1,\ldots,n$, the statement is satisfied, then

$$\begin{aligned}
& \mu\left(S^n(f_i\left(t_1,\ldots,t_n\right),s_1,\ldots,s_n)\right)\\
= & \mu\left(f_i\left(S^n(t_1,s_1\ldots,s_n),\ldots,S^n(t_n,s_1\ldots,s_n)\right)\right)\\
= & \{f_i\left(S^n(t_1,s_1\ldots,s_n),\ldots,S^n(t_n,s_1\ldots,s_n)\right)\}\\
= & \left\{f_i\left(r_1,\ldots,r_n\right)|r_j\in\widehat{S}^n\left(\{t_j\},\{s_1\},\ldots,\{s_n\}\right)\right\}\\
= & \widehat{S}^n\left(\{f_i\left(t_1,\ldots,t_n\right)\},\{s_1\},\ldots,\{s_n\}\right)\\
= & \widehat{S}^n\left(\mu(f_i\left(t_1,\ldots,t_n\right)),\mu(s_1),\ldots,\mu(s_n)\right).
\end{aligned}$$

Thus, μ is a monomorphism. Therefore, the superassociative $\left(W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n),S^n\right)$ is embeddable into $\left(P\left(W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)^n\right),\widehat{S}^n\right)$. Next, to prove the statement (2), we define the mapping $\lambda: W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)\to P\left(W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)\right)$ by $\lambda(t)=\{t\}$ for all $t\in W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)$. Obviously, λ is an injective mapping. Furthermore, it is not difficult to see that $\lambda(s+t)=\lambda(s)\widehat{+}\lambda(t)$. Finally, we prove that (3) holds. Let $(t_1,\ldots,t_n)$ be n-tuple of full terms that preserve a partition in $W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)^n$. The mapping $\beta: W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)^n\to P\left(W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)\right)^n$ is defined by $\beta\left((t_1,\ldots,t_n)\right)=(\{t_1\},\ldots,\{t_n\})$. Obviously, β is an injective mapping. It follows from the statement (1) that β is a homomorphism. □

3. Mappings whose images belong to $W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)$

This section aims to apply the notions of full terms preserving a partition and their languages to construct a class of equations in an additional step.

Recall from [15] that full hypersubstitution σ that preserves a partition on a finite set $\bar{n}$ is a mapping which taken from $\{f_i\mid i\in I\}$ to $W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)$. For instance, let $\tau_3=(3,3)$ with the corresponding ternary operation symbols f_1 and f_2. If $\sigma_1(f_1)=f_2(x_1,x_3,x_2)$ and $\sigma_2(f_2)=f_1(x_3,x_3,x_1)$, then both σ_1 and σ_2 belong to the set $Hyp^{T(\bar{3},\{\{1,2\},\{3\}\})}(\tau_3)$. In this situation, the notation σ_t stands for σ which takes operation symbols to a term t.

On the set $Hyp^{T(\bar{n},\mathcal{P})}(\tau_n)$, it is natural to define the binary operation $+_h$ by

$$(\sigma_1+_h\sigma_2)(f_i)=S^n(\sigma_1(f_i),\underbrace{\sigma_2(f_i),\ldots,\sigma_2(f_i)}_{n-\text{times}})$$

for every $\sigma_1,\sigma_2\in Hyp^{T(\bar{n},\mathcal{P})}(\tau_n)$.

Following the paper [16], a non-deterministic full hypersubstitution σ^{nd} was given. Infact, it is a mapping of the form $\sigma^{nd}:\{f_i|i\in I\}\to P(W^F_{\tau_n}(X_n))$. Thus, we restrict our attention to the subset $P(W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n))$ of $P(W^F_{\tau_n}(X_n))$.

Indeed, a non-deterministic full hypersubstitution σ^{nd} which preserves a partition is a mapping σ that sends any n-ary operation symbol to a set of full term preserving a partition. By the symbol $Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n)$, we denote the set of all such mappings. Moreover, by σ_A^{nd} we mean σ^{nd} which sends f_i to a subset A of $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$.

Using the non-deterministic operation $\widehat{S}^n$, we now define the binary operation $\widehat{+}_h$ on $Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n)$ by

$$(\sigma_1^{nd}\widehat{+}_h\sigma_2^{nd})(f_i) = \widehat{S}^n(\sigma_1^{nd}(f_i), \sigma_2^{nd}(f_i), \ldots, \sigma_2^{nd}(f_i)).$$

As a consequence, we prove the following theorem.

Theorem 3.1. *The following assertions are true:*

(1) $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), +_h)$ *and* $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), \widehat{+}_h)$ *are semigroups,*

(2) *there is a monomorphism from the semigroup* $(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n), +)$ *into* $(Hyp^{T(\bar{n},\mathcal{P})}(\tau_n), +_h)$,

(3) *there is a monomorphism from the semigroup* $(P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)), \widehat{+})$ *into* $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), \widehat{+}_h)$,

(4) $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), +_h)$ *can be isomorphically embedded into* $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), \widehat{+}_h)$.

Proof. First, we give a proof of (1). To do this, let σ_1, σ_2 and σ_3 be elements in $Hyp^{T(\bar{n},\mathcal{P})}(\tau_n)$. Due to the superassociativity of S^n on $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$, we have

$$\begin{aligned}
\Big((\sigma_1 +_h \sigma_2) +_h \sigma_3\Big)(f_i) &= S^n\Big((\sigma_1 +_h \sigma_2)(f_i), \sigma_3(f_i), \ldots, \sigma_3(f_i)\Big)\\
&= S^n\Big(S^n(\sigma_1(f_i), \sigma_2(f_i), \ldots, \sigma_2(f_i)), \sigma_3(f_i), \ldots, \sigma_3(f_i)\Big)\\
&= S^n\Big(\sigma_1(f_i), S^n(\sigma_2(f_i), \sigma_3(f_i), \ldots, \sigma_3(f_i)), \ldots,\\
&\qquad S^n(\sigma_2(f_i), \sigma_3(f_i), \ldots, \sigma_3(f_i))\Big)\\
&= S^n\Big(\sigma_1(f_i), (\sigma_2 +_h \sigma_3)(f_i), \ldots, (\sigma_2 +_h \sigma_3)(f_i)\Big)\\
&= \Big(\sigma_1 +_h (\sigma_2 +_h \sigma_3)\Big)(f_i),
\end{aligned}$$

which show that $+_h$ is associative over the set $Hyp^{T(\bar{n},\mathcal{P})}(\tau_n)$. The proof of a semigroup $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), \widehat{+}_h)$ is similar to $(Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n), +_h)$ but it depends on the fact that $\hat{S}^n$ on $P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ satisfies the superassociative law. To prove (3), for any t in $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$, we define $\xi : W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n) \to P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n))$ by $\xi(t) = \sigma_t$. Suppose that s and t are terms in $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. Then we have $\xi(s + t) = \xi(S^n(s, t, \ldots, t)) = \sigma_{S^n(s,t,\ldots,t)} = \sigma_s +_h \sigma_t = \xi(s) +_h \xi(t)$. Thus, ξ is a homomorphism. Clearly, ξ is an injection because from $\sigma_s = \sigma_t$, we have $\sigma_s(f_i) = \sigma_t(f_i)$ and that $s = t$. The proof of (3) can be done by setting a mapping $\bar{\xi} : P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)) \to Hyp_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ by $\bar{\xi}(A) = \sigma_A^{nd}$ for all subset A of $W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$. Finally, to prove that (4) holds, let f_i be an n-ary operation symbol. For any full hypersubstitution σ that preserves a partition of type τ_n, the mapping $\overline{\sigma^{nd}} : \{f_i \mid i \in I\} \to P(W_{\tau_n}^{T(\bar{n},P)}(X_n))$ can be defined by $\overline{\sigma^{nd}}(f_i) = \{\sigma(f_i)\}$. It is clear that $\overline{\sigma^{nd}}$ belongs to the set $Hyp_{nd}^{T(\bar{n},P)}(\tau_n)$. In order to prove that a monomorphism from $Hyp^{T(\bar{n},P)}(\tau_n)$ to $Hyp_{nd}^{T(\bar{n},P)}(\tau_n)$ exists, we construct the mapping $\phi : Hyp^{T(\bar{n},P)}(\tau_n) \to Hyp_{nd}^{T(\bar{n},P)}(\tau_n)$ by $\phi(\sigma) = \overline{\sigma^{nd}}$ for all $\sigma \in Hyp^{T(\bar{n},P)}(\tau_n)$. Obviously, the mapping ϕ is an injection. Next, we let σ_1, σ_2 be two mappings in $Hyp^{T(\bar{n},P)}(\tau_n)$. We first show that $\overline{\sigma_1 +_h \sigma_2} = \overline{\sigma_1} \widehat{+}_h \overline{\sigma_2}$. In fact, by Theorem 2.13, we have $(\overline{\sigma_1 +_h \sigma_2})(f_i) = \{(\sigma_1 +_h \sigma_2)(f_i)\} = \{S^n(\sigma_1(f_i), \sigma_2(f_i), \ldots, \sigma_2(f_i))\} = \widehat{S}^n(\{\sigma_1(f_i)\}, \{\sigma_2(f_i)\}, \ldots, \{\sigma_2(f_i)\}) = \{\sigma_1(f_i)\} \widehat{+}_h \{\sigma_2(f_i)\} = (\overline{\sigma_1} \widehat{+}_h \overline{\sigma_2})(f_i)$. As a result, ϕ is a homomorphism because $\phi(\sigma_1 +_h \sigma_2) = \overline{\sigma_1 +_h \sigma_2} = \overline{\sigma_1} \widehat{+}_h \overline{\sigma_2} = \phi(\sigma_1) \widehat{+}_h \phi(\sigma_2)$. □

Following the paper [16], any extension of non-deterministic full hypersubstitutions of type τ_n,

$$\hat{\sigma}^{nd} : P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)) \to P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)),$$

can be inductively defined as follows:

(1) $\widehat{\sigma}^{nd}[\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}] = (\sigma^{nd}(f_i))_\alpha$ where $\alpha \in T(\bar{n}, \mathcal{P})$,

(2) $\widehat{\sigma}^{nd}[\{f_i(t_1, \ldots, t_n)\}] = \widehat{S}^n(\sigma^{nd}(f_i), \widehat{\sigma}^{nd}[\{t_1\}], \ldots, \widehat{\sigma}^{nd}[\{t_n\}])$ and assume that each $\widehat{\sigma}^{nd}[\{t_j\}]$ is already defined for all $1 \leq j \leq n_i$,

(3) $\widehat{\sigma}^{nd}[T] = \bigcup_{t \in T} \widehat{\sigma}^{nd}[\{t\}]$ where $T \subseteq W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)$ and $|T| > 1$,

(4) $\widehat{\sigma}^{nd}[\emptyset] = \emptyset$.

An extension of non-deterministic full hypersubstitutions of type τ_n is an endomorphism of the algebra $(P(W^F_{\tau_n}(X_n)), \widehat{S}^n)$, which means that the following identity holds:

$$\widehat{\sigma}^{nd}[\widehat{S}^n(T, T_1, \ldots, T_n)] = \widehat{S}^n(\widehat{\sigma}^{nd}[T], \widehat{\sigma}^{nd}[T_1], \ldots, \widehat{\sigma}^{nd}[T_n])$$

for all $T, T_j \subseteq W^{T(\bar{n},\mathcal{P})}_{\tau_n}(X_n)$. Under the binary operation $\circ_{nd}$ defined on the set $Hyp^F(\tau_n)$ of all non-deterministic full hypersubstitutions of type τ_n given by $\sigma_1^{nd} \circ_{nd} \sigma_2^{nd} = \widehat{\sigma}_1^{nd} \circ \sigma_2^{nd}$ for all $\sigma_1^{nd}, \sigma_2^{nd} \in Hyp^F(\tau_n)$, then by the fact that any extension of non-deterministic full hypersubstitutions is an endomorphism on $(P(W^F_{\tau_n}(X_n)), \widehat{S}^n)$ it was proved that $(Hyp^F(\tau_n), \circ_{nd})$ forms a semigroup.

We now turn our concentration to a non-deterministic full hypersubstitution that preserves a partition. Similar to the definition of $\widehat{\sigma}^{nd}$ recalled above, in the case when we consider a singleton set of the form $\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}$ where α is a transformation on $T(\bar{n}, \mathcal{P})$, an extension of each σ^{nd} in $Hyp^{T(\bar{n},\mathcal{P})}_{nd}(\tau_n)$ may be defined in the same manner of (1) and others are not different.

Let us consider a type $\tau_3 = (3, 3)$ with operation symbols Δ, ∇ and $\alpha_j \in T(\bar{3}, \{\{1\}, \{2, 3\}\})$ for $j = 1, \ldots, 5$ given by

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Let $T = \{\Delta(x_{\alpha_1(1)}, x_{\alpha_1(2)}, x_{\alpha_1(3)})\}, T_1 = \{\nabla(x_{\alpha_2(1)}, x_{\alpha_2(2)}, x_{\alpha_2(3)}), \nabla(x_{\alpha_5(1)}, x_{\alpha_5(2)}, x_{\alpha_5(3)}\}, T_2 = \{\Delta(x_{\alpha_3(1)}, x_{\alpha_3(2)}, x_{\alpha_3(3)})\}, T_3 = \{\Delta(x_{\alpha_4(1)}, x_{\alpha_4(2)}, x_{\alpha_4(3)})\}$ be elements in $P(W^{T(\bar{3},\{\{1\},\{2,3\}\})}_{(3,3)}(X_3))$ and let

$$\sigma^{nd} : \{\Delta, \nabla\} \to P(W^{T(\bar{3},\{\{1\},\{2,3\}\})}_{(3,3)}(X_3))$$

be defined by

$$\Delta \mapsto \{\nabla(x_{\alpha_5(1)}, x_{\alpha_5(2)}, x_{\alpha_5(3)})\},$$

$$\nabla \mapsto \{\Delta(x_{\alpha_3(1)}, x_{\alpha_3(2)}, x_{\alpha_3(3)})\}.$$

Then $\widehat{\sigma}^{nd}[\widehat{S}^3(T, T_1, \ldots, T_n)]$

$$
\begin{aligned}
&= \widehat{\sigma}^{nd}[\{\Delta(\nabla(x_3,x_1,x_1),\Delta(x_2,x_2,x_2),\Delta(x_3,x_3,x_3)),\Delta(\nabla(x_1,x_2,x_3),\\
&\qquad \Delta(x_2,x_2,x_2),\Delta(x_3,x_3,x_3))\}]\\
&= \widehat{\sigma}^{nd}[\{\Delta(\nabla(x_3,x_1,x_1),\Delta(x_2,x_2,x_2),\Delta(x_3,x_3,x_3))\}]\\
&\quad \cup \widehat{\sigma}^{nd}[\{\Delta(\nabla(x_1,x_2,x_3),\Delta(x_2,x_2,x_2),\Delta(x_3,x_3,x_3))\}]\\
&= \widehat{S}^3(\sigma^{nd}(\Delta),\widehat{\sigma}^{nd}[\{\nabla(x_3,x_1,x_1)\}],\widehat{\sigma}^{nd}[\{\Delta(x_2,x_2,x_2)\}],\widehat{\sigma}^{nd}[\{\Delta(x_3,x_3,x_3)\}])\\
&\quad \cup \widehat{S}^3(\sigma^{nd}(\Delta),\widehat{\sigma}^{nd}[\{\nabla(x_1,x_2,x_3)\}],\widehat{\sigma}^{nd}[\{\Delta(x_2,x_2,x_2)\}],\widehat{\sigma}^{nd}[\{\Delta(x_3,x_3,x_3)\}])\\
&= \widehat{S}^3(\nabla(x_1,x_2,x_3),\nabla(x_1,x_3,x_3),\Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2))\\
&\quad \cup \widehat{S}^3(\nabla(x_1,x_2,x_3),\nabla(x_1,x_2,x_3),\Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2))\\
&= \{\nabla(\nabla(x_1,x_3,x_3),\Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2)),\nabla(\nabla(x_1,x_2,x_3),\\
&\quad \Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2))\}.
\end{aligned}
$$

On the opposite side, by the definition of $\widehat{\sigma}^{nd}$ and the operation σ^{nd} we obtain that

$$
\begin{aligned}
&\widehat{S}^3(\widehat{\sigma}^{nd}[T],\widehat{\sigma}^{nd}[T_1],\widehat{\sigma}^{nd}[T_2],\widehat{\sigma}^{nd}[T_3])\\
&\quad = \widehat{S}^3(\{\Delta(x_1,x_2,x_3)\},\{\nabla(x_1,x_3,x_3),\nabla(x_1,x_2,x_3)\},\{\Delta(x_3,x_3,x_3)\},\{\Delta(x_2,x_2,x_2)\})\\
&\quad = \{\Delta(\nabla(x_1,x_3,x_3),\Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2)),\Delta(\nabla(x_1,x_2,x_3),\\
&\qquad \Delta(x_3,x_3,x_3),\Delta(x_2,x_2,x_2))\}.
\end{aligned}
$$

From these processes, we note that the range of $\widehat{\sigma}^{nd}$, i.e., $\widehat{\sigma}^{nd}[\widehat{S}^3(T,T_1,\ldots,T_n)]$ and the result of composition $\widehat{S}^3(\widehat{\sigma}^{nd}[T],\widehat{\sigma}^{nd}[T_1],\widehat{\sigma}^{nd}[T_2],\widehat{\sigma}^{nd}[T_3])$ are different, which implies that in general an extension of each non-deterministic full hypersubstitution that preserves a partition does not satisfy an endomorphism property of the algebra $(P(W_{\tau_n}^{T(\bar{n},\mathcal{P})}(X_n)),\widehat{S}^n)$.

We remark here that the monoid $Hyp^{T(\bar{n},\mathcal{P})}(\tau_n)$ with respect to the composition $\circ_{nd}$ will not be obtained since its extension of each member does not preserve the operation. Moreover, in our perspective, full-solid non-deterministic varieties determined by a non-deterministic full hypersubstitution preserving a partition will not arise.

To address the absence of this property, the following problems are listed! First, what are the conditions under which $\widehat{\sigma}^{nd}$ is an endomorphism. If any, can we apply the operation $\circ_{nd}$ on the set $Hyp_{nd}^{T(\bar{n},\mathcal{P})}(\tau_n)$? If none, try to apply the concept of near homomorphism, as described in [5], to solve these difficulties. Another way to continue this work is to apply theory of conjugate pairs of additive closure operators, see [4], for describing a characterization of full-solid non-deterministic varieties of algebras via non-deterministic full hypersubstitutions that preserve a partition.

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