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Relative Rota-Baxter operators on a Jordan algebra with a representation and related structures

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Abstract. The purpose of this paper is to study $\mathcal{O}$-(dual-) Nijenhuis structures on a Jordan algebra with a representation. The notion of a (dual-)Nijenhuis pair is introduced and it can generate a trivial deformation of a Jordan algebra with representation. We introduce the notion of a $\mathcal{O}$-(dual-)Nijenhuis structure on a Jordan algebra with representation. Furthermore, we verify that relative Rota-Baxter operators and $\mathcal{O}$-(dual-)Nijenhuis structures can give rise to each other under some conditions. Finally, we study the notions of Rota-Baxter-Nijenhuis structures, r-matrix-Nijenhuis structures, ΩN-structures on a Jordan algebra and we investigate the relation between them.

1. Introduction

Jordan algebras were first studied in the 1930s in the context of axiomatic quantum mechanics and appeared in many areas of mathematics like differential geometry, Lie theory and analysis. A Jordan algebra can be regarded as an "opposite" of a Lie algebra in the sense that the commutator of an associative algebra is a Lie algebra and the anticommutator of an associative algebra is a Jordan algebra, although not every Jordan algebra is isomorphic to a subalgebra of the anticommutator of an associative algebra (such a Jordan algebra is called special, otherwise, it is called exceptional).

Kupershmidt introduced the notion of relative Rota-Baxter operators (also called $\mathcal{O}$-operators) of a Lie algebra to generalize (the operator form of) the famous classical Yang-Baxter equation in the Lie algebra [17]. Furthermore, a relative Rota-Baxter operator of a Lie algebra can give a solution

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of the classical Yang-Baxter equation in a larger Lie algebra [1]. Motivated by the notion of relative Rota-Baxter operator as a generalization of (the operator form of) the classical Yang-Baxter equation in [17, 1], D. Hou, X. Ni and C. M. Bai introduced the relative Rota-Baxter operator of a Jordan algebra [12].

Rota-Baxter operators, as a particular case of relative Rota-Baxter operators, were first introduced by Baxter in his study of fluctuation theory in probability [6]. They have been found useful in many contexts, for example in quantum analogue of Poisson geometry and so on, see [11] for more information. Thus it is very important to study the Rota-Baxter operators on Jordan algebras.

Nijenhuis operators on Lie algebras have been studied in [8] and [10]. In the perspective of deformations of Lie algebras, Nijenhuis operators canonically give rise to trivial deformations [22]. Nijenhuis operators have also been studied on pre-Lie algebras [29] and Poisson-Nijenhuis structures appeared in completely integrable systems [15] and were further studied in [16, 21]. The $r-n$ structure over a Lie algebra was studied in [25]. Recently, Hu, Liu and Sheng [13] studied the (dual) KN-structure as generalization of the $r-n$ structure. The associative, pre-Lie, Malcev, alternative and Leibniz analogues of Poisson-Nijenhuis structures have also been considered in [18, 28, 19, 20, 23, 24].

Inspired by these works, we consider the $\mathcal{O}$-(dual-)Nijenhuis structure on the Jordan algebra and characterize the relationships between Nijenhuis operators and relative Rota-Baxter operators.

The paper is organized as follows. In Section 2, we recall some necessary knowledge on Jordan algebras. In Section 3, we study infinitesimal deformations of a $\mathsf{JordanRep}$ pair, and introduce the notion of a Nijenhuis pair on a Jordan algebra and show that it generates a trivial deformation of a $\mathsf{JordanRep}$ pair. We also introduce the notion of a dual-Nijenhuis pair as the dual of a Nijenhuis pair. In Section 4, we introduce the notions of a $\mathcal{O}$-Nijenhuis structure and a $\mathcal{O}$-dual-Nijenhuis structure. Some properties of $\mathcal{O}$-(dual-)Nijenhuis structures are studied. In Section 5, we first give the relations between Nijenhuis operators and relative Rota-Baxter operators. Then, we prove that, on the one hand, $\mathcal{O}$-(dual-)Nijenhuis structures give rise to hierarchies of relative Rota-Baxter operators, which are pairwise compatible; on the other hand, compatible relative Rota-Baxter operators with a condition can give a $\mathcal{O}$-dual-Nijenhuis structure. In the final section, we study the notions of Rota-Baxter-Nijenhuis structures, PN-structures and ΩN-structures. Furthermore, we describe the relation between them.

Throughout this paper, all algebras are finite-dimensional and over a field $\mathbb{K}$ of characteristic 0.

2. Preliminaries and basics on Jordan algebras

A *Jordan algebra* is vector space J equipped with a commutative binary operation $\circ : A \otimes A \to A, (x, y) \mapsto x \circ y$ satisfying the following *Jordan identity*

$$as(x^2, y, x) = ((x \circ x) \circ y) \circ x - (x \circ x) \circ (y \circ x) = 0, \tag{1}$$

for any $x, y \in J$, where $x^2 = x \circ x$.

Remark 2.1.

(i) If $(A, *)$ is an associative algebra, then the operation given by $x \circ y = x * y + y * x$, defines a Jordan algebra structure on A. Such algebra and its subalgebras are called *special Jordan algebras*.

(ii) if $char(\mathbb{K}) \neq 2$ and 3, then the Jordan identity is equivalent to the following identity

$$\circlearrowleft_{x,y,z} as(x \circ y, u, z) = 0, \ \forall x, y, z, u \in J. \tag{2}$$

Definition 2.2. Let $(J, \circ)$ be a Jordan algebra and V be a vector space. A linear map $\rho : J \to gl(V)$ is called a *representation* (or a *module*) of $(J, \circ)$ if

$$\rho(x^2)\rho(x) - \rho(x)\rho(x^2) = 0, \tag{3}$$

$$2\rho(x \circ y)\rho(x) + \rho(x^2)\rho(y) - 2\rho(x)\rho(y)\rho(x) - \rho(x^2 \circ y) = 0 \tag{4}$$

for all $x, y \in J$. We denote it by (V, ρ) or simply ρ.

In fact, (V, ρ) is a representation of a Jordan algebra $(J, \circ)$ if and only if there exists a Jordan algebra structure on the direct sum $J \oplus V$ (the semi-direct sum) of the underlying vector spaces of J and$V$ given by

$$(x + u) \circ_\rho (y + v) = x \circ y + \rho(x)v + \rho(y)u, \ \forall x, y \in J, \ u, v \in V.$$

We denote it by $J \ltimes_\rho V$.

Example 2.3. Let $(J, \circ)$ be a Jordan algebra and $ad : J \to gl(J)$ be a linear map with $x \mapsto ad(x)$, where $ad(x)$ denote the left multiplication operator, that is , $ad(x)(y) = x \circ y$, for any $y \in J$. Then (ad, J) is a representation of $(J, \circ)$ which called the *adjoint representation*.

If we consider the Jordan algebra given by the identity (2), we get the following

Proposition 2.4. *Let $(J, \circ)$ be a Jordan algebra and V be a vector space. A linear map $\rho : J \to gl(V)$ is a representation of $(J, \circ)$ if and only if*

$$\circlearrowleft_{x,y,z} [\rho(x), \rho(y \circ z)] = 0, \tag{5}$$

$$\begin{aligned} &\rho(x)\rho(y)\rho(z) + \rho(z)\rho(y)\rho(x) + \rho((x \circ z) \circ y) \\ =&\rho(x)\rho(y \circ z) + \rho(y)\rho(z \circ x) + \rho(z)\rho(x \circ y), \end{aligned} \tag{6}$$

for all $x, y, z \in J$, where $[\cdot, \cdot]$ is the commutator in $gl(V)$.

Let (V, ρ) be a representation of a Jordan algebra $(J, \circ)$. Define $\rho^* : J \to gl(V^*)$ by

$$\langle \rho^*(x)\alpha, v\rangle = \langle \alpha, \rho(x)v\rangle, \forall x, \in J, v \in V, \alpha \in V^*.$$

Then (V^*, ρ^*) is a representation of $(J, \circ)$ which is called the *dual representation* of (V, ρ).

A linear map $T : V \to J$ is called a *relative Rota-Baxter operator* (or an $\mathcal{O}$*-operator*) associated to (V, ρ) if for all $u, v \in V$,

$$T(u) \circ T(v) = T(\rho(T(u))v + \rho(T(v))u). \tag{7}$$

A relative Rota-Baxter operator $R : J \to J$ associated to (J, ad) is called just a *Rota-Baxter operator* on J. That is, for any $x, y \in J$,

$$R(x) \circ R(y) = R(R(x) \circ y + x \circ R(y)). \tag{8}$$

A linear map $N : J \to J$ on a Jordan algebra $(J, \circ)$ is called a *Nijenhuis operator* on J if for any $x, y \in J$,

$$N(x) \circ N(y) = N(N(x) \circ y + x \circ N(y) - N(x \circ y)). \tag{9}$$

The deformed operation $\circ_N : J \otimes J \to J$ given by

$$x \circ_N y = N(x) \circ y + x \circ N(y) - N(x \circ y) \tag{10}$$

is a Jordan multiplication and N is a Jordan algebra homomorphism from $(J, \circ_N)$ to $(J, \circ)$.

Lemma 2.5. *Let $(J, \circ)$ be a Jordan algebra and N a Nijenhuis operator on J. Then we have, for all $k, l \in \mathbb{N}$,*

1. *$(J, \circ_{N^k})$ is a Jordan algebra.*
2. *N^l is also a Nijenhuis operator on the Jordan algebra $(J, \circ_{N^k})$.*
3. *The Jordan algebras $(J, (\circ_{N^k})_{N^l})$ and $(J, \circ_{N^{k+l}})$ coincide.*
4. *The Jordan algebras $(J, \circ_{N^k})$ and $(J, \circ_{N^l})$ are compatible, that is, any linear combination of $\circ_{N^k}$ and $\circ_{N^l}$ still makes J into an Jordan algebra.*
5. *N^l is an Jordan algebra homomorphism from $(J, \circ_{N^{k+l}})$ to $(J, \circ_{N^k})$.*

3. Infinitesimal deformations of a JordanRep pair

Let $(J, \circ)$ be a Jordan algebra and (V, ρ) be a representation. We say that we have a JordanRep pair and refer to it with the tuple $(J, \circ, V, \rho)$. Let $\omega : J \otimes J \to J$ and $\varpi : J \to gl(V)$ be two linear maps such that ω is commutative. Consider a t-parameterized family of operations and linear maps by

$$x \circ_t y = x \circ y + t\omega(x, y), \tag{11}$$

$$\rho_t(x) = \rho(x) + t\varpi(x), \ \forall\, x, y \in J. \tag{12}$$

If $(J, \circ_t, V, \rho_t)$ are JordanRep pairs for all t, we say that (ω, ϖ) generates a one-parameter infinitesimal deformation of the JordanRep pair $(J, \circ, V, \rho)$. We denote a one-parameter infinitesimal deformation of a JordanRep pair $(J, \circ, V, \rho)$ by $(J, \circ_t, V, \rho_t)$. By a direct computation, we can deduce that $(J, \circ_t, V, \rho_t)$ is a one-parameter infinitesimal deformation of a JordanRep pair $(J, \circ, V, \rho)$ if and only if

$$\omega(\omega(\omega(x, x), y), x) = \omega(\omega(x, x), \omega(y, x)), \tag{13}$$

$$\begin{aligned}&\omega(x^2 \circ y, x) + \omega(x^2, y) \circ x + (\omega(x, x) \circ y) \circ x \\ &\qquad = x^2 \circ \omega(y, x) + \omega(x, x) \circ (y \circ x) + \omega(x^2, y \circ x),\end{aligned} \tag{14}$$

$$\begin{aligned}&\omega(\omega(x^2, y), x) + \omega(\omega(x, x) \circ y, x) + \omega(\omega(x, x), y) \circ x \\ &\qquad = \omega(x, x) \circ \omega(y, x) + \omega(x^2, \omega(y, x)) + \omega(\omega(x, x), y \circ x),\end{aligned} \tag{15}$$

$$[\rho(x^2), \varpi(x)] + [\varpi(x^2), \rho(x)] + [\rho(\omega(x, x)), \rho(x)] = 0 \tag{16}$$

$$[\rho(\omega(x, x)), \varpi(x)] + [\varpi(x^2), \varpi(x)] + [\varpi(\omega(x, x)), \rho(x)] = 0, \tag{17}$$

$$\begin{aligned}
&\varpi(\omega(x,x))\varpi(x)-\varpi(x)\varpi(\omega(x,x))=0, && (18)\\
&2\rho(x\circ y)\varpi(x)+2\varpi(x\circ y)\rho(x)+2\rho(\omega(x,y))\rho(x)+\rho(x^2)\varpi(y)\\
&+\varpi(x^2)\rho(y)+\rho(\omega(x,x))\rho(y)-2\rho(x)\rho(y)\varpi(x)-2\rho(x)\varpi(y)\rho(x)\\
&-2\varpi(x)\rho(y)\rho(x)-\rho(\omega(x,x)\circ y)-\rho(\omega(x^2,y))-\varpi(x^2v\circ y)=0, && (19)\\
&2\varpi(x\circ y)\varpi(x)+2\rho(\omega(x,y))\varpi(x)+2\varpi(\omega(x,y))\rho(x)+\varpi(x^2)\varpi(y)\\
&+\rho(\omega(x,x))\varpi(y)+\varpi(\omega(x,x))\rho(y)-2\rho(x)\varpi(y)\varpi(x)-2\varpi(x)\rho(y)\varpi(x)\\
&-2\varpi(x)\varpi(y)\rho(x)-\rho(\omega(\omega(x,x),y))-\varpi(\omega(x,x)\circ y)-\varpi(\omega(x^2,y))=0, && (20)\\
&2\varpi(\omega(x,y))\varpi(x)+\varpi(\omega(x,x))\varpi(y)-2\varpi(x)\varpi(y)\varpi(x)-\varpi(\omega(\omega(x,x),y))=0. && (21)
\end{aligned}$$

Remark 3.1. Note that (13) means that (J,ω) is a Jordan algebra and (14) means that ω is a 2-cocycle of the Jordan algebra J with the coefficient in the adjoint representation. On the other hand, (15) means that $\circ$ is a 2-cocycle of the Jordan algebra (J,ω) with the coefficient in the adjoint representation. In addition, (18) and (21) mean that ϖ is a representation of the Jordan algebra (J,ω) on V. Furthermore, (16), (17), (19) and (20) mean that $\rho+\varpi$ is a representation of the Jordan algebra $(J,\circ+\omega)$ on V.

Definition 3.2. Two one-parameter infinitesimal deformation $(J,\circ_t,V,\rho_t)$ and $(J',\circ'_t,V,\rho'_t)$ of a JordanRep pair $(J,\circ,V,\rho)$ are equivalent if there exists an isomorphism (I_J+tN,I_V+tS) from $(J',\circ'_t,V,\rho'_t)$ to $(J,\circ_t,V,\rho_t)$, i.e.

$$(I_J+tN)(x\circ'_t y)=(I_J+tN)(x)\circ_t(I_J+tN)(y), \tag{22}$$
$$(I_V+tS)\rho'_t(x)u=\rho_t(I_J+tN)(x))(I_V+tS)u, \tag{23}$$

for any $x\in J$ and $u\in V$.

A one-parameter infinitesimal deformation of a JordanRep pair $(J,\circ,V,\rho)$ is said to be trivial if it is equivalent to $(J,\circ,V,\rho)$.

We can easily check that $(J,\circ_t,V,\rho_t)$ is a trivial deformation if and only if, for any $x,y\in J$,

$$\omega(x,y)=N(x)\circ y+x\circ N(y)-N(x\circ y), \tag{24}$$
$$N\omega(x,y)=N(x)\circ N(y), \tag{25}$$
$$\varpi(x)=\rho(N(x))+\rho(x)S-S\rho(x), \tag{26}$$
$$\rho(N(x))S=S\varpi(x). \tag{27}$$

It follows from (24) and (25) that N must be a Nijenhuis operator on the Jordan algebra $(J,\circ)$. Moreover, it follows from (26) and (27) that N and S should satisfy the following compatibility condition

$$\rho(N(x))S(v)=S(\rho(N(x))(v))+S(\rho(x)S(v))-S^2\rho(x)(v). \tag{28}$$

Definition 3.3. A pair (N, S), where $N \in gl(J)$ and $S \in gl(V)$, is called a *Nijenhuis pair* on a JordanRep pair $(J, \circ, V, \rho)$ if N is a Nijenhuis operator on the Jordan algebra $(J, \circ)$ and condition (28) holds.

We have seen that a trivial deformation of a JordanRep pair could give rise to a Nijenhuis pair. Conversely, a Nijenhuis pair can also generate a trivial deformation. By straightforward computations, we have the following theorem.

Theorem 3.4. *Let (N, S) be a Nijenhuis pair on a JordanRep pair $(J, \circ, V, \rho)$. Then a deformation $(J, \circ_t, V, \rho_t)$ of $(J, \circ, V, \rho)$ can be obtained by putting*

$$\omega(x, y) = N(x) \circ y + x \circ N(y) - N(x \circ y), \tag{29}$$

$$\varpi(x) = \rho(N(x)) + \rho(x)S - S\rho(x). \tag{30}$$

Furthermore, this deformation is trivial.

Similar to the definition of a Nijenhuis pair, we introduce the notion of a dual-Nijenhuis pair on a JordanRep.

Definition 3.5. A pair (N, S), where $N \in gl(J)$ and $S \in gl(V)$, is called a *dual-Nijenhuis pair* on the JordanRep pair $(J, \circ, V, \rho)$ if N is a Nijenhuis operator on the Jordan algebra $(J, \circ)$ and S satisfies

$$\rho(N(x))(S(v)) = S(\rho(N(x))(v)) + \rho(x)S^2(v) - S(\rho(x)S(v)) \tag{31}$$

for all $x \in J$ and $v \in V$.

Corollary 3.6. *Let (N, S) be a dual-Nijenhuis pair on the JordanRep pair $(J, \circ, V, \rho)$. Then for any $i \in \mathbb{N}$, (N^i, S^i) is a dual-Nijenhuis pair on the JordanRep pair $(J, \circ, V, \rho)$.*

Let (V^*, ρ^*) be the dual representation of a Jordan algebra $(J, \circ)$. In fact, there is a close relationship between a Nijenhuis pair and a dual-Nijenhuis pair.

Proposition 3.7. *(N, S) be a Nijenhuis pair on the JordanRep pair $(J, \circ, V, \rho)$ if and only if (N, S^*) is a dual-Nijenhuis pair on the JordanRep pair $(J, \circ, V^*, \rho^*)$.*

Proof. It follows from, for $x \in J$ and $\alpha \in J^*$

$$\begin{aligned}
&\langle \rho(N(x))S(v) - S(\rho(N(x))(v)) - S(\rho(x)S(v)) + S^2\rho(x)(v), \alpha\rangle \\
&= \langle v, S^*\rho^*(N(x))(\alpha) - \rho^*(N(x))(S^*(\alpha)) - S^*\rho^*(x)(S^*(\alpha)) + \rho^*(x)((S^*)^2(\alpha))\rangle.
\end{aligned}$$

□

Example 3.8. Let N be a Nijenhuis operator on a JordanRep pair $(J,\circ,V,\rho)$. Then (N,N^*) is a dual-Nijenhuis pair on a JordanRep pair $(J,\circ,J^*,ad^*)$, where (J^*,ad^*) is the dual representation of the adjoint representation (J,ad).

Let (N,S) be a Nijenhuis pair on a JordanRep pair $(J,\circ,V,\rho)$. If in addition (N,S) is a dual-Nijenhuis pair, then we can obviously obtain

$$2S(\rho(x)S(v)) = \rho(x)S^2(v) + S^2(\rho(x)(v)),\ \forall x\in J,\ v\in V. \tag{32}$$

Definition 3.9. A Nijenhuis pair (N,S) on the JordanRep pair $(J,\circ,V,\rho)$ is called *perfect* if the identity (32) holds.

A Nijenhuis pair gives rise to a Nijenhuis operator on the semidirect product Jordan algebra

Proposition 3.10. *Let (N,S) be a Nijenhuis pair on a JordanRep pair $(J,\circ,V,\rho)$, then $N+S$ is a Nijenhuis operator on the Jordan algebra $J\ltimes_\rho V$. Furthermore, if (N,S) is a perfect Nijenhuis pair, then $N+S^*$ is also a Nijenhuis operator on the Jordan algebra $J\ltimes_{\rho^*}V^*$.*

Proof. Let $x,y\in J$ and $u,v\in V$. Then

$$\begin{aligned}&(N+S)(x+u)\circ_\rho(N+S)(y+v) = (N(x)+S(u))\circ_\rho(N(y)+S(v))\\&= N(x)\circ N(y)+\rho(N(x))S(v)+\rho(N(y))S(x)\\&= N(N(x)\circ y)+N(x\circ N(y))-N^2(x\circ y)+\rho(N(x))S(v)+\rho(N(y))S(x).\end{aligned}$$

Moreover,

$$\begin{aligned}(N+S)((N+S)(x+u)\circ_\rho(y+v)) &= N(N(x)\circ y)+S(\rho(N(x))v)+S(\rho(y)S(u)),\\(N+S)((x+u)\circ_\rho(N+S)(y+v)) &= N(x\circ N(y))+S(\rho(x)S(v))+S(\rho(N(y))u).\end{aligned}$$

and

$$(N+S)^2((x+u)\circ_\rho(y+v)) = N^2(x\circ y)+S^2(\rho(x)v)+S^2(\rho(y)u).$$

Using (9) and (28), we can easily deduce that $N+S$ is a Nijenhuis operator on the Jordan algebra $J\ltimes_\rho V$.

By a similar computation, it is not hard to check that if (N,S) is a perfect Nijenhuis pair, then $N+S^*$ is also a Nijenhuis operator on the Jordan algebra $J\ltimes_{\rho^*}V^*$. Hence the proof. □

Now, define $\overline{\rho} : J \to gl(V)$ and $\widetilde{\rho} : J \to gl(V)$, respectively, as follow $(\forall x \in J)$

$$\overline{\rho}(x) = \rho(N(x)) + [\rho(x), S]. \tag{33}$$

$$\widetilde{\rho}(x) = \rho(N(x)) + [S, \rho(x)]. \tag{34}$$

Corollary 3.11. *(i) If (N, S) is a Nijenhuis pair on a* JordanRep *pair $(J, \circ, V, \rho)$, then we deduce that $\overline{\rho}$ is a representation of the Jordan algebra $(J, \circ_N)$ on V.*
(ii) If (N, S) is a dual-Nijenhuis pair on a JordanRep *pair $(J, \circ, V, \rho)$, then we deduce that $\widetilde{\rho}$ is a representation of the Jordan algebra $(J, \circ_N)$ on V.*

Proof. (i). According to Proposition 3.10, $N + S$ is a Nijenhuis operator on $J \ltimes_\rho V$. Then $(J \ltimes_\rho V, (\circ_\rho)_{N+S})$ is a Jordan algebra. For any $x, y \in J$ and $u, v \in V$, we have

$$\begin{aligned}
(x+u)(\circ_\rho)_{N+S}(y+v) &= (N+S)(x+u) \circ_\rho (y+v) + (x+u) \circ_\rho (N+S)(y+v) \\
&\quad - (N+S)((x+u) \circ_\rho (y+v)) \\
=& N(x) \circ + \rho(N(x))v + \rho(y)S(u) + x \circ N(y) + \rho(x)S(v) + \rho(N(y))u \\
&\quad - N(x \circ y) - S\rho(x)(v) - S\rho(y)(u) \\
=& x \circ_N y + \big(\rho(N(x)) + \rho(x)S - S\rho(x)\big)(v) + \big(\rho(N(y)) + \rho(y)S - S\rho(y)\big)(u) \\
=& x \circ_N y + \big(\rho(N(x)) + [\rho(x), S]\big)(v) + \big(\rho(N(y)) + [\rho(y), S]\big)(u) \\
=& x \circ_N y + \overline{\rho}(x)(v) + \overline{\rho}(y)(u),
\end{aligned}$$

then we see that $\overline{\rho}$ is a representation of the Jordan algebra $(J, \circ_N)$ on V.
(ii). Take $v \in V$ and $\alpha \in V^*$, we obtain

$$\langle \rho^*(N(x))(\alpha) + [\rho^*(x), S^*](\alpha), v\rangle = \langle \alpha, (\rho(N(x)) + [S, \rho(x)])(v)\rangle,$$

then the dual map $\widetilde{\rho^*}$ of $\widetilde{\rho}$ is given by

$$\widetilde{\rho^*}(x) = \rho^*(N(x)) + [\rho^*(x), S^*], \quad \forall x \in J. \tag{35}$$

In addition, If (N, S) is a dual-Nijenhuis pair on a JordanRep pair $(J, \circ, V, \rho)$, by Proposition 3.7, then (N, S^*) is a Nijenhuis pair with the representation ρ^* on V^*. Using the above calculation in (**i**), we check that $\widetilde{\rho^*}$ is a representation of $(J, \circ_N)$ on V^* and thus $\widetilde{\rho}$ is a representation of the Jordan algebra $(J, \circ_N)$ on V. □

4. $\mathcal{O}$-(dual-) Nijenhuis structures

In this section, we introduce the notion of an $\mathcal{O}$-(dual-) Nijenhuis structure on a JordanRep pair. Jordan algebras and pre-Jordan algebras are closely related via relative Rota-Baxter operators. Recall that a pre-Jordan algebra $(A, \bullet)$ is such that

$$\begin{aligned}&(x \circ x) \bullet (x \bullet y) = x \bullet ((x \circ x) \bullet y),\\&2(x \circ y) \bullet (x \bullet z) + (x \circ x) \bullet (y \bullet z) = 2x \bullet (y \bullet (x \bullet z)) + ((x \circ x) \circ y) \bullet z,\end{aligned}$$

for all $x, y, z \in A$, where $x \circ y = x \bullet y + y \bullet x$. Note that $(A, \circ)$ is a Jordan algebra which is called the *associated Jordan algebra* of $(A, \bullet)$ (see [12] for more details).

The following result establishes the connection between relative Rota-Baxter operators and pre-Jordan algebras given in [12].

Proposition 4.1. *Let $T : V \to J$ be a relative Rota-Baxter operator on a* JordanRep *pair $(J, \circ, V, \rho)$. Then the product*

$$u \bullet^T v = \rho(T(u))v, \forall u, v \in V, \tag{36}$$

defines a pre-Jordan algebra structure on V. Denote by $(V, \circ^T)$ the associated Jordan algebra of $(V, \bullet^T)$, then T is a Jordan algebra homomorphism from $(V, \circ^T)$ to $(J, \circ)$. Note that

$$u \circ^T v = \rho(T(u))v + \rho(T(v))u, \ \forall u, v \in V. \tag{37}$$

Now, let (N, S) be a (dual)-Nijenhuis pair on a JordanRep pair $(J, \circ, V, \rho)$ and $T : V \to J$ be a relative Rota-Baxter operator on J with respect to ρ. We define a deformed product of $\circ^T$ by S as follows

$$u \circ^T_S v = S(u) \circ^T v + u \circ^T S(v) - S(u \circ^T v), \forall u, v \in V. \tag{38}$$

The following definition introduce the notion of a $\mathcal{O}$-(dual)-Nijenhuis structure on a JordanRep pair.

Definition 4.2. A triple (T, S, N) is called a *$\mathcal{O}$-(dual)-Nijenhuis structure* on a JordanRep pair $(J, \circ, V, \rho)$ if T is a relative Rota-Baxter operator on J with respect to ρ and (N, S) is a (dual)-Nijenhuis pair on J such that the following compatibility conditions hold

$$NT = TS, \tag{39}$$

$$u \circ^{NT} v = u \circ^T_S v. \tag{40}$$

If (39) and (40) hold, we say that T and (N, S) are compatible.

Note that if (N, S) is a perfect Nijenhuis pair, then a $\mathcal{O}$-Nijenhuis structure is also a $\mathcal{O}$-dual-Nijenhuis structure. Define two other products similar to (37), using the representations (33) and (34) as follow

$$u \circ^T_{\overline{\rho}} v = \overline{\rho}(T(u))v + \overline{\rho}(T(v))u, \tag{41}$$

$$u \circ^T_{\widetilde{\rho}} v = \widetilde{\rho}(T(u))v + \widetilde{\rho}(T(v))u, \ \forall u, v \in V. \tag{42}$$

Lemma 4.3. *(i) Let (T, S, N) be a $\mathcal{O}$-Nijenhuis structure. Then we have*

$$u \circ^T_S v = u \circ^T_{\overline{\rho}} v.$$

(ii) Let (T, S, N) be a $\mathcal{O}$-dual-Nijenhuis structure. Then we have

$$u \circ^T_S v = u \circ^T_{\widetilde{\rho}} v.$$

Proof. (i). For any $u, v \in V$, we have

$$\begin{aligned}
u \circ^T_S v &= S(u) \circ^T v + u \circ^T S(v) - S(u \circ^T v) \\
&= \rho(TS(u))v + \rho(T(v))S(u) + \rho(T(u))S(v) + \rho(TS(v))u \\
&- S(\rho(T(u))v) - S(\rho(T(v))u) \\
&= \rho(NT(u))v + [\rho(T(u)), S]v + \rho(NT(v))u + [\rho(T(v)), S]u \\
&= \overline{\rho}(T(u))v + \overline{\rho}(T(v))u \\
&= u \circ^T_{\overline{\rho}} v.
\end{aligned}$$

(ii). Let $u, v \in V$. Then

$$\begin{aligned}
u \circ^T_S v + u \circ^T_{\widetilde{\rho}} v &= 2\big(\rho(NT(u))v + \rho(NT(v))u\big) \\
&= 2u \circ^{NT} v \circ \\
&= 2u \circ^T_S v,
\end{aligned}$$

which implies that $u \circ^T_S v = u \circ^T_{\widetilde{\rho}} v$. ☐

Thus, if (T, S, N) is a $\mathcal{O}$-(dual-)Nijenhuis structure, then the three brackets $\circ^T_S$, $\circ^T_{\overline{\rho}}$ $(\circ^T_{\widetilde{\rho}})$ and $\circ^{NT}$ are the same. Contrary to $\circ^T$, the products $\circ^T_S$, $\circ^T_{\overline{\rho}}$ and $\circ^T_{\widetilde{\rho}}$ don't satisfy, in general, the Jordan identity. A naturally question is: under what condition, they are Jordan products? A partially answer is given in the following proposition

Proposition 4.4. *Let (T, S, N) be a $\mathcal{O}$-(dual)-Nijenhuis structure. Then S is a Nijenhuis operator on the pre-Jordan algebra $(V, \bullet^T)$ and then on its associated Jordan algebra $(V, \circ^T)$. Therefore $\circ^T_{\overline{\rho}}$ $(\circ^T_{\widetilde{\rho}})$, $\circ^T_S$ and $\circ^{NT}$ are all Jordan products.*

Proof. For the $\mathcal{O}$-Nijenhuis structure (T, S, N), by (28) and substituting x by $T(u)$, we get

$$\begin{aligned} 0 &= \rho(NT(u))S(v) - S(\rho(NT(u))(v)) - S(\rho(T(u))S(v)) + S^2(\rho(T(u))(v)) \\ &= \rho(TS(u))S(v) - S(\rho(TS(u))(v)) - S(\rho(Tu)S(v)) + S^2(\rho(Tu)(v)) \\ &= S(u) \bullet^T S(v) - S\Big(S(u) \bullet^T v + u \bullet^T S(v) - S(u \bullet^T v)\Big), \end{aligned}$$

which implies that S is a Nijenhuis operator on the pre-Jordan algebra $(V, \bullet^T)$. Thus S is a Nijenhuis operator on the sub-adjacent Jordan algebra $(V, \circ^T)$.

For the $\mathcal{O}$-dual-Nijenhuis structure (T, S, N), the proof is not direct. In fact, let $T : V \to J$ be a relative Rota-Baxter operator on a JordanRep pair $(J, \circ, V, \rho)$. The product

$$u \bullet^T v = \rho(T(u))v, \ \forall u, v \in V$$

defines a pre-Jordan algebra structure on V. Denote by $(V, \circ^T)$ the associated Jordan algebra of $(V, \bullet^T)$. Then

$$u \circ^T v = \rho(T(u))v + \rho(T(v))u, \ \forall u, v \in V.$$

Let (N, S) be a (dual)-Nijenhuis pair on a JordanRep pair $(J, \circ, V, \rho)$. A deformed product of $\circ^T$ by S is defined as follows

$$u \circ_S^T v = S(u) \circ^T v + u \circ^T S(v) - S(u \circ^T v), \ \forall u, v \in V.$$

Then

$$\begin{aligned} u \circ_S^T v = &\rho(TS(u))v + \rho(T(v))S(u) + \rho(T(u))S(v) \\ &+\rho(TS(v))u - S(\rho(T(u))v) - S(\rho(T(v))u). \end{aligned}$$

For the $\mathcal{O}$-dual-Nijenhuis structure (T, S, N) we have $u \circ^{TS} v = u \circ_S^T v$ for all $u, v \in V$. Therefore,

$$\begin{aligned} \rho(TS(u))v + \rho(TS(v))u =&\rho(TS(u))v + \rho(T(v))S(u) + \rho(T(u))S(v) \\ &+\rho(TS(v))u - S(\rho(T(u))v) - S(\rho(T(v))u). \end{aligned}$$

Then

$$S(\rho(T(u))v) + S(\rho(T(v))u) \quad = \quad \rho(T(u))S(v) + \rho(T(v))S(u). \tag{43}$$

Replacing u by $S(u)$ in (43), we obtain

$$S(\rho(TS(u))v) + S(\rho(T(v))S(u)) \quad = \quad \rho(TS(u))S(v) + \rho(T(v))S^2(u). \tag{44}$$

Applying S to the both sides (43), we get

$$S^2(\rho(T(u))v) + S^2(\rho(T(v))u) \quad = \quad S(\rho(T(u))S(v)) + S(\rho(T(v))S(u)). \tag{45}$$

Since (N,S) is a dual pair then for all $x\in J,\ u\in V$

$$\rho(x)S^2(u)-\rho(N(x))S(u)=S(\rho(x)S(u))-S(\rho(N(x))u).$$

Replacing in this equality x by $T(v)$, where $v\in V$ we get

$$\rho(T(v))S^2(u)-\rho(N(T(v)))S(u)=S(\rho(T(v))S(u))-S(\rho(N(T(v)))u).$$

Since $NT=TS$ then

$$\rho(T(v))(S^2(u))-\rho(TS(v))(S(u))=S(\rho(T(v))(S(u)))-S(\rho(TS(v))(u)). \tag{46}$$

According to (44)-(46), we have

$$\begin{aligned}
& S(u)\circ^T S(v)-S(u\circ_S^T v)\\
= \ & \rho(TS(u))(S(v))+\rho(TS(v))(S(u))+S^2(\rho(T(u))(v))+S^2(\rho(T(v))(u))\\
& -S(\rho(T(u))(S(v)))-S(\rho(TS(v))(u))-S(\rho(TS(u))(v))-S(\rho(T(v))(S(u)))\\
= \ & [S^2(\rho(T(u))(v))+S^2(\rho(T(v))(u))-S(\rho(T(u))(S(v)))-S(\rho(T(v))(S(u)))]\\
& +\rho(TS(u))(S(v))+\rho(TS(v))(S(u))-S(\rho(TS(u))(v))-S(\rho(TS(v))(u))\\
\overset{(45)}{=} \ & \rho(TS(u))(S(v))-S(\rho(TS(u))(v))+[\rho(TS(v))(S(u))-S(\rho(TS(v))(u))]\\
\overset{(46)}{=} \ & \rho(TS(u))(S(v))-S(\rho(TS(u))(v))+\rho(T(v))(S^2(u))-S(\rho(T(v))(S(u)))\\
\overset{(44)}{=} \ & 0.
\end{aligned}$$

Thus S is a Nijenhuis operator on the Jordan algebra $(V,\circ^T)$. □

Theorem 4.5. *Let (T,S,N) be a $\mathcal{O}$-Nijenhuis (resp. $\mathcal{O}$-dual-Nijenhuis) structure. Then*

- (i) *T is a relative Rota-Baxter operator on the deformed* JordanRep *pair $(J,\circ_N,V,\overline{\rho})$ (resp. $(J,\circ_N,V,\widetilde{\rho})$).*
- (ii) *NT is a relative Rota-Baxter operator on the* JordanRep *pair $(J,\circ,V,\rho)$.*

Proof. We only prove the theorem for the $\mathcal{O}$-Nijenhuis structure. The other one can be proved similarly.
(i). Since T is a relative Rota-Baxter operator on a JordanRep pair $(J,\circ,V,\rho)$ and $TS=NT$, we have

$$\begin{aligned}
T(u)\circ_N T(v) &= NT(u)\circ T(v)+T(u)\circ NT(v)-NT(u\circ^T v)\\
&= TS(u)\circ T(v)+T(u)\circ TS(v)-TS(u\circ^T v)\\
&= T(S(u)\circ^T v+u\circ^T S(v)-S(u\circ^T v))\\
&= T(u\circ_S^T v)=T(u\circ_{\overline{\rho}}^T v).
\end{aligned}$$

Thus, T is a relative Rota-Baxter operator on the deformed JordanRep pair $(J, \circ_N, V, \bar{\rho})$.
(ii). By (40), we have

$$NT(u \circ^{NT} v) = NT(u \circ_S^T v) = N(T(u) \circ_N T(v)) = NT(u) \circ NT(v),$$

which implies that NT is a relative Rota-Baxter operator on the JordanRep pair $(J, \circ, V, \rho)$. □

The following result proves that the $\mathcal{O}$-Nijenhuis structure can give a $\mathcal{O}$-dual-Nijenhuis structure under some condition.

Theorem 4.6. *Let (T, S, N) be a $\mathcal{O}$-Nijenhuis structure. If T is invertible, then (T, S, N) is a $\mathcal{O}$-dual-Nijenhuis structure.*

Proof. We only need to prove that the Nijenhuis pair (S, N) is also a dual-Nijenhuis pair. By (40), we have

$$\begin{aligned} 0 &= u \circ_S^T v - u \circ^{TS} v \\ &= \rho(T(u))(S(v)) + \rho(T(v))(S(u)) - S\Big(\rho(T(u))(v) + \rho(T(v))(u)\Big), \end{aligned}$$

which implies that

$$\rho(T(u))(S(v)) + \rho(T(v))(S(u)) = S\Big(\rho(T(u))(v) + \rho(T(v))(u)\Big), \tag{47}$$

substituting $S(u)$ for u in (47) we have that

$$\rho(TS(u))(S(v)) + \rho(T(v))(S^2(u)) = S\big(\rho(TS(u))(v)\big) + S\big(\rho(T(v))(S(u))\big). \tag{48}$$

Since S is a Nijenhuis operator on the Jordan algebra $(V, \circ^T)$ and $u \circ_S^T v = u \circ^{TS} v$, we have

$$S(u \circ^{TS} v) = S(u) \circ^T S(v),$$

which means that

$$S\Big(\rho(TS(u))(v) + \rho(TS(v))(u)\Big) = \rho(TS(u))(S(v)) + \rho(TS(v))(S(u)). \tag{49}$$

Combining (48) and (49), we have

$$\begin{aligned} 0 &= S\big(\rho(T(v))(S(u))\big) - \rho(T(v))(S^2(u)) - S\big(\rho(TS(v))(u)\big) + \rho(TS(v))(S(u)) \\ &= S\big(\rho(T(v))(S(u))\big) - \rho(T(v))(S^2(u)) - S\big(\rho(NT(v))(u)\big) + \rho(NT(v))(S(u)). \end{aligned}$$

Since T is invertible and let $x = T(v)$, we have

$$S\big(\rho(x)(S(u))\big) - \rho(x)(S^2(u)) - S\big(\rho(N(x))(u)\big) + \rho(N(x))(S(u)) = 0.$$

Thus the Nijenhuis pair (S, N) is a dual-Nijenhuis pair. We finish the proof. □

5. Compatible relative Rota-Baxter operators and $\mathcal{O}$-(dual)-Nijenhuis structures

In this section, we introduce compatible relative Rota-Baxter operators and $\mathcal{O}$-Nijenhuis structures and we mainly characterize the relationship between compatible relative Rota-Baxter operators and $\mathcal{O}$-(dual)-Nijenhuis structures.

5.1. Compatible relative Rota-Baxter operators on Jordan algebras

Definition 5.7. Let $T_1, T_2 : V \longrightarrow J$ be two relative Rota-Baxter operators on a JordanRep pair $(J, \circ, V, \rho)$. If for all $k_1, k_2 \in \mathbb{K}$, $k_1T_1 + k_2T_2$ is still a relative Rota-Baxter operator, then T_1 and T_2 are called *compatible*.

The following result follows from a direct computation.

Proposition 5.8. *Let $T_1, T_2 : V \longrightarrow J$ be two relative Rota-Baxter operators on a JordanRep pair $(J, \circ, V, \rho)$. Then T_1 and T_2 are compatible if and only if the following equation holds:*

$$\begin{aligned} T_1(u) \circ T_2(v) + T_2(u) \circ T_1(v) =& T_1\Big(\rho(T_2(u))(v) + \rho(T_2(v))(u)\Big) \\ &+ T_2\Big(\rho(T_1(u))(v) + \rho(T_1(v))(u)\Big), \end{aligned} \tag{50}$$

for all $u, v \in V$.

Using a relative Rota-Baxter operator and a Nijenhuis operator, we can construct a pair of compatible relative Rota-Baxter operators .

Proposition 5.9. *Let $T : V \longrightarrow J$ be a relative Rota-Baxter operator on a JordanRep pair $(J, \circ, V, \rho)$ and N a Nijenhuis operator on $(J, \circ)$. Then NT is a relative Rota-Baxter operator on the JordanRep pair $(J, \circ, V, \rho)$ if and only if for all $u, v \in V$, the following equation holds:*

$$\begin{aligned} &N\Big(NT(u) \circ T(v) + T(u) \circ NT(v)\Big) \\ =& N\Big(T\big(\rho(NT(u))(v) + \rho(NT(v))(u)\big) + NT\big(\rho(T(u))(v) + \rho(T(v))(u)\big)\Big). \end{aligned} \tag{51}$$

In this case, if in addition N is invertible, then T and NT are compatible. More explicitly, for any relative Rota-Baxter operator T, if there exists an invertible Nijenhuis operator N such that NT is also a relative Rota-Baxter operator, then T and NT are compatible.

Proof. Since N is a Nijenhuis operator, we have

$$NT(u)\circ NT(v) = N\Big(NT(u)\circ T(v) + T(u)\circ NT(v)\Big) - N^2(T(u)\circ T(v)).$$

Note that

$$T(u)\circ T(v) = T\Big(\rho(T(u))(v) + \rho(T(v))(u)\Big).$$

Then

$$NT(u)\circ NT(v) = NT\Big(\rho(NT(u))(v) + \rho(NT(v))(u)\Big)$$

if and only if (51) holds.

If NT is a relative Rota-Baxter operator and N is invertible, then we have

$$\begin{aligned} NT(u)\circ T(v) + T(u)\circ NT(v) =& T\big(\rho(NT(u))(v) + \rho(NT(v))(u)\big) \\ &+NT\big(\rho(T(u))(v) + \rho(T(v))(u)\big), \end{aligned}$$

which is exactly the condition that NT and T are compatible. □

A pair of compatible relative Rota-Baxter operators can also give rise to a Nijenhuis operator under some conditions.

Proposition 5.10. *Let $T_1, T_2 : V \longrightarrow J$ be two relative Rota-Baxter operators on a* JordanRep *pair $(J,\circ,V,\rho)$. Suppose that T_2 is invertible. If T_1 and T_2 are compatible, then $N = T_1T_2^{-1}$ is a Nijenhuis operator on the Jordan algebra $(J,\circ)$.*

Proof. For all $x, y \in J$, there exist $u, v \in V$ such that $T_2(u) = x, T_2(v) = y$. Hence $N = T_1T_2^{-1}$ is a Nijenhuis operator if and only if the following equation holds:

$$NT_2(u)\circ NT_2(v) = N(NT_2(u)\circ T_2(v) + T_2(u)\circ NT_2(v)) - N^2(T_2(u)\circ T_2(v)).$$

Since $T_1 = NT_2$ is an relative Rota-Baxter operator, the left hand side of the above equation is

$$NT_2(\rho(NT_2(u))(v) + \rho(NT_2(v))(u)).$$

Since T_2 is a relative Rota-Baxter operator which is compatible with $T_1 = NT_2$, we have

$$\begin{aligned} & NT_2(u)\circ T_2(v) + T_2(u)\circ NT_2(v) \\ = \;& T_2(\rho(NT_2(u))(v) + \rho(NT_2(v))(u)) + NT_2(\rho(T_2(u))(v) + \rho(T_2(v))(u)) \\ = \;& T_2(\rho(NT_2(u))(v) + \rho(NT_2(v))(u)) + N(T_2(u)\circ T_2(v)). \end{aligned}$$

Let N act on both sides, we get the conclusion. □

By Proposition 5.9 and 5.10, we have

Corollary 5.11. *Let $T_1, T_2 : V \longrightarrow J$ be two relative Rota-Baxter operators on a* JordanRep *pair $(J, \circ, V, \rho)$. Suppose that T_1 and T_2 are invertible. Then T_1 and T_2 are compatible if and only if $N = T_1 T_2^{-1}$ is a Nijenhuis operator.*

In particular, as a direct application, we have the following conclusion.

Corollary 5.12. *Let $(J, \circ)$ be a Jordan algebra. Suppose that R_1 and R_2 are two invertible Rota-Baxter operators. Then R_1 and R_2 are compatible in the sense that any linear combination of R_1 and R_2 is still a Rota-Baxter operator if and only if $N = R_1 R_2^{-1}$ is a Nijenhuis operator.*

5.2. Hierarchy of relative Rota-Baxter operators

In the following, first we construct compatible relative Rota-Baxter operators from $\mathcal{O}$-(dual-)Nijenhuis structures. Given a $\mathcal{O}$-(dual-)Nijenhuis structure (T, S, N), by Theorem 4.5, T and TS are relative Rota-Baxter operators. In fact, they are compatible.

Proposition 5.13. *Let (T, S, N) be a $\mathcal{O}$-(dual-)Nijenhuis structure. Then T and $TS = NT$ are compatible relative Rota-Baxter operators.*

Proof. We only prove the conclusion for the $\mathcal{O}$-Nijenhuis structure. The other one can be proved similarly. It is sufficient to prove that $T + TS$ is a relative Rota-Baxter operator. It is obvious that

$$u \circ^{T+TS} v = u \circ^T v + u \circ^{TS} v = u \circ^T v + u \circ_S^T v.$$

Thus, we have

$$\begin{aligned}
&(T+TS)(u \circ^{T+TS} v) \\
=\,& T(u \circ^T v) + TS(u \circ_S^T v) + TS(u \circ^T v) + T(u \circ_S^T v) \\
=\,& T(u \circ^T v) + TS(u \circ_S^T v) + T \circ S(u \circ^T v) \\
&+T(S(u) \circ^T v + u \circ^T S(v) - S(u \circ^T v)) \\
=\,& T(u \circ^T v) + TS(u \circ_S^T v)) + T(S(u) \circ^T v + u \circ^T S(v)) \\
=\,& T(u) \circ T(v) + TS(u) \circ TS(v) + TS(u) \circ T(v) + T(u) \circ TS(v) \\
=\,& (T+TS)(u) \circ (T+TS)(v),
\end{aligned}$$

which means that $T + TS$ is a relative Rota-Baxter operator. □

Lemma 5.14. *Let (T, S, N) be a $\mathcal{O}$-(dual-)Nijenhuis structure. Then for all $k, i \in \mathbb{N}$, we have*

$$T_k(u \circ^T_{S^{k+i}} v) = T_k(u) \circ_{N^i} T_k(v), \tag{52}$$

where $T_k = TS^k = N^k T$ and set $T_0 = T$.

Proof. Since T is a relative Rota-Baxter operator and $TS = NT$, we have

$$\begin{aligned} T(u \circ^T_{S^i} v) &= T\big(S^i(u) \circ^T v + u \circ^T S^i(v) - S^i(u \circ^T v)\big) \\ &= N^i(T(u)) \circ T(v) + T(u) \circ N^i(T(v)) - N^i(T(u) \circ T(v)) \\ &= T(u) \circ_{N^i} T(v). \end{aligned} \tag{53}$$

Since S is a Nijenhuis operator on the Jordan algebra $(V, \circ^T)$, we have

$$S^k(u \circ^T_{S^{k+i}} v) = S^k(u) \circ^T_{S^i} S^k(v). \tag{54}$$

Then by (53) and (54), we have

$$\begin{aligned} T_k(u \circ^T_{S^{k+i}} v) = TS^k(u \circ^T_{S^{k+i}} v) &= T(S^k(u) \circ^T_{S^i} S^k(v)) \\ &= T(S^k(u)) \circ_{N^i} T(S^k(v)). \end{aligned}$$

The proof is finished. □

The proof of the following lemma is similar to the proof of Proposition 5.1 in [15].

Lemma 5.15. *Let (T, S, N) be a $\mathcal{O}$-(dual-)Nijenhuis structure. Then for all $k, i \in \mathbb{N}$ such that $i \leq k$,*

$$u \circ^{T_k} v = u \circ^T_{S^k} v = S^{k-i}(u \circ^{T_i} v), \tag{55}$$

where $T_k = TS^k = N^k T$ and set $T_0 = T$.

Proposition 5.16. *Let (T, S, N) be a $\mathcal{O}$-(dual-)Nijenhuis structure on a JordanRep pair $(J, \circ, V, \rho)$. Then all $T_k = N^k T$ are relative Rota-Baxter operators on a JordanRep pair $(J, \circ, V, \rho)$ and for all $k, l \in \mathbb{N}$, T_k and T_l are compatible.*

Proof. We only prove the conclusion for the $\mathcal{O}$-Nijenhuis structure. The other one can be proved similarly.

By (52) and (55) with $i=0$, we have

$$T_k(u \circ^{T_k} v) = T_k(u) \circ T_k(v),$$

which implies that T_k is a relative Rota-Baxter operator on a JordanRep pair $(J, \circ, V, \rho)$.

For the second conclusion, we need to prove that $T_k + T_{k+i}$ is a relative Rota-Baxter operator for all $k, i \in \mathbb{N}$. By (55), we have

$$u \circ^{T_k + T_{k+i}} v = u \circ^{T_k} v + u \circ^{T_{k+i}} v = u \circ^{T_k} v + u \circ^{T_k}_{S^i} v.$$

Thus, we have

$$\begin{aligned}
&(T_k + T_{k+i})(u \circ^{T_k + T_{k+i}} v)\\
&= T_k(u \circ^{T_k} v) + T_k(u \circ^{T_k}_{S^i} v) + T_{k+i}(u \circ^{T_k} v) + T_{k+i}(u \circ^{T_k}_{S^i} v)\\
&= T_k(u \circ^{T_k} v) + T_{k+i}(u \circ^{T_k} v) + T_{k+i}(u \circ^{T_k}_{S^i} v)\\
&\quad + T_k(S^i(u) \circ^{T_k} v + u \circ^{T_k} S^i(v) - S^i(u \circ^{T_k} v))\\
&= T_k(u \circ^{T_k} v) + T_{k+i}(u \circ^{T_k}_{S^i} v) + T_k(S^i(u) \circ^{T_k} v) + T_k(u \circ^{T_k} S^i(v))\\
&= T_k(u) \circ T_k(v) + T_{k+i}(u) \circ T_{k+i}(v) + T_{k+i}(u) \circ T_k(v) + T_k(u) \circ T_{k+i}(v)\\
&= (T_k + T_{k+i})(u) \circ (T_k + T_{k+i})(v).
\end{aligned}$$

Thus $T_k + T_{k+i}$ is a relative Rota-Baxter operator on a JordanRep pair $(J, \circ, V, \rho)$, that is, T_k and T_{k+i} are compatible. In other words, T_k and T_l are compatible for all positive integers k, l. We finish the proof. □

Compatible relative Rota-Baxter operators can give rise to $\mathcal{O}$-dual-Nijenhuis structures.

Proposition 5.17. *Let $T, T_1 : V \longrightarrow J$ be two relative Rota-Baxter operators on a JordanRep pair $(J, \circ, V, \rho)$. If T and T_1 are compatible with T invertible, then*

- *(i) $(T, S = T^{-1}T_1, N = T_1T^{-1})$ is a $\mathcal{O}$-dual-Nijenhuis structure;*
- *(ii) $(T_1, S = T^{-1}T_1, N = T_1T^{-1})$ is a $\mathcal{O}$-dual-Nijenhuis structure.*

Proof. (i). The proof of (N, S) being a dual-Nijenhuis pair is similar to the proof of Theorem 4.6. We omit the details. It is obvious that $TS = NT$. Thus we only need to prove that the compatibility condition (40) holds. By the compatibility condition of T and T_1 and Proposition 5.10, $N = T_1T^{-1}$

is a Nijenhuis operator on the Jordan algebra J. By Proposition 5.8, we also have

$$T(u)\circ T_1(v)+T_1(u)\circ T(v)=T\Big(\rho(T_1(u))(v)+\rho(T_1(v))(u)\Big) \\ +T_1\Big(\rho(T(u))(v)+\rho(T(v))(u)\Big),\quad \forall u,v\in V.$$

Substituting T_1 with TS, then we have

$$T(u)\circ TS(v)+TS(u)\circ T(v)=T\Big(\rho(TS(u))(v)+\rho(TS(v))(u)\Big) \\ +TS\Big(\rho(T(u))(v)+\rho(T(v))(u)\Big). \tag{56}$$

Since T is a relative Rota-Baxter operator, we have

$$T(u)\circ TS(v)+TS(u)\circ T(v)=T\Big(\rho(T(u))(S(v))+\rho(TS(v))(u) \\ +\rho(TS(u))(v)+\rho(T(v))(S(u))\Big).$$

Since T is invertible, (56) is equivalent to

$$S\Big(\rho(T(u))(v)+\rho(T(v))(u)\Big)=\rho(T(u))(S(v))+\rho(T(v))(S(u)). \tag{57}$$

On the other hand, we have

$$u\circ_S^T v-u\circ^{TS}v=\rho(T(u))(S(v))+\rho(T(v))(S(u))-S\Big(\rho(T(u))(v)+\rho(T(v))(u)\Big).$$

Thus, (57) implies that $u\circ_S^T v=u\circ^{TS}v$. Therefore, $(T,S=T^{-1}T_1,N=T_1T^{-1})$ is a $\mathcal{O}$-dual-Nijenhuis structure.

Furthermore, since T and TS are relative Rota-Baxter operators, thus

$$TS(u\circ^{TS}v)=TS(u)\circ TS(v)=T(S(u)\circ^T S(v)).$$

As T is invertible, we have

$$S(u\circ^{TS}v)=S(u)\circ^T S(v).$$

(ii) By direct calculation, we have

$$\begin{aligned}&u\circ_S^{T_1}v-u\circ^{T_1S}v\\ =&\rho(T_1(u))(S(v))+\rho(T_1(v))(S(u))-S\Big(\rho(T_1(u))(v)+\rho(T_1(v))(u)\Big)\end{aligned}$$

$$= \rho(TS(u))(S(v)) + \rho(TS(v))(S(u)) - S\Big(\rho(TS(u))(v) + \rho(TS(v))(u)\Big)$$
$$= S(u) \circ^T S(v) - S(u \circ^{TS} v) = 0.$$

Thus, $(T_1, S = T^{-1}T_1, N = T_1T^{-1})$ is also a $\mathcal{O}$-dual-Nijenhuis structure. □

Proposition 5.18. *Let (T, N, S) be an $\mathcal{O}$-dual-Nijenhuis structure on a JordanRep pair $(J, \circ, V, \rho)$. If T is invertible, then (T, N^k, S^k) and $(T_k = N^kT, N^k, S^k)$ are $\mathcal{O}$-dual-Nijenhuis structures for all $k \in \mathbb{N}$.*

Proof. Since (T, N, S) is an $\mathcal{O}$-dual-Nijenhuis structure on a JordanRep pair $(J, \circ, V, \rho)$, by Proposition 5.16, T and $T_k = N^kT$ are compatible relative Rota-Baxter operators. Then by the condition that T is invertible and Proposition 5.17, the conclusions follow immediately. □

6. PN- and ΩN-structures on Jordan algebras

In this section, we introduce the notions of Rota-Baxter-Nijenhuis structures on Jordan algebra and Poisson-Nijenhuis structures on Jordan algebra which is also called r-matrix-Nijenhuis structures in some references (for example, [13]). Furthermore, we introduce the notion of an ΩN-structure on Jordan algebra , which consists of a symplectic structure and a Nijenhuis operator satisfying some compatibility conditions. The relations among these structures are given.

6.1. PN-structures, RBN-structures on a Jordan algebra

As a Jordan analog of the classical Yang-baxter equation, Zhelyabin, V. N. introduced the Jordan Yang-Baxter equation (JYBE). Let $(J, \circ)$ be a Jordan algebra and an element $\pi = \sum_i x_i \otimes y_i \in J \otimes J$ is called a Jordan r-matrix if it satisfies the classical Jordan Yang-Baxter equation (JYBE):

$$\pi_{12} \circ \pi_{13} + \pi_{13} \circ \pi_{23} - \pi_{12} \circ \pi_{23} = 0,$$

where $\pi_{12} = \sum_i x_i \otimes y_i \otimes 1 \in J^{\otimes 3}$ etc. See ([31, 32, 33]) for more details.

Lemma 6.1. [12] *Let $(J, \circ)$ be a Jordan algebra and $\pi \in J \otimes J$. Then π is a skew-symmetric solution of the JYBE if and only if $\pi^\sharp : J^* \to J$ defined by*

$$\langle \pi^\sharp(\xi), \eta\rangle = \pi(\xi, \eta), \ \forall \xi, \eta \in J^*, \tag{58}$$

is a relative Rota-Baxter operator with respect to the dual representation (ad^*, J^*).

Definition 6.2. Let π be an Jordan r-matrix and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. A pair (π, N) is a PN-structure (r-matrix-Nijenhuis structure) on the Jordan algebra if for any $x, y \in J$ and $\alpha, \beta \in J^*$, they satisfy

$$N\pi^\sharp = \pi^\sharp N^*, \tag{59}$$

$$\alpha \circ^{N\pi^\sharp} \beta = \alpha \circ^{\pi^\sharp}_{N^*} \beta, \tag{60}$$

where $\pi^\sharp : J^* \to J$ is a linear map induced by $\langle \pi^\sharp(\alpha), \beta\rangle = \pi(\alpha, \beta)$, (60) is given by (40) with $S = N^*$, $T = \pi^\sharp$ and the representation $\rho = ad^*$.

Theorem 6.3. *Let π be an Jordan r-matrix and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. Then (π, N) is a PN-structure on $(J, \circ)$ if and only if $(\pi^\sharp, S = N^*, N)$ is is a $\mathcal{O}$-dual-Nijenhuis structure on the* JordanRep *pair $(J, \circ, J^*, ad^*)$.*

Proof. It is straightforward. $\square$

By Proposition 5.13 and Theorem 6.3, we have

Corollary 6.4. *Let $\pi \in \wedge^2 J$ be a Jordan r-matrix and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. If (π, N) is a PN-structure on $(J, \circ)$, then π and π_N are compatible Jordan r-matrices in the sense that any linear combination of π and π_N is still a Jordan r-matrix, where $\pi_N \in \wedge^2 J$ is given by $\pi_N(\alpha, \beta) = \langle N\pi^\sharp(\alpha), \beta\rangle$, for all $\alpha, \beta \in J^*$.*

By Proposition 5.16 and Theorem 6.3, we have

Corollary 6.5. *Let (π, N) be a PN-structure on a Jordan algebra $(J, \circ)$. Then for all $k \in \mathbb{N}$, $\pi_k \in \wedge^2 J$ defined by $\pi_k(\alpha, \beta) = \langle N^k \pi^\sharp(\alpha), \beta\rangle$ for all $\alpha, \beta \in J^*$, are pairwise compatible Jordan r-matrices.*

Similar to the PN-structure, we give the definition of Rota-Baxter-Nijenhuis structure on a Jordan algebra.

Definition 6.6. Let $(J, \circ)$ be a Jordan algebra. Let $\mathcal{R} : J \to J$ be a Rota-Baxter operator and $N : J \to J$ a Nijenhuis operator on the Jordan algebra J. A pair $(\mathcal{R}, N)$ is a *Rota-Baxter-Nijenhuis structure* on $(J, \circ)$ if for any $x, y \in J$, they satisfy

$$N\mathcal{R} = \mathcal{R}N, \tag{61}$$

$$x \circ^{N\mathcal{R}} y = x \circ_N^{\mathcal{R}} y, \tag{62}$$

where (62) is given by (40) with $T = \mathcal{R}$ and the representation $\rho = ad$.

It is obvious that if $(\mathcal{R}, N)$ is Rota-Baxter-Nijenhuis structure, then the triple $(\mathcal{R}, S = N, N)$ is a $\mathcal{O}$-Nijenhuis structure on the JordanRep pair $(J, \circ, J, ad)$.

In the following, we study the relation between Rota-Baxter-Nijenhuis structure and r-matrix- Nijenhuis structure. First we recall some notions which was given in the article of Hou, Ni and Bai [12].

Let $(J, \circ)$ be a Jordan algebra with an invariant, non-degenerate, symmetric bilinear form $B \in J \otimes J$. Then B induces a bijective linear map $B^\sharp : J^* \to J$ given by

$$\langle B^\sharp(\alpha), \beta \rangle = B(\alpha, \beta), \ \forall \alpha, \beta \in J^*. \tag{63}$$

In view of invariance of B, for all $x \in J$, $\alpha \in J^*$, we have

$$B^\sharp(ad_x^*(\alpha)) = ad_x(B^\sharp(\alpha)) = x \circ B^\sharp(\alpha). \tag{64}$$

A skew-symmetric endomorphism of (J, B) is a linear map $\mathcal{R}$ from J to J such that $\mathcal{R}B^\sharp : J^* \to J$ is skew-symmetric.

In the following, we consider the relation between the Rota-Baxter-Nijenhuis structure and Jordan r-matrix-Nijenhuis structure.

Theorem 6.7. *Let $\mathcal{R}$ be a skew-symmetric endomorphism of (J, B), $N : J \to J$ a Nijenhuis operator, and set $\pi^\sharp = \mathcal{R}B^\sharp$. Assume that B and N are compatible, i.e.*

$$B^\sharp N^* = NB^\sharp \tag{65}$$

If $(\mathcal{R}, N)$ is a Rota-Baxter-Nijenhuis structure on the Jordan algebra J, then (π, N) is an r-matrix- Nijenhuis structure on the Jordan algebra J. Conversely, let (π, N) be a r-matrix-Nijenhuis structure on the Jordan algebra J with an invariant, non-degenerate, symmetric bilinear form B. Then $(\mathcal{R} = \pi^\sharp(B^\sharp)^{-1}, N)$ is a Rota-Baxter-Nijenhuis structure on the Jordan algebra J.

Proof. According to Corollary 3.4 [12], π is an Jordan r-matrix is equivalent to that $\pi^\sharp(B^\sharp)^{-1} = \mathcal{R}$ is a Rota-Baxter operator.
$(\Rightarrow)$ Since $B^\sharp$ is bijective, then for any $\alpha, \beta \in J^*$, there exist $x, y \in J$, such

that $\alpha = (B^\sharp)^{-1}(x)$, $\beta = (B^\sharp)^{-1}(y)$. By (61) and (65), it is obvious that $N\pi^\sharp = \pi^\sharp N^*$. In the sequel, we will verify that

$$\alpha \circ^{N\pi^\sharp} \beta = \alpha \circ^{\pi^\sharp}_{N^*} \beta.$$

By (64), we have

$$\begin{aligned}
\alpha \circ^{\pi^\sharp} \beta =& ad^*(\pi^\sharp(\alpha))\beta + ad^*(\pi^\sharp(\beta))\alpha \\
=& ad^*(\pi^\sharp((B^\sharp)^{-1}(x))(B^\sharp)^{-1}(y) + ad^*(\pi^\sharp((B^\sharp)^{-1}(y))(B^\sharp)^{-1}(x) \\
=& ad^*(\mathcal{R}(x))(B^\sharp)^{-1}(y) + ad^*(\mathcal{R}(y))(B^\sharp)^{-1}(x) \\
=& (B^\sharp)^{-1}\big(\mathcal{R}(x)\circ y + x \circ \mathcal{R}(y)\big) \\
=& (B^\sharp)^{-1}(x \circ^{\mathcal{R}} y),
\end{aligned}$$

which implies that

$$\alpha \circ^{\pi^\sharp} \beta = (B^\sharp)^{-1}(x \circ^{\mathcal{R}} y). \tag{66}$$

Thus by (65) and (66), one has

$$\begin{aligned}
& \alpha \circ^{N\pi^\sharp} \beta - \alpha \circ^{\pi^\sharp}_{N^*} \beta \\
=& \alpha \circ^{N\pi^\sharp} \beta - N^*(\alpha) \circ^{\pi^\sharp} \beta - \alpha \circ^{\pi^\sharp} N^*(\beta) + N^*(\alpha \circ^{\pi^\sharp} \beta) \\
=& (B^\sharp)^{-1}(x \circ^{N\mathcal{R}} y) - N^*(B^\sharp)^{-1}(x) \circ^{\pi^\sharp} (B^\sharp)^{-1}(y) \\
& - (B^\sharp)^{-1}(x) \circ^{\pi^\sharp} N^*(B^\sharp)^{-1}(y) + N^*((B^\sharp)^{-1}(x) \circ^{\pi^\sharp} (B^\sharp)^{-1}(y)) \\
=& (B^\sharp)^{-1}(x \circ^{N\mathcal{R}} y) - (B^\sharp)^{-1}N(x) \circ^{\pi^\sharp} (B^\sharp)^{-1}(y) \\
& - (B^\sharp)^{-1}(x) \circ^{\pi^\sharp} (B^\sharp)^{-1}N(y) + N^*((B^\sharp)^{-1}(x) \circ^{\pi^\sharp} (B^\sharp)^{-1}(y)) \\
=& (B^\sharp)^{-1}(x \circ^{N\mathcal{R}} y) - (B^\sharp)^{-1}(N(x) \circ^{\mathcal{R}} y) - (B^\sharp)^{-1}(x \circ^{\mathcal{R}} N(y)) + N^*(B^\sharp)^{-1}(x \circ^{\mathcal{R}} y) \\
=& (B^\sharp)^{-1}(x \circ^{N\mathcal{R}} y) - (B^\sharp)^{-1}(N(x) \circ^{\mathcal{R}} y) - (B^\sharp)^{-1}(x \circ^{\mathcal{R}} N(y)) + (B^\sharp)^{-1}N(x \circ^{\mathcal{R}} y) \\
=& (B^\sharp)^{-1}(x \circ^{N\mathcal{R}} y) - (B^\sharp)^{-1}\Big(N(x) \circ^{\mathcal{R}} y + x \circ^{\mathcal{R}} N(y) - N(x \circ^{\mathcal{R}} y)\Big) \\
=& (B^\sharp)^{-1}\Big(x \circ^{N\mathcal{R}} y - x \circ^{\mathcal{R}}_N y\Big) \\
=& 0.
\end{aligned}$$

Hence (π, N) is an Jordan r-matrix-Nijenhuis structure.
($\Leftarrow$) By (59) and (65), we have

$$N\mathcal{R} = \mathcal{R}N$$

The proof of the remaining part is similar to the case of the converse. We finish the proof. $\square$

6.2. ΩN-structures on Jordan algebra

Definition 6.8. [32] A symplectic form on a Jordan algebra $(J, \circ)$ is a skew-symmetric bilinear form $\omega \in \wedge^2 J^*$ satisfying

$$\omega(x \circ y, z) + \omega(y \circ z, x) + \omega(z \circ x, y) = 0, \quad \forall x, y, z \in J. \tag{67}$$

An element $\omega \in \wedge^2 J^*$ induces a linear map $\omega^\sharp : J \to J^*$ by

$$\langle \omega^\sharp(x), y \rangle = \omega(x, y), \quad \forall x, y \in J.$$

We say that $\omega \in \wedge^2 J^*$ is non-degenerate if $\omega^\sharp$ is an isomorphism.

Remark 6.9. In some references (for example, [5]), a symplectic form on a Jordan algebra is assumed to be non-degenerate.

Lemma 6.10. [32] *Let $(J, \circ)$ be a Jordan algebra and $\omega \in \wedge^2 J^*$ a non-degenerate bilinear form. Then ω is a symplectic form if and only if $(\omega^\sharp)^{-1} : J^* \to J$ is a relative Rota-Baxter operator on a* JordanRep *pair $(J, \circ, J^*, ad^*)$.*

By Lemma 6.1 and Lemma 6.10, we obtain

Corollary 6.11. *Let $(J, \circ)$ be a Jordan algebra. Then a non-degenerate $\pi \in \wedge^2 J$ is a Jordan r-matrix if and only if $\omega \in \wedge^2 J^*$ defined by*

$$\omega(x, y) = \langle (\pi^\sharp)^{-1}(x), y \rangle, \quad \forall x, y \in J, \tag{68}$$

is a symplectic form on J.

Definition 6.12. Let ω be a symplectic form and N a Nijenhuis operator on an Jordan algebra $(J, \circ)$. Then (ω, N) is called an *ΩN-structure* on the Jordan algebra $(J, \circ)$ if for all $x, y \in J$,

$$\omega(N(x), y) = \omega(x, N(y)), \tag{69}$$

and $\omega_N : \otimes^2 J \to J$ defined by $\omega_N(x, y) = \omega(N(x), y)$ is also a symplectic form, i.e.

$$\omega(N(x \circ y), z) + \omega(N(y \circ z), x) + \omega(N(z \circ x), y) = 0. \tag{70}$$

Example 6.13. Let J be the 4-dimensional Jordan algebra with basis $\{e_1, e_2, e_3, e_4\}$ and multiplication

$$e_1 \circ e_1 = 2e_1, \quad e_3 \circ e_1 = e_1 \circ e_3 = e_3.$$

Then J is a Jordan algebra. Let $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ be the dual basis and ω a non-degenerate skew-symmetric bilinear form given by

$$\omega = ke_1^* \wedge e_3^* + le_2^* \wedge e_4^*, \quad kl \neq 0.$$

Then ω is a symplectic structure on the Jordan algebra $(J, \circ)$. It is straightforward to verify that all

$$N = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

are Nijenhuis operators on the Jordan algebra $(J, \circ)$. By direct calculation, we have $\omega(N(x), y) = \omega(x, N(y))$ and

$$\omega(N(x \circ y), z) + \omega(N(y \circ z), x) + \omega(N(z \circ x), y) = 0, \quad \forall x, y, z \in J.$$

Thus, (ω, N) is an ΩN-structure on the Jordan algebra $(J, \circ)$.

There is a close relationship between ΩN-structures and $\mathcal{O}$-dual-Nijenhuis structures.

Theorem 6.14. *Let $\omega \in \wedge^2 J^*$ be a non-degenerate bilinear form and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. Then (ω, N) is an ΩN-structure on $(J, \circ)$ if and only if $((\omega^\sharp)^{-1}, S = N^*, N)$ is a $\mathcal{O}$-dual-Nijenhuis structure on the* JordanRep *pair $(J, \circ, J^*, ad^*)$.*

Proof. It is obvious that (69) is equivalent to that $(\omega^\sharp)^{-1}N^* = N(\omega^\sharp)^{-1}$. By Lemma 6.10, $\omega \in \wedge^2 J^*$ is a symplectic form if and only if $(\omega^\sharp)^{-1}$ is a relative Rota-Baxter operator on a JordanRep pair $(J, \circ, J^*, ad^*)$. Thus we only need to show that (70) is equivalent to

$$\alpha \circ^{N(\omega^\sharp)^{-1}} \beta = \alpha \circ^{(\omega^\sharp)^{-1}}_{N^*} \beta, \quad \forall \alpha, \beta \in J^*. \tag{71}$$

Assume that $\alpha = (\omega^\sharp)(x)$ and $\beta = (\omega^\sharp)(y)$ for some $x, y \in J$. For all $z \in J$, by (69) and the fact that ω is a symplectic form, we have

$$\begin{aligned} &\langle \alpha \circ^{N(\omega^\sharp)^{-1}} \beta - \alpha \circ^{(\omega^\sharp)^{-1}}_{N^*} \beta, z\rangle \\ &= \langle ad^*_{(\omega^\sharp)^{-1}(\alpha)} N^*\beta + ad^*_{(\omega^\sharp)^{-1}(\beta)} N^*\alpha - N^*\Big(ad^*_{(\omega^\sharp)^{-1}(\alpha)}\beta + ad^*_{(\omega^\sharp)^{-1}(\beta)}\alpha\Big), z\rangle \\ &= \langle N^*\omega^\sharp(y), x \circ z\rangle + \langle N^*\omega^\sharp(x), y \circ z\rangle - \langle \omega^\sharp(y), x \circ N(z)\rangle - \langle \omega^\sharp(x), y \circ N(z)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \omega(y, N(x \circ z)) + \omega(x, N(y \circ z)) - \omega(y, x \circ N(z)) - \omega(x, y \circ N(z)) \\
&= \omega(y, N(x \circ z)) + \omega(x, N(y \circ z)) + \omega(N(z), x \circ y) \\
&= \omega(y, N(z \circ x)) + \omega(x, N(y \circ z)) + \omega(z, N(x \circ y)) \\
&= 0,
\end{aligned}
$$

which implies that (71) holds. □

By Theorem 6.3 and Theorem 6.14, we obtain the following result which gives the relation between a PN-structure and an ΩN-structure.

Corollary 6.15. *Let $\omega \in \wedge^2 J^*$ be a non-degenerate bilinear form and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. Then (ω, N) is an ΩN-structure on $(J, \circ)$ if and only if (ω^{-1}, N) is a PN-structure on $(J, \circ)$, where $\omega^{-1} \in \wedge^2 J$ is defined by*

$$\omega^{-1}(\alpha, \beta) = \langle (\omega^\sharp)^{-1}(\alpha), \beta \rangle, \quad \forall \alpha, \beta \in J^*.$$

Example 6.16. Consider the ΩN-structure (ω, N) given by Example 6.13. Then (π, N) given by

$$r = -\frac{1}{k} e_1 \wedge e_3 - \frac{1}{l} e_2 \wedge e_4, \; kl \neq 0, \quad N = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

is a PN-structure on $(J, \circ)$.

By Proposition 5.13 and Theorem 6.14, we have

Corollary 6.17. *Let $\omega \in \wedge^2 J^*$ be a non-degenerate bilinear form and $N : J \to J$ a Nijenhuis operator on a Jordan algebra $(J, \circ)$. If (ω, N) is an ΩN-structure on $(J, \circ)$, then $(\omega^\sharp)^{-1}$ and $N(\omega^\sharp)^{-1}$ are compatible relative Rota-Baxter operators on a* JordanRep *pair $(J, \circ, J^*, ad^*)$.*

By Proposition 5.18, Theorem 6.14 and Corollary 6.15, we have

Corollary 6.18. *Let (ω, N) be an ΩN-structure on a Jordan algebra $(J, \circ)$. If ω is non-degenerate, then for all $k \in \mathbb{N}$, ω_k are symplectic structures, in which $\omega_k \wedge^2 J^*$ is defined by $\omega_k(x, y) = \omega(N^k(x), y)$. Furthermore, for all $k, l \in \mathbb{N}$, ω_k and ω_l are compatible in the sense that any linear combination of ω_k and ω_l are still symplectic structures.*

By Proposition 5.17 and Theorem 6.14, we have

Corollary 6.19.

1. *Let ω_1 and ω_2 be non-degenerate symplectic structures on a Jordan algebra $(J, \circ)$. If ω_1 and ω_2 are compatible, then $(\omega_1, N = \omega_1^\sharp(\omega_2^\sharp)^{-1})$ and $(\omega_2, N = \omega_1^\sharp(\omega_2^\sharp)^{-1})$ are ΩN-structures on the Jordan algebra $(J, \circ)$.*

2. *Let π be an Jordan r-matrix and ω a non-degenerate symplectic structure on a Jordan algebra $(J, \circ)$. If π and $(\omega)^{-1}$ are compatible r-matrices, then $(\pi, N = \pi^\sharp(\omega^\sharp)^{-1})$ is a PN-structure and $(\omega, N = \pi^\sharp(\omega^\sharp)^{-1})$ is an ΩN-structure on the Jordan algebra $(J, \circ)$.*

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