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Completely regular R-semirings and completely regular L-semirings

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Abstract. In [6] it was shown that a semiring is completely regular semiring if and only if it is a b-lattice of completely simple semirings. In this paper, we generalize this concept and introduce completely regular R-semiring, completely regular L-semiring and we show that a semiring is completely regular semiring if and only if it is both a completely regular L-semiring and a completely regular R-semiring. Moreover, we show that a semiring is a completely regular R-semiring (completely regular L-semiring) if and only if it is a b-lattice of completely simple R-semirings (completely simple L-semirings).

1. Introduction and preliminaries

Structure of a regular semigroup is well known and Green's equivalence relations [3] have vital role in the determination of this structure. Completely regular semirings were introduced and characterized in [6]. A semiring is completely regular semiring if and only if it is disjoint union of skew-rings and also if and only if it is a b-lattice of completely simple semirings (see [6]). Due to their rich structure, it is natural to search for classes of semirings close to completely regular semirings. In this paper, we introduce new classes of semirings by imposing some conditions on the elements of a semiring whose additive reduct is a regular semigroup. These new classes of semirings can be considered as the generalization of completely regular semirings [6].

By a *semiring* we mean here an algebraic structure $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations '$+$' and '$\cdot$' (called addition and multiplication respectively) defined on S such that both the reducts $(S, +)$ and $(S, \cdot)$ are semigroups and all elements $a, b, c \in S$ satisfy

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$a(b+c) = ab+ac$ and $(a+b)c = ac+bc$. For briefness, we sometimes write only S instead of $(S, +, \cdot)$ and we simply write ab instead of writing $a \cdot b$ for any two elements a, b in a semiring S.

A semiring S is said to be *additively commutative* if $a + b = b + a$ for all $a, b \in S$. A semiring $(S, +, \cdot)$ is called an *additively regular* semiring if for every element $a \in S$, there exists an element $x \in S$ such that $a + x + a = a$. Additively regular semirings were first studied by J. Zeleznekow [7].

Let a be an element in a semiring S. Then an element $x \in S$ satisfying $a + x + a = a$ and $x + a + x = x$ is said to be an *additive inverse* of a. A semiring $(S, +, \cdot)$ is said to be a *completely regular semiring* [6] if for each element $a \in S$, there exists an element $x \in S$ such that $a + x + a = a$, $a + x = x + a$ and $a(a + x) = a + x$. A semiring $(S, +, \cdot)$ is called an *idempotent semiring* if both the reducts $(S, +)$ and $(S, \cdot)$ are bands, i.e. $a + a = a = a \cdot a$ for all $a \in S$. According to Grillet [2], a *skew-ring* is a semiring $(S, +, \cdot)$ such that the additive reduct $(S, +)$ is a group. If a semiring S is such that the additive reduct $(S, +)$ is a semilattice and the multiplicative reduct $(S, \cdot)$ is a band, then S is called a *b-lattice*.

A non-empty subset I of a semiring S is a *left ideal* of S if $a + b, ra \in I$ for all $a, b \in I$ and for all $r \in S$. A right ideal is defined dually. A non-empty subset I of a semiring S is said to be an *ideal* of S if it is a left ideal of S as well as a right ideal of S. If I is a left ideal of a semiring S such that either $a+x \in I$ or $x+a \in I$, where $a \in I$ and $x \in S$, imply $x \in I$, then I is called a *left k-ideal* of S. A right k-ideal is defined dually. An ideal I of a semiring S is said to be a *k-ideal* of S, if either $a + x \in I$ or $x + a \in I$, where $a \in I$ and $x \in S$, imply $x \in I$. A mapping $\Phi : S \longrightarrow T$ between two semirings S and T is called a semiring homomorphism if $(a+b)\Phi = a\Phi + b\Phi$ and $(a \cdot b)\Phi = a\Phi \cdot b\Phi$ for all $a, b \in S$.

Throughout this paper, $E^+(S)$ denotes the set of all additive idempotents of the semiring S and the set of all additive inverses of an element $a \in S$, if exists, is denoted by $V^+(a)$. Also, for all $a \in S$ and for any $n \in \mathbb{N}$, we write $na = \underbrace{a + a + \cdots + a}_{n-copies}$.

As usual, we denote the Green's relations on the semiring $(S, +, \cdot)$ by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{H}$ and correspondingly, the $\mathcal{L}$-relation, $\mathcal{R}$-relation, $\mathcal{D}$-relation, $\mathcal{J}$-relation and $\mathcal{H}$-relation on $(S, +)$ are denoted by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$ respectively. In fact, the relations $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$ are all congruence relations on the multiplicative reduct $(S, \cdot)$. Thus if any one of these happens to be a congruence on $(S, +)$, it will be a semiring congruence on the semiring $(S, +, \cdot)$. A congruence ρ on a semiring S is called a *b-lattice congruence* if S/ρ is a b-lattice. A semiring S is called a *b-lattice* Y of semirings S_α $(\alpha \in Y)$ if S admits a b-lattice congruence ρ such that $Y = S/\rho$ and each S_α is a ρ-class. For any $a \in S$, we let H_a^+ be the $\mathcal{H}^+$-class in $(S, +)$ containing a. For other notations and symbols not given in this paper, the reader is referred to Golan [1], Howie [3], Petrich and Reilly [5].

2. Completely regular R-semirings and completely regular L-semirings

In this section, we introduce completely regular R-semirings and completely regular L-semirings and study their properties.

Definition 2.1. We call a semiring $(S, +, \cdot)$ a *completely regular R-semiring* if for each $a \in S$, there exists $x \in S$ such that $a + x + a = a$ and $a(a + x) = (a + x)a = a + x = 2a + 2x$.

There are plenty of examples of completely regular R-semirings. For example, every skew-ring, every ring, every distributive lattice and every idempotent semiring are completely regular R-semirings. Moreover, from the definition, it is clear that every completely regular semiring is a completely regular R-semiring, but the converse may not be true in general. This follows from the following example.

Example 2.2. Suppose $S = 2\mathbb{Z} \times 2\mathbb{Z}$ such that $(a, b) + (c, d) = (c, b + d)$ and $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in S$. Then S is an additively regular semiring with respect to the given operations. For any $p = (a, b) \in S$, there is an element $q = (0, -b) \in V^+(p)$ such that (i) $p + q + p = p$ and (ii) $p(p + q) = (p + q)p = p + q = 2p + 2q$. Hence S is a completely regular R-semiring. Note that for $p = (a, b) \in S$, we have to choose $q = (0, -b)$ so that the properties (i) and (ii) hold. But then $p + q \neq q + p$. Therefore, S is not a completely regular semiring. Also note that S is not a quasi completely regular semiring [4], not an idempotent semiring, not a ring, not a skew-ring and not even a distributive lattice. Now, for $p = (a, b) \in S$, if we choose $z = (a, -b)$, then $p + z + p = p$ and $p + z = z + p$, but z does not satisfy the property $p(p + z) = (p + z)p = p + z = 2p + 2z$. From this, it at once follows that $(S, +)$ is a completely regular semigroup but the semiring $(S, +, \cdot)$ is not a completely regular semiring. Therefore, a semiring whose additive reduct is a completely regular semigroup may not be a completely regular semiring.

The following theorem characterizes completely regular R-semirings.

Theorem 2.3. *A semiring S is a completely regular R-semiring if and only if for each $a \in S$, there exists an element $y \in V^+(a)$ such that $a(a+y) = (a+y)a = a+y = 2a+2y$.*

Proof. First suppose that S is a completely regular R-semiring. Then for any element $a \in S$, there exists an element $x \in S$ such that $a+x+a=a$ and $a(a+x) = (a+x)a = a+x = 2a+2x$. Let $y = x+a+x$. Then clearly $y \in V^+(a)$. Now, $a(a+y) = a(a+x+a+x) = a(a+x) = a+x = a+(x+a+x) = a+y$. Similarly, $(a+y)a = a+y$. Finally, $2a+2y = 2a+2(x+a+x) = a+(a+x+a)+x+x+a+x = 2a+2x+a+x = a+x+a+x = a+y$. Therefore, $a(a+y) = (a+y)a = a+y = 2a+2y$.

The converse is obvious. □

Remark 2.4. In a semiring S, if for an element $a \in S$ there exists an element $x \in S$ such that $x \in V^+(a)$, then both $a+x$ and $x+a \in E^+(S)$. But the additive idempotent $a+x$ mainly plays the crucial role in the definition of completely regular R-semiring. Instead of $a+x$, if we consider the additive idempotent $x+a$ in a similar way, then we call the semiring a completely regular L-semiring.

Definition 2.5. A semiring S is said to be a *completely regular L-semiring* if for each $a \in S$, there exists an element $x \in S$ such that $a+x+a = a, a(x+a) = (x+a)a = x+a = 2x+2a$.

Similar to Theorem 2.3, we can prove the following result.

Theorem 2.6. *A semiring S is a completely regular L-semiring if and only if for each $a \in S$, there exists an element $y \in V^+(a)$ such that $a(y+a) = (y+a)a = y+a = 2y+2a$.*

Remark 2.7. For an element a in a semiring S, we denote

$$I_L(a) = \{x \in S : x \in V^+(a),\ a(x+a) = (x+a)a = x+a = 2x+2a\}$$

and

$$I_R(a) = \{x \in S : x \in V^+(a),\ a(a+x) = (a+x)a = a+x = 2a+2x\}.$$

Then S is completely regular L-semiring if and only if $I_L(a) \neq \emptyset$ and completely regular R-semiring if and only if $I_R(a) \neq \emptyset$ for all $a \in S$.

Remark 2.8. It is worth mentioning that $I_L(a)$ or $I_R(a)$ may contain more than one element. For instance, if we consider the semiring $S = \mathbb{Z} \times \mathbb{Z}$ with the operations given as $(a, b) + (c, d) = (c, b + d)$ and $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in S$, then for any $a \in \mathbb{Z}$, we have $I_R((1, a)) = \{(x, -a) : x \in \mathbb{Z}\}$.

3. Properties of completely regular R-semirings and completely regular L-semirings

Here we mainly discuss the structure and properties of completely regular R-semiring and completely regular L-semiring.

First we state the following results from [5, Lemma I.7.9].

Proposition 3.1. *For any $\mathcal{H}$-class H of a semigroup S, the following are equivalent:*

(i) *H is a group.*

(ii) *H contains an idempotent.*

(iii) *There exist elements $a, b \in H$ with $ab \in H$.*

(iv) *$HH \subseteq H$.*

Using the above results, we now establish some properties of completely regular R-semiring.

Theorem 3.2. *If S is a completely regular R-semiring, then the following properties hold:*

(i) *Every $\mathcal{R}^+$-class of S is a subsemiring.*

(ii) *Every $\mathcal{H}^+$-class is a subgroup of $(S, +)$. Hence $(S, +)$ is a completely regular semigroup.*

Proof. (i). Let a be an element in a completely regular R-semiring S. Then there exists an element $y \in V^+(a)$ such that $a + y + a = a$ and $a(a + y) = (a + y)a = a + y = 2a + 2y$. Now, $a = a + y + a = a(a + y) + a = a^2 + ay + a$ and $a^2 = a(a+y+a) = a(a+y)+a^2 = a+y+a^2$ imply that $a^2 \,\mathcal{R}^+\, a$. Again, $2a + 2y + a = a + y + a = a$ imply $2a \,\mathcal{R}^+\, a$. Let $b, c \in R^+_a$, where R^+_a is the $\mathcal{R}^+$-class containing $a \in S$. Then $b = a + s_1, a = b + s_2, c = a + s_3, a = c + s_4$ for some $s_1, s_2, s_3, s_4 \in S$. Now, $a + b = 2a + s_1, 2a = a + b + s_2$ imply that $(a + b) \,\mathcal{R}^+\, 2a \,\mathcal{R}^+ a$ and thus $a + b \in R^+_a$. Similarly, we can show that $b + c \in R^+_b = R^+_a$. Again, $bc = (a + s_1)(a + s_3) = a^2 + as_3 + s_1a + s_1s_3$ and

$a^2 = (b + s_2)(c + s_4) = bc + bs_4 + s_2c + s_2s_4$ imply that $bc\,\mathcal{R}^+\,a^2\,\mathcal{R}^+\,a$ and thus $bc \in R_a^+$. Therefore, each $\mathcal{R}^+$-class is a subsemiring of S.

(ii). Let a be an element in a completely regular R-semiring S. Then there exists an element $y \in V^+(a)$ such that $a + y + a = a$ and $a(a + y) = (a + y)a = a + y = 2a + 2y$. It is easy to verify that $a\,\mathcal{R}^+\,2a$, $y\,\mathcal{R}^+\,(y + a)$. Similarly, for the element $y \in S$, we can show that $y\,\mathcal{R}^+\,2y$. Therefore, we have $(y + a)\,\mathcal{R}^+\,2y$. Let $e = 2y + 2a$. Then $e + e = (2y + 2a) + (2y + 2a) = 2y + (2a + 2y + a) + a = 2y + 2a = e$ and thus $e \in E^+(S)$. Again, $e + 2y = 2y + 2a + 2y = 2y + a + y = 2y$ implies $e\,\mathcal{R}^+\,2y\,\mathcal{R}^+\,(y + a)$ and thus $y + a = e + u$ for some $u \in S$. From this, we have $e + y + a = y + a$, i.e., $y + a = e + y + a = 2y + 2a + y + a = 2y + 2a = e$, i.e., $2y + 2a = e = y + a$.

Again, let $f = a + 2y + a$. Then $f + f = a + 2y + a + a + 2y + a = a + 2y + 2a + 2y + a = a + y + a + 2y + a = a + 2y + a = f$, hence $f \in E^+(S)$. Now $a + f = 2a + 2y + a = a$ implies $a\,\mathcal{L}^+\,f$. Also, $f + a = (a + 2y + a) + a = a + 2y + 2a = a + y + a = a$ implies $a\,\mathcal{R}^+\,f$. Therefore, $a\,\mathcal{H}^+\,f$, i.e., H_a^+, the $\mathcal{H}^+$-class containing the element a contains an idempotent of the semigroup $(S, +)$. Hence by Proposition 3.1, it follows that H_a^+ is a subgroup of $(S, +)$ and consequently, $(S, +)$ is a completely regular semigroup. □

Corollary 3.3. *If S is a completely regular R-semiring and $a \in S$, then $a + 2y + a = a + 2z + a \in E^+(S)$ for any two elements $y, z \in I_R(a)$.*

Proof. Let $a \in S$ and $y \in I_R(a)$. Then from the proof of Theorem 3.2, it follows that $f = a + 2y + a \in E^+(S)$ is the identity element of the subgroup $(H_a^+, +)$. Similarly, for any other element $z \in I_R(a)$, we can prove that $a + 2z + a \in E^+(S)$ is also an identity element of the subgroup $(H_a^+, +)$. From the uniqueness of the identity element, it follows that $a + 2y + a = a + 2z + a$. □

Remark 3.4. If S is a completely regular R-semiring, then for every $a \in S$ and $y \in I_R(a)$, the unique element $a + 2y + a \in E^+(S)$ is denoted by 0_a. Therefore, for $a \in S$ and for any $y \in I_R(a)$, $0_a = a + 2y + a$ is the identity element of the group $(H_a^+, +)$.

Similarly as Theorem 3.2, we can prove the following result for completely regular L-semiring.

Theorem 3.5. *If S is a completely regular L-semiring, then the following properties hold:*

(i) Every $\mathcal{L}^+$-class of S is a subsemiring.

(ii) Every $\mathcal{H}^+$-class is a subgroup of $(S, +)$. Hence $(S, +)$ is a completely

regular semigroup.

Remark 3.6. From Theorem 3.2 and Theorem 3.5, we conclude that in a completely regular R-semiring each $\mathcal{R}^+$-class and in a completely regular L-semiring each $\mathcal{L}^+$-class are subsemirings. Moreover, for both of these semirings the additive reduct is a completely regular semigroup. The names of our newly defined semirings are chosen according to their structures. Since the additive reduct is a completely regular semigroup, it follows that in a completely regular R-semiring and also in a completely regular L-semiring, $\mathcal{D}^+ = \mathcal{J}^+$.

Theorem 3.7. *If S is a completely regular R-semiring, then the relation $\mathcal{J}^+$ is a b-lattice congruence on S and each $\mathcal{J}^+$–class is a subsemiring of S.*

Proof. Since S is a completely regular R-semiring, so by Theorem 3.2, it follows that $(S,+)$ is a completely regular semigroup and hence we find from the Theorem II.1.4 in [5] that $\mathcal{J}^+$ is a semilattice congruence on $(S,+)$ and obviously each $\mathcal{J}^+$-class is a subsemigroup of $(S,+)$. To complete the proof, it remains to show that $\mathcal{J}^+$ is a band congruence on $(S,\cdot)$ and each $\mathcal{J}^+$-class is a subsemigroup of $(S,\cdot)$. Clearly, $\mathcal{J}^+$ is a congruence on $(S,\cdot)$. Let $a \in S$. Then there exists an element $y \in V^+(a)$ such that $a(a+y) = (a+y)a = a+y = 2a+2y$. Now, $a = a+y+a = (a+y)+(a+y)+a = (a+y)+a(a+y)+a = (a+y)+a^2+(ay+a)$ and $a^2 = a(a+y+a+y+a) = a(a+y)+a(a+y)+a^2 = (a+y)+a+(y+a^2)$ imply that $a^2\,\mathcal{J}^+\,a$. Hence $\mathcal{J}^+$ is a band congruence on $(S,\cdot)$ and therefore, $\mathcal{J}^+$ is a b-lattice congruence on the semiring S. Finally, to show each $\mathcal{J}^+$-class is a subsemigroup of $(S,\cdot)$, let $b,c \in J_a^+$, where J_a^+ is the $\mathcal{J}^+$-class containing an element $a \in S$. Then $b\,\mathcal{J}^+\,a$ and $c\,\mathcal{J}^+\,a$. Since $\mathcal{J}^+$ is a band congruence on $(S,\cdot)$, it follows that $bc\,\mathcal{J}^+\,a^2\mathcal{J}^+\,a$ and therefore, $bc \in J_a^+$. Thus each $\mathcal{J}^+$-class is a subsemigroup of $(S,\cdot)$ and consequently, each $\mathcal{J}^+$-class is a subsemiring of S. □

Similarly as Theorem 3.7, we can prove the following result for completely regular L-semiring.

Theorem 3.8. *If S is a completely regular L-semiring, then each $\mathcal{J}^+$-class is a subsemiring of S and the relation $\mathcal{J}^+$ is a b-lattice congruence on S.*

Now we state some results for completely regular R-semirings and it can be verified in similar ways that analogous results hold for completely regular L-semirings.

Lemma 3.9. *Let a be an element in a completely regular R-semiring. Then the following properties hold:*

(*i*) $0_e = e$ *for all* $e \in E^+(S)$.

(*ii*) $E^+(S) = \{0_a : a \in S\}$.

Proof. (*i*). Since 0_e is the unique identity element of the group $(H_e^+, +)$, so clearly $0_e = e$ for all $e \in E^+(S)$.

(*ii*). Obviously, $0_a \in E^+(S)$ for all $a \in S$ as 0_a is the identity of the group $(H_a^+, +)$. Again, by (i), we have $0_e = e$ for all $e \in E^+(S)$. Hence $E^+(S) = \{0_a : a \in S\}$. □

Lemma 3.10. *If S is a completely regular R-semiring, then for all $a \in S$ and $y \in I_R(a)$, the following properties hold:*

(*i*) $y(y + a) = (y + a)y$.

(*ii*) *for all* $n \in \mathbb{N}$, $na + ny = a + y$ *and* $ny + na = y + a$.

(*iii*) $(0_a)^2 = 0_{a^2} = a0_a = 0_a a$.

(*iv*) $0_y = y + 2a + y$.

(*v*) $0_a + 0_y = 0_{a+y}$ *and* $0_y + 0_a = 0_{y+a}$.

(*vi*) $0_y \in I_R(0_a)$.

Proof. (*i*). Suppose $a \in S$ and $y \in I_R(a)$. Then $y^2 = (y + a + y)y = (y^2+ay)+y^2$ implies $(y^2+ay)\,\mathcal{R}^+\,y^2\mathcal{R}^+\,y$. Also, $y^2+ay = y^2+a(y+a+y) = (y^2 + ay + a) + y$ and $y = y + a + y = y + a(a + y) = y + a^2 + ay = y+a^2+(a+y+a)y = (y+a^2+ay)+y^2+ay$ imply that $(y^2+ay)\mathcal{L}^+\,y$. Hence $(y^2 + ay)\mathcal{H}^+\,y$. Since $(H_y^+, +)$ is a group and $y^2 + ay = (y + a)y \in E^+(S)$, it follows that the identity of $(H_y^+, +)$ is $0_y = y^2 + ay = (y+a)y$. Similarly, we can show that $0_y = y(y + a)$. Consequently, $y(y + a) = (y + a)y$.

(*ii*). Since $y \in I_R(a)$, we have $2a + 2y = a + y$. Now, $3a + 3y = a + (2a + 2y) + y = a + (a + y) + y = 2a + 2y = a + y$. Similarly, by induction, we can prove that $na + ny = a + y$ for all $n \in \mathbb{N}$.

Again, from the proof of Theorem 3.2, it follows that $2y + 2a = y + a$ and hence similar to $na + ny = a + y$, we can prove that $ny + na = y + a$.

(*iii*). Clearly, $\mathcal{H}^+$ is a congruence on $(S, \cdot)$. Again, for any $a \in S$, we have $a\,\mathcal{H}^+\,0_a$. This implies $a^2\,\mathcal{H}^+\,a0_a\,\mathcal{H}^+\,0_a a\,\mathcal{H}^+\,(0_a)^2$. Since each $\mathcal{H}^+$-class contains unique additive idempotent, therefore, we must have $(0_a)^2 = 0_{a^2} = a0_a = 0_a a$.

(*iv*). Let $a \in S$ and $y \in I_R(a)$. Then it is easy to verify that $e = y + 2a + y \in E^+(S)$. Moreover, $e + y = y + 2a + 2y = y + a + y = y$ and

$y+e=2y+2a+y=y+a+y=y$ imply that $y\,\mathcal{H}^+\,e$. Therefore, e is the identity element of the group $(H^+_y,+)$ and hence $0_y=e=y+2a+y$.

(v). Now, $0_a+0_y=(a+2y+a)+(y+2a+y)=a+2y+(a+y+a)+a+y=a+2y+2a+y=a+y+a+y=a+y$. Since $a+y\in E^+(S)$, so $0_{a+y}=a+y$. Therefore, $0_a+0_y=0_{a+y}$. Similarly, we can prove that $0_y+0_a=0_{y+a}$.

(vi). For $a\in S$, let $y\in I_R(a)$. Then $0_a+0_y+0_a=a+y+0_a=a+y+(a+2y+a)=a+2y+a=0_a$ and $0_y+0_a+0_y=y+a+0_y=y+a+(y+2a+y)=y+2a+y=0_y$. Therefore, $0_y\in V^+(0_a)$. Also, $0_a(0_a+0_y)=(a+2y+a)(a+y)=a(a+y)+2y(a+y)+a(a+y)=a(a+y)+y(a+y)+a(a+y)=a(a+y)=a+y=0_a+0_y$. Similarly, we can show that $(0_a+0_y)0_a=0_a+0_y$ and obviously $0_a+0_y=20_a+20_y$. Therefore, $0_y\in I_R(0_a)$. □

Theorem 3.11. *Let S be a completely regular R-semiring. Then S is a completely regular semiring if and only if $(0_a)^2=0_a$, for all $a\in S$.*

Proof. First we assume that S is a completely regular R-semiring such that $(0_a)^2=0_a$, for all $a\in S$. Since S is a completely regular R-semiring, then by Theorem 3.2 (ii), it follows that H^+_x is a subgroup of $(S,+)$, for all $x\in S$. Since $0_{a^2}=(0_a)^2=0_a$, for all $a\in S$, we must have $a^2\,\mathcal{H}^+\,a$, for all $a\in S$. Let $b,c\in H^+_a$. Then $b\mathcal{H}^+\,c\,\mathcal{H}^+\,a$. Since $\mathcal{H}^+$ is a congruence on $(S,\cdot)$, so $bc\,\mathcal{H}^+\,a^2\,\mathcal{H}^+\,a$ and thus $bc\in H^+_a$. Hence H^+_a is a skew-ring and therefore, S is disjoint union of skew-rings. Consequently, by [6, Theorem 3.6], it follows that S is a completely regular semiring.

The converse statement is obvious. □

Combining Lemma 3.9 (ii) and Theorem 3.11, we have the following corollary.

Corollary 3.12. *A completely regular R-semiring is a completely regular semiring if and only if every additive idempotent is also a multiplicative idempotent.*

Theorem 3.13. *Let S be a completely regular R-semiring. Then S is a skew-ring if and only if $0_a=0_b$, for all $a,b\in S$.*

Proof. Since S is a completely regular R-semiring, so $(S,+)$ is a regular semigroup. Again, since $0_a=0_b$, for all $a,b\in S$, it follows that $E^+(S)$ is a singleton set. This implies $(S,+)$ is a regular semigroup with only one additive idempotent element. Therefore, $(S,+)$ is a group and hence S is a skew-ring.

The second part is obvious. □

Corollary 3.14. *A completely regular R-semiring is a skew-ring if and only if it contains exactly one additive idempotent.*

Theorem 3.15. *Let $\psi : S \longrightarrow T$ be a semiring homomorphism from a completely regular R-semiring S into a semiring T. Then*

(*i*) *$S\psi$ is also a completely regular R-semiring.*

(*ii*) *$0_s\psi = 0_{s\psi}$, for all $s \in S$.*

Proof. (*i*). Suppose $t \in S\psi$. Then there exists some $s \in S$ such that $t = s\psi$. As S is a completely regular R-semiring, so there exists $x \in V^+(s) \subseteq S$ such that $s(s + x) = (s + x)s = s + x = 2s + 2x$. Let $x\psi = y$. Then one can easily prove that $y \in V^+(t)$ such that $t(t + y) = (t + y)t = t + y = 2t + 2y$. Consequently, $S\psi$ is a completely regular R-semiring.

(*ii*). For any element $s \in S$, there exists an element $x \in V^+(s)$ such that $s(s + x) = (s + x)s = s + x = 2s + 2x$. Then $0_s = s + 2x + s$. Let $s\psi = t$ and $x\psi = y$. Then it is easy to verify that $y \in V^+(t)$ such that $t(t+y) = (t+y)t = t+y = 2t+2y$. Then by definition $0_t = t+2y+t$ and thus $0_{s\psi} = s\psi + 2x\psi + s\psi = (s + 2x + s)\psi = 0_s\psi$. Therefore, $0_s\psi = 0_{s\psi}$, for all $s \in S$. □

Corollary 3.16. *If ρ be a congruence on a completely regular R-semiring S such that $a\,\rho\,b$ for some $a, b \in S$, then $0_a\,\rho\,0_b$.*

Proof. Consider the natural epimorphism $\psi : S \longrightarrow S/\rho$ defined by $a\psi = a\rho$, for all $a \in S$. Now, $0_a\rho = 0_a\psi = 0_{a\psi} = 0_{a\rho} = 0_{b\rho} = 0_{b\psi} = 0_b\psi = 0_b\rho$ implies $0_a\,\rho\,0_b$. □

Proposition 3.17. *Every left (right) k-ideal of a completely regular R-semiring is also a completely regular R-semiring. Hence every k-ideal of a completely regular R-semiring is also a completely regular R-semiring.*

Proof. Suppose S be a completely regular R-semiring, K be a left k-ideal of S and $a \in K$. Then there exists an element $x \in V^+(a) \subseteq S$ such that $a(a + x) = (a + x)a = a + x = 2a + 2x$. Now, $a \in K$ implies $a^2, xa \in K$, i.e., $a + x = (a + x)a = a^2 + xa \in K$. Since K is a left k-ideal of S, so $a, a + x \in K$ implies $x \in K$. Consequently, K is a completely regular R-semiring. Similarly, we can show that every right k-ideal of S and every k-ideal of S are also completely regular R-semirings. □

Proposition 3.18. *Let $(S, +, \cdot)$ be a semiring such that $(S, +)$ is a semilattice. Then S is a completely regular R-semiring if and only if S is a completely regular semiring.*

Proof. First suppose S is a completely regular R-semiring such that $(S, +)$ is a semilattice. Then for any $a \in S$, there exists an element $x \in V^+(a)$ such that $a(a + x) = (a + x)a = a + x$. Since $(S, +)$ is a semilattice, so we have $a = a + x + a = a + a + x = a + x$. Hence from $a(a + x) = a + x$ we have $a^2 = a$. Therefore, S is a b-lattice and thus S is a completely regular semiring.

The converse part is obvious. □

Proposition 3.19. *Let $(S, +, \cdot)$ be a completely regular R-semiring such that $(S, \cdot)$ is a band. Then S is a completely regular semiring.*

Proof. Since S is a completely regular R-semiring such that $a^2 = a$ for all $a \in S$, so it follows that $a^2 \, \mathcal{H}^+ \, a$ and hence similar to the proof of Theorem 3.11, we can conclude that each $\mathcal{H}^+$-class is a skew-ring. Therefore, S is a completely regular semiring. □

We now recall a result from Petrich and Reilly [5, Lemma II.1.6].

Lemma 3.20. *The following conditions on a semigroup S are equivalent:*

- (i) *S is a rectangular band.*
- (ii) *S is regular and satisfies the identity $ab = axb$ for all $a, b, x \in S$.*
- (iii) *S is a completely simple band.*

Theorem 3.21. *Let $(S, +, \cdot)$ be a completely regular R-semiring whose multiplicative reduct $(S, \cdot)$ is a completely simple semigroup. Then $(S, \cdot)$ is a rectangular band if and only if S is a completely regular semiring.*

Proof. If $(S, \cdot)$ is a rectangular band, then by Proposition 3.19, we conclude that S is a completely regular semiring.

Conversely, if S is a completely regular semiring, then $E^+(S) \neq \emptyset$. Clearly, $E^+(S)$ is an ideal of $(S, \cdot)$. Since $(S, \cdot)$ is simple, so $S = E^+(S)$. Therefore, every element of S is an additive idempotent and hence $(S, \cdot)$ is a band [6, Lemma 2.5]. Thus $(S, \cdot)$ is a completely simple band and so by Lemma 3.20, it follows that $(S, \cdot)$ is a rectangular band. □

Definition 3.22. A completely regular R-semiring (L-semiring) is said to be a *completely simple R-semiring* (*L-semiring*) if $\mathcal{J}^+ = S \times S$.

Theorem 3.23. *The following conditions on a semiring S are equivalent:*

(i) *S is a completely regular R-semiring.*

(ii) *S is a b-lattice of completely simple R-semirings.*

Proof. $(i) \Rightarrow (ii)$. Since S is a completely regular R-semiring, so by Theorem 3.2, it follows that $(S,+)$ is a completely regular semigroup and therefore, $\mathcal{J}^+$ is a semilattice congruence on $(S,+)$. One can easily show that $\mathcal{J}^+$ is a congruence on $(S,\cdot)$. Moreover, for any $a \in S$, $(a^2,a) \in \mathcal{R}^+ \subseteq \mathcal{J}^+$ implies $a^2 \,\mathcal{J}^+\, a$ and thus $\mathcal{J}^+$ is a b-lattice congruence on the semiring S. Again, as $(S,+)$ is a completely regular semigroup, it follows that each $\mathcal{J}^+$-class is a completely simple subsemigroup of $(S,+)$. To show each $\mathcal{J}^+$-class is a subsemiring of S, let $b,c \in J_a^+$, where J_a^+ is the $\mathcal{J}^+$-class containing an element $a \in S$. Since b, $c \in J_a^+$, there exist x, y, u, v, x_1, y_1, u_1, $v_1 \in S$ such that $b = x+a+y$, $\quad a = u+b+v$, $\quad c = x_1+a+y_1$, $\quad a = u_1+c+v_1$. Now, $bc = (x+a+y)(x_1+a+y_1) = (xx_1+xa+xy_1+ax_1)+a^2+(ay_1+yx_1+ya+yy_1)$ and $a^2 = (u+b+v)(u_1+c+v_1) = (uu_1+uc+uv_1+bu_1)+bc+(bv_1+vu_1+vc+vv_1)$. Hence $bc \,\mathcal{J}^+\, a^2 \,\mathcal{J}^+\, a$. This shows that every $\mathcal{J}^+$-class is a subsemiring of the semiring S. Finally, since S is a completely regular R-semiring, so for each element $u \in S$, there exists an element $v \in V^+(u)$ such that $u(u+v) = (u+v)u = u+v = 2u+2v$. One can easily verify that $u \,\mathcal{J}^+\, v$ and hence it follows that each $\mathcal{J}^+$-class is also a completely regular R-semiring. Consequently, S is a b-lattice of completely simple R-semirings.

$(ii) \Rightarrow (i)$. This is obvious. □

Similaly as Theorem 3.23, we can prove the following result.

Theorem 3.24. *The following conditions on a semiring S are equivalent:*

(i) *S is a completely regular L-semiring.*

(ii) *S is a b-lattice of completely simple L-semirings.*

Proposition 3.25. *A semiring is completely regular semiring if and only if it is both a completely regular L-semiring as well as a completely regular R-semiring.*

Proof. If S is completely regular semiring, then from the definition it is clear that S is both a completely regular L-semiring as well as a completely regular R-semiring.

Conversely, suppose S is both completely regular L-semiring as well as completely regular R-semiring. Then every $\mathcal{L}^+$-class and every $\mathcal{R}^+$-class of S are semirings. Again, since $(S,+)$ is a completely regular semigroup, so each $\mathcal{H}^+$-class is a subgroup of $(S, +)$. To show each $\mathcal{H}^+$-class is a subsemigroup of $(S, \cdot)$, let $b, c \in H_a^+$, where H_a^+ is the $\mathcal{H}^+$-class containing an element $a \in S$. Then $b\,\mathcal{H}^+\,a\,\mathcal{H}^+\,c$. This implies $b\,\mathcal{R}^+\,a\,\mathcal{R}^+\,c$ and $b\,\mathcal{L}^+\,a\,\mathcal{L}^+\,c$. Therefore, $bc\,\mathcal{R}^+\,a^2\,\mathcal{R}^+\,a$ and $bc\,\mathcal{L}^+\,a^2\,\mathcal{L}^+\,a$ and thus $bc\,\mathcal{H}^+\,a$. Therefore, $bc \in H_a^+$ and hence each $\mathcal{H}^+$-class is a semiring. Thus each $\mathcal{H}^+$-class is a skew-ring and hence by [6, Theorem 3.6], it follows that S is a completely regular semiring. □

References

[1] **J.S. Golan**, *The theory of semirings with applications in mathematics and theoretical computer science*, Pitman Monographs and Surveys in Pure and Applied Mathematics **54** (1992), Longman Scientific.

[2] **M.P. Grillet**, *Semirings with a completely simple additive semigroup*, J. Austral. Math. Soc. **20** (Series A) (1975), 257 − 267.

[3] **J.M. Howie**, *Fundamentals of semigroup theory*, Clarendon Press, Oxford (1995).

[4] **S.K. Maity, R. Ghosh**, *On quasi completely regular semirings*, Semigroup Forum **89** (2014), 422 − 430.

[5] **M. Petrich, N.R. Reilly**, *Completely regular semigroups*, Wiley, New York (1999).

[6] **M.K. Sen, S.K. Maity, K.P. Shum**, *On completely regular semirings*, Bull. Cal. Math. Soc. **98** (2006), 319 − 328.

[7] **J. Zeleznekow**, *Regular semirings*, Semigroup Forum **23** (1981), 119 − 136.

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