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Solvable and nilpotent ultra-groups

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Abstract We propose several characterizations of solvable ultra-groups and investigate the Jordan-Hölder Theorem and the Zassenhaus Lemma, in ultra-groups. We also define nilpotent ultra-groups by using the center of ultra-groups. Finally, we establish the relation between nilpotent and solvable ultra-groups. Our results aim to serve as a bridge between groups and ultra-groups.

1. Introduction

Solvable and nilpotent groups are two fundamental classes of groups in algebra, and they play a critical role in the study of Lie groups, Galois groups and others [2, 4]. In this paper, we focus on exploring the concepts solvability and nilpotency and their relation for ultra-groups. Ultra-groups are defined on the use of transversals in groups. Moghaddasi et al. built upon the concept of hypergroups and transversals to introduce the concept ultra-groups [7, 8]. Tolue et al. [9] introduced the category of ultra-group and investigated some properties of this category.

The organization of the paper is as follows. We first present some basic definitions and results in ultra-groups. Next, we introduce the concept of solvable ultra-groups and discuss some general concepts such as composition series, subnormal series. Finally, we characterize the Zassenhaus Lemma and generalize the Jordan-Hölder theorem for ultra-groups. In Section 3 we describe nilpotent ultra-groups and establish the relationship between solvable and nilpotent ultra-group.

Our results can be used as a bridge between groups and ultra-groups.

The notation in this paper is as in [7] and [9].

A pair (A, B) of subsets of a group G is called *transversal* if the equality $ab = a'b'$ implies $a = a'$ and $b = b'$, where $a, a' \in A$, $b, b' \in B$. It is not hard

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to deduce that a pair (H, M) of subgroups of a group G is transversal if and only if $H \cap M = \{e\}$, where e is the identity of the group G. Furthermore, for a subgroup H and a subset M of a group G we conclude that the pair (H, M) is a transversal if and only if $M \cap Hg$ contains at most one element, for all $g \in G$. A subset M of a group G is called (*right unitary*) *complementary set* with respect to a subgroup H, if for any elements $m \in M$ and $h \in H$ there exist unique elements $h' \in H$ and $m' \in M$ such that $mh = h'm'$. We denote h' and m' by ${}^{m}h$ and m^h, respectively. Similarly for any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1m_2 = (m_1, m_2)[m_1, m_2]$. There are $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$, since $G = HM$.

Definition 1.1. [7] Let M be a transversal set of a subgroup H over a group G. The set M together with a binary operation $\alpha : M \times M \longrightarrow M$ and a family of unary operations $\beta_h : M \longrightarrow M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$ is called a *right ultra-group*. An ultra-group M is called abelian, if for all elements a, b in M, $[a, b] = [b, a]$.

We use the notation ${}_HM$ to represent the right ultra-group of subgroup H which we briefly display with the notation M.

Definition 1.2. [7] Let M be an ultra-group of a subgroup H of a group G. A subset $K \subseteq M$ which contains the identity element of the group G, is called a *subultra-group* of M, if K is closed under operations α and β_h. This is denoted by $K < M$.

Proposition 1.3. [7] *Let M be an ultra-group of a subgroup H over a group G. Then we have the following properties:*

(*i*) $a^{hh'} = (a^h)^{h'}$,

(*ii*) $[a, b]^h = [a^{({}^bh)}, b^h]$,

(*iii*) $[[a, b], c] = [a^{(b,c)}, [b, c]]$,

(*iv*) $e^h = e, \quad a^e = a$,

(*v*) $[e, a] = a = [a, e]$,

(*vi*) $[a^{[-1]}, a] = e = [a^{(a^{(-1)})}, a^{[-1]}]$,

for $a, b, c \in M$ and $h, h' \in H$.

Definition 1.4. [8] A subultra-group N of an ultra-group M is called *normal* if $[N, [a, b]] = [a, [N, b]]$ for all $a, b \in M$ and is denoted by $N \lhd M$.

According to the definition, every ultra-group M, has normal subultra-group $\{e\}$. We note that an ultra-group M is normal subultra-group of itself, whenever the left cancellation law be established for M (see [7]).

Lemma 1.5. [7] *Let K be a subultra-group of an ultra-group M. Then for $a, b \in M$ the following conditions are equivalent.*

(*i*) $a \in [K, b]$,

(*ii*) $[K, a] = [K, b]$,

(*iii*) $[a^{(b^{(-1)})}, b^{[-1]}] \in K$.

Theorem 1.6. *An ultra-group N is a normal subultra-group of M, if and only if $[[N, a], [N, b]] = [N, [a, b]]$, for every $a, b \in M$.*

Proof. If N is a normal subultra-group then $[[N, a], [N, b]] = [N, [a, b]]$ for every $a, b \in M$ by Lemma 2.5 in [7]. Conversely, let $[[N, a], [N, b]] = [N, [a, b]]$, for $b = e$ we have $[N, a] = [N, [a, N]]$. So $\left[[a, N]^{a^{(-1)}}, a^{[-1]}\right] \in N$ and since $\mid \left[[a, N]^{a^{(-1)}}, a^{[-1]}\right] \mid = \mid N \mid$ we have $\left[[a, N]^{a^{(-1)}}, a^{[-1]}\right] = N$. Thus $\left[\left[[a, N]^{a^{(-1)}}, a^{[-1]}\right], a\right] = [N, a]$ and $\left[[a, N]^{\overbrace{a^{(-1)}(a^{[-1]}, a)}^{e}}, \underbrace{\left[a^{[-1]}, a\right]}_{e}\right]$ $= [N, a]$. So $[a, N] = [N, a]$ and $[a, [N, b]] = [a, [b, N]] = \left[\left[a^{(b,N)^{-1}}, b\right], N\right] =$ $\left[N, \left[\left[a^{(b,N)^{-1}}, b\right], N\right]\right] = [N, [a, [b, N]]] = \left[[N, a], \underbrace{[N, [b, N]]}_{[N,b]}\right] = [[N, a], [N, b]]$. So $[a, [N, b]] = [N, [a, b]]$, hence we get the result. □

Definition 1.7. [7] Let ${}_{H_i}M_i$ be an ultra-group of a subgroup H_i over group G_i, $i = 1, 2$ and φ be a group homomorphism between two subgroups H_1 and H_2. A function f: ${}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an *ultra-group homomorphism* provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$:

(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,

(ii) $(f(m))^{\varphi(h)} = f(m^h)$.

Theorem 1.8. (First isomorphism theorem) [7] *Let f be a surjective ultra-group homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ and θ a congruence over ${}_{H_1}M_1$ such that $\theta \subseteq Kerf$. If $\pi : {}_{H_1}M_1 \longrightarrow {}_{H_1}M_1/\theta$ is a canonical homomorphism then there exists a homomorphism $g\colon M_1/\theta \to M_2$ satisfying $g\pi = f$.*

Theorem 1.9. (Second isomorphism theorem)[7] *If N, N' are normal subultra-groups of an ultra-group M such that $N \subseteq N'$, then*

$$\frac{\frac{M}{N}}{\frac{N'}{N}} \cong \frac{M}{N'}.$$

Theorem 1.10. (Third isomorphism theorem) [7] *If K is a subultra-group of an ultra-group M and N is a normal subultra-group of M, then*

$$\frac{K}{K \cap N} \cong \frac{[N, K]}{N}.$$

2. Solvable ultra-groups

First, we present some definition similar that to what we have in group theory and refer the readers to [4, 6].

Definition 2.1. A sequence $M_0, M_1, \ldots, M_n$ of subultra-groups of M is called *subnormal series* if $M_n \lhd \ldots \lhd M_1 \lhd M_0 = M$. If all M_i are normal in M, then the series is called *normal*.

Every ultra-group M has normal series $\{e\} \lhd M$. A subnormal series of ultra-groups need not be normal. Let $D_8 = \langle a, b \mid a^4 = b^2 = e, (ab)^2 = e \rangle$ and $H = \{e\}$. The series $D_8 > \{e, b, a^2, a^2b\} > \{e, b\} > \{e\}$ is subnormal but it is not normal, since $\{e, b\} \ntriangleleft D_8$.

Definition 2.2. Let $M = M_0 > M_1 > \ldots > M_n$ be a subnormal series of ultra-groups. Each series

$M = M_0 > M_1 > \ldots > M_i > N > M_{i+1} > \ldots > M_n$ or

$M = M_0 > M_1 > \ldots > M_i > \ldots > M_n > N$

is called a *one-step refinement* of this series if N is a normal subultra-group of M_i and if $i < n$, M_{i+1} is normal in N. A refinement of a subnormal series is subnormal series that obtained from the finite number of one-step refinement.

Definition 2.3. An ultra-group M is called *simple* if it has just the normal subultra-group $\{e\}$. A subnormal series $M = M_0 > M_1 > \ldots > M_n = \{e\}$ of an ultra-group is called a *composition series* if each quotient ultra-group $\frac{M_i}{M_{i+1}}$ is simple for every $0 \leqslant i \leqslant n-1$.

Definition 2.4. A subnormal series $M = M_0 > M_1 > \ldots > M_n = \{e\}$ of ultra-group M is called a *solvable series* if each factor $\frac{M_i}{M_{i+1}}$ is abelian.

Definition 2.5. An ultra-group M is called *solvable* if it has a subnormal series $M = M_0 > M_1 > \ldots > M_n = \{e\}$ such that M_{i+1} is normal in M_i for every $0 \leqslant i \leqslant n-1$ and $\frac{M_i}{M_{i+1}}$ is an abelian ultra-group.

Theorem 2.6. *Every subultra-group and every quotient ultra-group of a solvable ultra-group is solvable.*

Proof. The proof of the first part is similar to groups and we omit it. Now let N is a normal subultra-group of a solvable ultra-group M. Hence M has a solvable series as follow $M = M_0 > M_1 > ... > M_n = \{e\}$. Since N is a normal subultra-group, $[M,N]$ is a subultra-group of M and $N \subseteq [M_i, N]$ for every $0 \leqslant i \leqslant n$. Now consider the series:

$$\frac{M}{N} = \frac{[M,N]}{N} > \frac{[M_1,N]}{N} > ... > \frac{[M_n,N]}{N} = N$$

such that every M_i is normal in M_{i-1}. We have:

$$\begin{aligned}\left[[N,m_i], \left[[N,m_{i-1}], [N,m'_{i-1}]\right]\right] &= \left[[N,m_i], \left[N, \left[m_{i-1}, m'_{i-1}\right]\right]\right] \\ &= \left[N, \left[m_i, \left[m_{i-1}, m'_{i-1}\right]\right]\right] \\ &= \left[N, \left[m_{i-1}, \left[m_i, m'_{i-1}\right]\right]\right] \\ &= \left[[N,m_{i-1}], \left[N, \left[m_i, m'_{i-1}\right]\right]\right] \\ &= \left[[N,m_{i-1}], \left[[N,m_i], [N,m'_{i-1}]\right]\right].\end{aligned}$$

Therefore $\frac{[M_i,N]}{N}$ is normal in $\frac{[M_{i-1},N]}{N}$.
Now, by Second and Third isomorphism theorems, for ultra-groups:

$$\frac{\frac{[M_{i-1},N]}{N}}{\frac{[M_i,N]}{N}} \simeq \frac{[M_{i-1},N]}{[M_i,N]} = \frac{[M_{i-1},[M_i,N]]}{[M_i,N]} \simeq \frac{M_{i-1}}{M_{i-1}\cap[M_i,N]}$$

$$\simeq \frac{\frac{M_{i-1}}{M_i}}{\frac{M_{i-1}\cap[M_i,N]}{M_i}}.$$

Since every $\frac{M_{i-1}}{M_i}$ is abelian and $\frac{M_{i-1}\cap[M_i,N]}{M_i}$ is a normal subultra-group of $\frac{M_{i-1}}{M_i}$ we see that $\frac{\frac{M_{i-1}}{M_i}}{\frac{M_{i-1}\cap[M_i,N]}{M_i}}$ is abelian. Hence, every $\frac{\frac{[M_{i-1},N]}{N}}{\frac{[M_i,N]}{N}}$ is abelian and $\frac{M}{N}$ is solvable. □

Theorem 2.7. *Let N be a normal subultra-group of an ultra-group M. If N and $\frac{M}{N}$ are solvable, then M is solvable.*

Proof. It is similar to what we have for groups. □

Theorem 2.8. *Let K and N be normal and solvable subultra-groups of M. Then $[K,N]$ is a solvable subultra-group of M.*

Proof. We have $\frac{[K,N]}{N} \simeq \frac{K}{K\cap N}$ by Third isomorphism theorem for ultra-groups. Since K is solvable, therefore the quotient $\frac{K}{K\cap N}$ is solvable by Theorem 2.6 and consequently $\frac{[K,N]}{N}$ is solvable. On the other hand N and $\frac{[K,N]}{N}$ are solvable, therefore $[K,N]$ is solvable by Theorem 2.7. □

In [7] Moghaddasi et al. proved that if N is a normal subultra-group of an ultra-group M, then every subultra-group of $\frac{M}{N}$ is of the form $\frac{K}{N}$, where K is a subultra-group of M containing N. On the other hand, $\frac{K}{N}$ is a normal subultra-group of $\frac{M}{N}$ if and only if K is a normal subultra-group of M. Thus, when $M \neq N$, $\frac{M}{N}$ is simple if and only if N is maximal in the set of all normal subultra-groups L of M with $L \neq M$.

Theorem 2.9. *Let M be an ultra-group of a subgroup H over a group G. Then*

(i) *Each finite ultra-group M has composition series.*

(ii) *Each refinement of a solvable series of ultra-group is a solvable series.*

(iii) *A subnormal series of an ultra-group M is a composition series if and only if it has no proper refinements.*

Proof. (i). Let M_1 be a maximal normal subultra-group of M, then $\dfrac{M}{M_1}$ is simple. Let M_2 be a maximal normal subultra-group of M_1 and so on. Since M is finite, this process must be stopped. Let $M_n = \{e\}$, then $M = M_0 > M_1 > \ldots > M_n = \{e\}$ is a composition series.

(ii). If $\dfrac{M_i}{M_{i+1}}$ is abelian and $M_{i+1} \lhd M' \lhd M_i$, then $\dfrac{M'}{M_{i+1}}$ is abelian since it is a subultra-group of an ultra-group $\dfrac{M_i}{M_{i+1}}$ and an ultra-group $\dfrac{M_i}{M'}$ is abelian since it is isomorphic to the quotient ultra-group $\dfrac{\frac{M_i}{M_{i+1}}}{\frac{M'}{M_{i+1}}}$ by the Second isomorphism theorem.

(iii). Let $M = M_0 > M_1 > \ldots > M_n = \{e\}$ be a composition series. Let $M = M_0 > M_1 > \ldots > M_i > M' > M_{i+1} > \ldots > M_n = \{e\}$ be a refinement of this series. Since $M_{i+1} \lhd M' \lhd M_i$, then $\dfrac{M'}{M_{i+1}}$ is a normal subultra-group of $\dfrac{M_i}{M_{i+1}}$ and every proper normal subultra-group of $\dfrac{M_i}{M_{i+1}}$ has this form. Hence the result is obtained from the fact that in this case $\dfrac{M_i}{M_{i+1}}$ is not simple and therefore is not composition series. Conversely if

$$M = M_0 > M_1 > \ldots > M_n = \{e\} \tag{1}$$

is a subnormal series then it has no proper refinement. Suppose this series is not composition series. Thus there exist a subultra-group M_i such that M_i is not maximal subultra-group of M_{i-1} and therefore there exist a subultra-group M_j such that $M_{i-1} \neq M_j \neq M_i$ and M_j a normal subultra-group of M_{i-1} and M_i is a normal subultra-group of M_j. This is a proper refinement of this series. A contradiction. Thus (1) is a composition series. □

Definition 2.10. Let M be an ultra-group. Two subnormal series S and T are called equivalent if there is a one to one correspondence between their factors such that corresponding factors are isomorphic ultra-groups.

Lemma 2.11. *Let S be a composition series of an ultra-group M, then any refinement of S is equivalent to S.*

Proof. According to Theorem 2.9, the proof is similar for groups. □

Before we prove the Zassenhaus Lemma, we need to establish some necessary lemmas.

Lemma 2.12. *Let M be an ultra-group of a subgroup H of a group G.*

(i) *For every $a, b, c \in M$ and every subultra-group K of M if $[a, b] = c$ and $a, c \in K$ then $b \in K$.*

(ii) *For every $a, b, c \in M$ and every subultra-group K of M if $[a, b] = c$ and $b, c \in K$ then $a \in K$.*

Proof. (i). Let $[a, b] = c$ be such that $a, c \in K$. Therefore
$\left[(a^{[-1]})^{(a,b)^{-1}}, [a, b]\right] = \left[\underbrace{(a^{[-1]})^{(a,b)^{-1}}}_{\in K}, c\right] = k \in K$, hence $\left[\left[a^{[-1]}, a\right], b\right] = k$.
Thus $b = k \in K$.
(ii) Let $[a, b] = c$ be such that $b, c \in K$. Therefore

$$\begin{aligned}
[a, b]^{b(-1)} = c^{b(-1)} &\Rightarrow \left[a^{{}^b b(-1)}, b^{b(-1)}\right] = c^{b(-1)} \\
&\Rightarrow \left[\left[a^{{}^b b(-1)}, b^{b(-1)}\right], b^{[-1]}\right] = \left[c^{b(-1)}, b^{[-1]}\right] \\
&\Rightarrow \left[a^{({}^b b(-1))(b^{b(-1)}, b^{[-1]})}, \left[b^{b^{(-1)}}, b^{[-1]}\right]\right] = \left[c^{b(-1)}, b^{[-1]}\right] \\
&\Rightarrow a^{({}^b b(-1))(b^{b(-1)}, b^{[-1]})} = \left[c^{b(-1)}, b^{[-1]}\right]
\end{aligned}$$

Now from $\left[c^{b(-1)}, b^{[-1]}\right] \in K$ and $({}^b b^{(-1)})(b^{b(-1)}, b^{[-1]}) = e$, $\left[b^{b(-1)}, b^{[-1]}\right] = e$, $a^e = a$ hence $a \in K$. □

Theorem 2.13. *Let N, K be two subultra-groups of an ultra-group M and $N \lhd M$. Then $N \cap K \lhd K$.*

Proof. Let $a, b \in K$ and $x \in [a, [N \cap K, b]]$, thus there exists $c \in N \cap K$ such that $x = [a, [c, b]]$. Since $N \lhd M$, there exists $c_1 \in N$ such that $x = [a, [c, b]] = [c_1, [a, b]]$. As K is a subultra-group, there exists $m \in K$

such that $x = [m, [a, b]]$. Now from $x = [m, [a, b]] = [c_1, [a, b]]$ and the right cancellation for a binary operation, $m = c_1$ hence, $c_1 \in N \cap K$. Therefore $[a, [N \cap K, b]] \subseteq [N \cap K, [a, b]]$.

Now suppose $a, b \in K$ and $x \in [N \cap K, [a, b]]$. Thus there exists $c \in N \cap K$ such that $x = [c, [a, b]]$. Since $N \lhd M$, there exists $c_1 \in N$ such that $x = [c, [a, b]] = [a, [c_1, b]]$. By the above Lemma (i) and $x, a, b \in K$, we have $c_1 \in K$ and so $c_1 \in N \cap K$. Thus $[N \cap K, [a, b]] \subseteq [a, [N \cap K, b]]$. Hence the assertion follows. □

Lemma 2.14. *Let N, K be subultra-groups of an ultra-group M. Then*

$$[N \cup K, N \cup K] = [N, K] \cup [K, N] .$$

Proof. Assume that $x \in [N \cup K, N \cup K]$, then $x \in [K, N]$ or $[N, K]$ or $[K, K] = K$ or $[N, N] = N$. Considering $N, K \subseteq [N, K], [K, N]$, thus $x \in [N, K] \cup [K, N]$. Therefore $[N \cup K, N \cup K] \subseteq [N, K] \cup [K, N]$. Conversely, it is clear that $[N, K], [K, N] \subseteq [N \cup K, N \cup K]$. Thus $[N, K] \cup [K, N] \subseteq [N \cup K, N \cup K]$. □

Notation 2.15. For subultra-groups N, K of an ultra-group M we denote

$$N \overset{n}{\cup} K = [N \cup K, N \cup K]^n$$

In particular, for $n = 2$, $N \overset{2}{\cup} K = [[N \cup K, N \cup K], N \cup K]$ and for $n = 3$, $N \overset{3}{\cup} K = [[[N \cup K, N \cup K], N \cup K], N \cup K]$.

Lemma 2.16. *If N, K are two subultra-groups of an ultra-group M, then $N \overset{n}{\cup} K$ is a subultra-group of M.*

Proof. $N \overset{n}{\cup} K = [N \cup K, N \cup K]^n$. Thus by Lemma 2.14

$$\begin{aligned} [N \cup K, N \cup K]^n &= [\ldots[[[N \cup K, N \cup K], N \cup K], \ldots], N \cup K] \\ &= \ldots = [N, K] \cup [K, N] \cup [[N, K], N] \cup [[K, N], K] . \end{aligned}$$

Therefore $N \overset{n}{\cup} K = [N, K] \cup [K, N] \cup [[N, K], N] \cup [[K, N], K]$. Clearly $e \in [N \cup K, N \cup K]^n$. Let $x, y \in [N \cup K, N \cup K]^n$. Then there exist $n_1, n_2 \in N$ such that $x = [N \cup K, N \cup K]^{n_1}$ and $y = [N \cup K, N \cup K]^{n_2}$. Also there exist $\alpha_i, \beta_i, \gamma_i \in N \cup K$ for $i = 1, 2$ such that $x = [[\ldots[\alpha_1, \beta_1], \gamma_1], \ldots]$ and $y = [[\ldots[\alpha_2, \beta_2], \gamma_2], \ldots]$. By considering the fact that $\alpha_i, \beta_i, \gamma_i$ are in $N \cup K$ we conclude they are in N or K thus for every $h \in H$, $(\alpha_i)^h, (\beta_i)^h, (\gamma_i)^h$ are in N or K and therefore in $N \overset{n}{\cup} K$ that is $[x, y] \in N \overset{n}{\cup} K$. Also, by this fact $N \overset{n}{\cup} K$ is closed with operation β. Thus, $N \overset{n}{\cup} K$ is a subultra-group of M. □

Lemma 2.17. *If N, K are two normal subultra-groups of M, then for every $n \in N$, $N \overset{n}{\cup} K = [N, K]$.*

Proof. Since $[N \cup K, N \cup K] = [N, N] \cup [N, K] \cup [K, K] \cup [K, N] = [N, K] \cup [K, N]$ and also since N, K are normal subultra-groups we have $[N, K] = [K, N]$ hence $N \overset{n}{\cup} K = [N, K]$. □

The join of two ultra-groups N, K is denoted by $N \vee K$.

Lemma 2.18. *Let N, K be two subultra-groups of ultra-group M of a subgroup H over a group G. Then $N \vee K = N \overset{n}{\cup} K$.*

Proof. If $K = \{e\}$, then for every $n \in N$, $[N \cup K, N \cup K]^n = N$ then $N \subseteq [N \cup K, N \cup K]^n$. Similarly $K \subseteq [N \cup K, N \cup K]^n$, thus $[N \cup K, N \cup K]^n$ is an upper bound for N, K. If C is an upper bound for N, K then $[N, K] \subseteq C$, $[K, N] \subseteq C$. Therefore $[[N, K], N] \subseteq C$ and $[[K, N], K] \subseteq C$. Consequently $[N, K] \cup [K, N] \cup [[N, K], N] \cup [[K, N], K] \subseteq C$ and hence by Lemma 2.16, $N \overset{n}{\cup} K \subseteq C$. □

Theorem 2.19. *If N, K are two subultra-groups of M such that $N \lhd M$, then $N \vee K = [N, K]$.*

Proof. Since $N \lhd M$ therefore for every $x \in M$ we have, $[N, x] = [x, N]$. Hence $[N, K] = [K, N]$ and $[N \cup K, N \cup K]^n = [N, K]^n = [N, K]$. Thus $N \vee K = [N, K]$. □

Now by Lemma 2.5 in [7] and above theorem we have:

Proposition 2.20. *If N, K are normal subultra-groups of M then $N \vee K \lhd M$.*

The next lemma that we consider it as Zassenhaus Lemma is quite technical. Its value will be immediately apparent in the proof of Theorem 2.22.

Lemma 2.21. *Let K^*, N^*, N, K be subultra-groups of an ultra-group M such that K^* is normal in K and N^* is normal in N. Then*

(i) $[N^*, N \cap K^*] \lhd [N^*, N \cap K]$,

(ii) $[K^*, N^* \cap K] \lhd [K^*, N \cap K]$,

(iii) $\dfrac{[N^*, N \cap K]}{[N^*, N \cap K^*]} \cong \dfrac{[K^*, N \cap K]}{[K^*, N^* \cap K]}$.

Proof. Since K^* is normal in K, $N \cap K^* = (N \cap K) \cap K^*$ is normal in $N \cap K$ (Theorem 2.13). Similarly, $N^* \cap K$ is normal in $N \cap K$. Consequently, by Theorem 2.19 and Proposition 2.20, $D = [N^* \cap K, K^* \cap N]$ is normal in $N \cap K$. We define

$$f : [N^*, N \cap K] \longrightarrow \tfrac{N \cap K}{D}, \quad f([a, c]) = [D, c]\,.$$

f is well define because if $[a, c] = [b, d]$ then

$$\left[(a^{[-1]})^{(a,c)}, [a, c]\right] = \left[(a^{[-1]})^{(a,c)}, [b, d]\right]$$
$$\left[\underbrace{\left[a^{[-1]}, a\right]}_{e}, c\right] = \left[\underbrace{(a^{[-1]})^{(a,c)}}_{a_2}, [b, d]\right].$$

Therefore $c^{d^{(-1)}} = [a_2, [b, d]]^{d^{(-1)}}$

$$\Longrightarrow c^{d^{(-1)}} = \left[\underbrace{a_2^{[b,d]d^{(-1)}}}_{a_3}, [b, d]^{d^{(-1)}}\right] = \left[a_3, \left[b^{d(d^{(-1)})}, d^{d^{(-1)}}\right]\right]$$

$$\Longrightarrow \left[c^{d^{(-1)}}, d^{[-1]}\right] = \left[\left[\underbrace{\left[a_3^{(b^{d(d^{(-1)})}, d^{d^{(-1)}})^{-1}}, b^{d(d^{(-1)})}\right]}_{a_4}, d^{d^{(-1)}}\right], d^{[-1]}\right]$$

$$\Longrightarrow \left[c^{d^{(-1)}}, d^{[-1]}\right] = \left[a_4, \left[\underbrace{d^{d^{(-1)}}, d^{[-1]}}_{e}\right]\right]$$

$$\Longrightarrow \underbrace{\left[c^{d^{(-1)}}, d^{[-1]}\right]}_{\in N \cap K} = a_4 \in N^*$$

$$\Longrightarrow \left[c^{d^{(-1)}}, d^{[-1]}\right] \in N \cap K \cap N^* = N^* \cap K \subseteq D = [N^* \cap K, K^* \cap N]$$

$$\Longrightarrow \left[c^{d^{(-1)}}, d^{[-1]}\right] \in D \Longrightarrow [D, c] = [D, d] \quad \text{by ([7], Lemma 2.1)}$$

The map f is an ultra-groups homomorphism since

$$\begin{aligned} f\left[[a,c],[b,d]\right] &= f\left[\underbrace{a^{(c,[b,d])}}_{a_1},[c,[b,d]]\right] \xlongequal{b\in N^*\lhd N,\exists b_1\in N^*} f\left[a_1,[b_1,[c,d]]\right] \\ &= f\left[\underbrace{\left[a_1^{(b_1,[c,d])},b_1\right]}_{\in N^*},\underbrace{[c,d]}_{\in N\cap K}\right] = [D,[c,d]] = [[D,c],[D,d]] \\ &= [f[a,c],f[b,d]]. \end{aligned}$$

Also for every $m=[a,c]\in[N^*,N\cap K]$ and $h\in H$ we have $f(m^h)=f[a,c]^h=f\left[a^{c^h},c^h\right]=\left[D,c^h\right]=[D,c]^h=[D,c]^{id(h)}=(f(m))^{id(h)}$ where $id:H\to H$ is a group homomorphism. This implies that f is an ultra-group homomorphism. The map f is clearly surjective. If $[a,c]\in ker(f)$ then, $f([a,c])=[D,c]=D$. Now by ([7], Lemma 2.5), since D is normal in $N\cap K$, it follows that $c\in D$ that is $c=[a_1,c_1]$ where $a_1\in N^*\cap K$ and $c_1\in N\cap K^*$. Therefore

$$[a,c]=[a,[a_1,c_1]]=[[a^{(a_1,c_1)^{-1}},a_1],c_1]\in[N^*,N\cap K^*].$$

So $ker(f)=[N^*,(N\cap K^*)]$. Thus $[N^*,N\cap K^*]\lhd[N^*,N\cap K]$. A symmetric argument shows that $[K^*,N^*\cap K]$ is normal in $[K^*,N\cap K]$. By the First isomorphism theorem for ultra-groups we have

$$\frac{[N^*,N\cap K]}{[N^*,N\cap K^*]}\cong\frac{N\cap K}{D}=\frac{N\cap K}{[N^*\cap K,K\cap N^*]}$$

which completes the proof. □

Theorem 2.22. *Let M be an ultra-group. Any two subnormal (normal) series of M have subnormal (normal) refinements that are equivalent.*

Proof. Let $M=M_0>M_1>\dots M_n$ and $M=N_0>N_1>\dots N_m$ be a subnormal (resp. normal) series. Let $M_{n+1}=\{e\}=N_{m+1}$ and for every $0\leqslant i\leqslant n$ consider $M_i=[M_{i+1},M_i\cap N_0]>[M_{i+1},M_i\cap N_1]>\dots[M_{i+1},M_i\cap N_j]>[M_{i+1},M_i\cap N_{j+1}]>\dots>[M_{i+1},M_i\cap N_m]>[M_{i+1},M_i\cap N_{m+1}]=M_{i+1}$ for every $0\leqslant j\leqslant m$. The Zassenhaus Lemma applied to M_{i+1},M_i,N_{j+1},N_j shows that $[M_{i+1},M_i\cap N_{j+1}]$ is normal in $[M_{i+1},M_i\cap N_j]$. Inserting these ultra-groups between every M_i and M_{i+1}, which we denoting $[M_{i+1},M_i\cap N_j]$ by $M(i,j)$. Therefore gives a subnormal refinement of the series $M=M_0>M_1>\dots M_n$:
$M=M(0,0)>M(0,1)>\dots>M(0,m)>M(1,0)>M(1,1)>$

$M(1,2) > \ldots > M(1,m) > M(2,0) > \ldots > M(n-1,m) > M(n,0) > \ldots > M(n,m)$ where $M(i,0) = M_i$. Now this refinement has $(n+1)(m+1)$ (not necessarily distinct) terms. A symmetric argument shows that there is a refinement of $M = N_0 > N_1 > \ldots N_m$ (where $N(i,j) = [N_{j+1}, M_i \cap N_j]$ and $N(0,j) = N_j$) as follows:

$$M = N(0,0) > N(1,0) > \ldots > N(n,0) > N(0,1) > N(1,1) > \ldots > N(n,1) > N(0,2) > \ldots > N(n,m-1) > N(0,m) > \ldots > N(n,m).$$

This refinement also has $(n+1)(m+1)$ terms. For every pair (i,j) where $0 \leqslant i \leqslant n, 0 \leqslant j \leqslant m$. There is by Zassenhaus Lemma (applied to $M_{i+1}, M_i, N_{j+1}, N_j$) an isomorphism

$$\frac{M(i,j)}{M(i,j+1)} = \frac{[M_{i+1}, M_i \cap N_j]}{[M_{i+1}, M_i \cap N_{j+1}]} \cong \frac{[N_{j+1}, M_i \cap N_j]}{[N_{j+1}, M_{i+1} \cap N_j]} = \frac{N(i,j)}{N(i+1,j)}.$$

This completes the proof. □

We close this section by the following theorem which gives the Jordan-Hölder Theorem for ultra-groups.

Theorem 2.23. *Any two composition series of an ultra-group M are equivalent. Therefore, every ultra-group having a composition series determines a unique list of simple ultra-groups.*

Proof. The proof follows from Lemma 2.11 and Theorem 2.22. □

3. Nilpotent ultra-groups

In this section, firstly we define the center of an ultra-group and the upper central series. Next, we describe nilpotent ultra-groups and define commutators to construct the drived series. Finally, we present some results for solvable and nilpotent ultra-groups.

Definition 3.1. Let M be an ultra-group of a subgroup H of a group G. The *center* of ultra-group M is defined as

$$Z(M) = \left\{ z \in M \mid z^h = z, [z,[a,b]] = [a,[z,b]] \text{, for every } a,b \in M, h \in H \right\}.$$

Lemma 3.2. *$Z(M)$ is a normal subultra-group of an ultra-group M.*

Proof. Clearly $e \in Z(M)$. Let $z_1, z_2 \in Z(M)$, we have $[z_i,[a,b]] = [a,[z_i,b]]$ for $i = 1,2$. Consequently, $[[z_1,z_2],[a,b]] = [z_1,[z_2,[a,b]]] = [z_1,[a,[z_2,b]]] =$

$[a, [z_1, [z_2, b]]] = [a, [[z_1, z_2], b]]$ and $[z_1, z_2]^h = [z_1, z_2]$, thus $Z(M)$ is a subultra-group of M. Also $[Z(M), [a, b]] = [a, [Z(M), b]]$ for every $a, b \in M$, therefore $Z(M)$ is a normal subultra-group of M. □

Let M be an ultra-group of a subgroup H of a group G and let $Z(M)$ be the center of M and $\pi : M \longrightarrow \frac{M}{Z(M)}$ be the canonical epimorphism. Since $Z(\frac{M}{Z(M)})$ is a normal subultra-group of $\frac{M}{Z(M)}$, by the corresponding theorem for ultra-groups (see[7]) we have $\pi^{-1}(Z(\frac{M}{Z(M)})) \lhd M$. Hence, by induction, $Z_1(M) = Z(M)$ and $Z_i(M) = \pi^{-1}(Z(\frac{M}{Z_{i-1}(M)}))$ for $i > 1$ where $\pi : M_i \longrightarrow \frac{M_i}{Z(M)}$. Therefore we obtain the sequence

$$\{e\} = Z_0(M) < Z_1(M) < Z_2(M) < \dots$$

of normal subultra-groups of M, which is called the *upper central series* of an ultra-group M.

Definition 3.3. An ultra-group M is called a *nilpotent ultra-group* if there exists a natural number n such that $Z_n(M) = M$.

By the definition of the product of a family $(f_i)_{i \in I}$ of morphisms in each category (see Proposition 1.7 in [5] and D efinition 10.34 in [1]) , we see if $f_i : M_i \to M'_i$ is a family of ultra-group homomorphisms then $f = \Pi f_i : \Pi M_i \to \Pi M'_i$ is an ultra-group homomorphism furthermore $ker(f) = \Pi ker(f_i)$.

Lemma 3.4. *Let $\{M_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be a family of ultra-groups such that for every $i \in I$, N_i is normal subultra-group of M_i. Then ΠN_i is a normal subultra-group of ΠM_i and $\frac{\Pi M_i}{\Pi N_i} \cong \Pi \frac{M_i}{N_i}$.*

Proof. Let $\pi_i : M_i \to \frac{M_i}{N_i}$ be the canonical epimorphism. By the above paragraph $\Pi \pi_i : \Pi M_i \to \Pi \frac{M_i}{N_i}$ is an epimorphism with the kernel ΠN_i. Now by First isomorphism theorem for ultra-groups $\frac{\Pi M_i}{\Pi N_i} \cong \Pi \frac{M_i}{N_i}$. □

Theorem 3.5. *Direct product of finite nilpotent ultra-groups is nilpotent.*

Proof. Let M_i be ultra-groups over the group G_i; $i = 1, 2$. This is sufficient to prove this result for the direct product of two ultra-groups. The proof for most factors is similar. Let $M = M_1 \times M_2$. We prove by induction $Z_i(M) = Z_i(M_1) \times Z_i(M_2)$. The proof for $i = 1$ is clear. Suppose π_{M_j} is the canonical epimorphism $M_j \longrightarrow \frac{M_j}{Z_i(M_j)}$ then π is the canonical epimorphism from $M = M_1 \times M_2$ onto $\frac{M_1}{Z_i(M_1)} \times \frac{M_2}{Z_i(M_2)}$ by the above paragraph. Now consider the ultra-group homomorphism ψ from $\frac{M_1}{Z_i(M_1)} \times \frac{M_2}{Z_i(M_2)}$ to $\frac{M_1 \times M_2}{Z_i(M_1) \times Z_i(M_2)}$ that it is equal to $\frac{M_1 \times M_2}{Z_i(M_1 \times M_2)} = \frac{M}{Z_i(M)}$ by Lemma 3.4. Consider an ultra-group epimorphism $\varphi : M \longrightarrow \frac{M}{Z_i(M)}$ as a composition of two ultra-group homomorphisms ψ and π. As a result,

$$\begin{aligned} Z_{i+1}(M) = \varphi^{-1}(Z(\frac{M}{Z_i(M)})) &= \pi^{-1}\psi^{-1}(Z(\frac{M}{Z_i(M)})) \\ &= \pi^{-1}(Z(\frac{M_1}{Z_i(M_1)} \times \frac{M_2}{Z_i(M_2)})) \\ &= \pi^{-1}(Z(\frac{M_1}{Z_i(M_1)}) \times Z(\frac{M_2}{Z_i(M_2)})) \\ &= \pi_{M_1}^{-1}(Z(\frac{M_1}{Z_i(M_1)})) \times \pi_{M_2}^{-1}(Z(\frac{M_2}{Z_i(M_2)})) \\ &= Z_{i+1}(M_1) \times Z_{i+1}(M_2). \end{aligned}$$

Thus for every i, $Z_i(M) = Z_i(M_1) \times Z_i(M_2)$. Since M_1, M_2 are nilpotent, there exists $n \in \mathbb{N}$ such that $Z_n(M_1) = M_1$ and $Z_n(M_2) = M_2$. Due to this $Z_n(M) = M_1 \times M_2 = M$ and thus M is nilpotent. □

Let M be an ultra-group of subgroup H over a group G. We define the *commutator* of M as the subultra-group generated by the set $\left\{ \left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]} \right] \mid a,b \in M \right\}$ and denoted it by M'. The element $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]} \right]$ is called commutator of a, b and denoted by $\left[\widehat{a,b} \right]$. Now let C be the subultra-group generated by the commutators elements of M. We show that C is a normal subultra-group. First for every $a, b \in M$

$$\begin{aligned} \left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]} \right] &= c_1 \in C \\ \Rightarrow \left[\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]} \right], [b,a] \right] &= [c_1, [b,a]] \\ \Rightarrow \left[[a,b]^{([b,a]^{(-1)})^{([b,a]^{[-1]},[b,a])}}, \left[[b,a]^{[-1]}, [b,a] \right] \right] &= [c_1, [b,a]] \\ \Rightarrow [a,b] = [c_1, [b,a]] \,. \quad (*) \end{aligned}$$

So for every $a, b \in M$ there exists $c_1 \in C$ such that $[a,b] = [c_1, [b,a]]$. Now, using Theorem 1.6, we will prove that C is a normal subultra-group of M. Note that for every $x, y \in M$ and by $(*)$ we have $[c_1, [x,y]] = [c_1, [c_2, [y,x]]] = \left[\left[c_1^h, c_2 \right], [y,x] \right] = [c_3, [y,x]]$, where $h = (c_2, [y,x])^{-1}$, $c_3 =$

$\left[c_1^h, c_2\right]$, thus $[C,[x,y]] \subseteq [C,[y,x]]$. The inverse of this equation is proved in the same way. Therefore

$$[C,[x,y]] = [C,[y,x]]. \quad (**)$$

For every $x, y \in_H M$ and every $c_1, c_2 \in C$:

$$\begin{aligned}
[[c_1,x],[c_2,y]] &= \left[c_1^h, [x,[c_2,y]]\right] && h = (x,[c_1,y]) \\
&= \left[c_1^h, [c_3,[[c_2,y],x]]\right] && by(*) \\
&= \left[\left[(c_1^h)^{(c_3,[[c_2,y],x])^{-1}}, c_3\right], [[c_2,y],x]\right] \\
&= [c_4,[[c_2,y],x]] && c_4 = \left[(c_1^h)^{(c_3,[[c_2,y],x])^{-1}}, c_3\right] \\
&= \left[c_4, \left[c_2^{(y,x)}, [y,x]\right]\right] \\
&= [c_5,[y,x]] && c_5 = \left[c_4^{(c_2^{(y,x)},[y,x])^{-1}}, c_2^{(y,x)}\right]
\end{aligned}$$

Therefore $[[c_1,x],[c_2,y]] \subseteq [c_5,[y,x]]$ and hence $[[C,x],[C,y]] \subseteq [C,[y,x]]$. Now by $(**)$, $[c_1,[x,y]] = [c_2,[y,x]] = \left[\left[c_2^h, y\right], x\right]$ which $h = (y,x)^{-1}$. Then by $(*)$ we have, $\left[\left[c_2^h, y\right], x\right] = \left[c_3, \left[x, \left[c_2^h, y\right]\right]\right] = \left[\left[c_3^{h'}, x\right], \left[c_2^h, y\right]\right]$, where $h' = (x, \left[c_2^h, y\right])^{-1}$. Thus $[C,[y,x]] \subseteq [[C,x],[C,y]]$. These show that C is a normal subultra-group of M.

Theorem 3.6. *Let M be an ultra-group of a subgroup H over a group G. Then an ultra-group M is abelian if and only if $M' = \{e\}$.*

Proof. If ultra-group M is abelian then for every $a, b \in M$, $[a,b] = [b,a]$ so $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] = \left[\widehat{a,b}\right] = e$ by considering the fact for every $a \in M$ we have $\left[a^{a^{(-1)}}, a^{[-1]}\right] = e$. Conversely if for every $a, b \in M$ we have $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] = e$ then in view of Proposition 1.3 and the right cancellation law $[a,b]^{[b,a]^{(-1)}} = [b,a]^{[b,a]^{(-1)}}$ and thus $[a,b] = [b,a]$. □

Theorem 3.7. *Let M be an ultra-group of subgroup H over a group G and M' be the commutator subultra-group of M. Then $\frac{M}{M'}$ is abelian.*

Proof. For every $a, b \in M$, $\left[\widehat{a,b}\right] \in M'$ if and only if $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] \in M'$. Thus $[M',[a,b]] = [M',[b,a]]$. Since M' is a normal subultra-group of M, we have $[[M',a],[M',b]] = [[M',b],[M',a]]$, so $\frac{M}{M'}$ is abelian. □

Theorem 3.8. *Let M be an ultra-group of a subgroup H over a group G and N be a normal subultra-group of M. Then $\frac{M}{N}$ is abelian if and only if $M' < N$.*

Proof. Let $\frac{M}{N}$ be abelian ultra-group. It is sufficient to show that N contains the generators of ultra-group M'. For x belongs to the generators of M', there exist $a, b \in M$ such that $x = \left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right]$. On the other hand $\frac{M}{N}$ is abelian, therefore for every $a, b \in M$ we have $[[N,a], [N,b]] = [[N,b], [N,a]]$. Thus by normality of N we have, $[N, [a,b]] = [N, [b,a]]$ (by Theorem 1.6). So $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] \in N$. This implies that $x \in N$ and therefore $M' < N$.

Conversely if $M' < N$, then $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] \in M'$ for every $a, b \in M$. Since $M' < N$, $\left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] \in N$. Thus $[N, [a,b]] = [N, [b,a]]$. Since N is normal, $[[N,a], [N,b]] = [[N,b], [N,a]]$. □

Lemma 3.9. *If X is a generating set of an ultra-group M then M' is generated by the set of commutators of elements of X.*

Proof. Let K be a normal subultra-group generated by the commutators of elements of X. By definition of M' we have $K < M'$. On the other hand, the set $\frac{X}{K}$ generates the quotient ultra-group $\frac{M}{K}$. Now $[[K,x_1], [K,x_2]] = [[K,x_2], [K,x_1]]$ if and only if $\left[[x_1,x_2]^{[x_2,x_1]^{(-1)}}, [x_2,x_1]^{[-1]}\right] = \left[\widehat{x_1, x_2}\right] \in K$. So $\frac{M}{K}$ is abelian by Theorem 3.8 and hence $M' < K$. □

Let N and K be two normal subultra-groups of an ultra-group M. Then $\left[\widehat{N, K}\right]$ is a subultra-group of M generated by $\{\left[\widehat{n, k}\right] | n \in N, k \in K\}$.

Let M be an ultra-group of a subgroup H over the group G and let A, B be two normal subultra-group of M. Then for every $a, b \in M$ we have:

$$\begin{aligned} \left[\widehat{a, b}\right] &= \left[[a,b]^{[b,a]^{(-1)}}, [b,a]^{[-1]}\right] \\ &= \left[\left[a^{\left(b^{[b,a]^{(-1)}}\right)}, b^{[b,a]^{[-1]}}\right], \left[(a^{[-1]})^{b^{(-1)}}, b^{[-1]}\right]^{[b,a]}\right] \\ &= \left[\left[a^{h_1}, b^{h_2}\right], \left[(a^{[-1]})^{h_3}, (b^{[-1]})^{h_4}\right]\right] \\ &= \left[\left[\left[a^{h'_1}, b^{h'_2}\right], a^{[-1]^{h_3}}\right], (b^{[-1]})^{h_4}\right] \end{aligned} \tag{2}$$

$$= \left[\left[b', \left[a^{h'_1}, (a^{[-1]})^{h_3}\right]\right], (b^{[-1]})^{h_4}\right] \quad \text{by } B \lhd M$$
$$= \left[a', \left[b', (b^{[-1]})^{h_4}\right]\right] \in [A, B] \quad \text{by } A \lhd M$$

where $h_1 = ({}^{b}[b,a]^{(-1)})$, $h_2 = [b,a]^{[-1]}$, $h_3 = b^{(-1)}({}^{b^{(-1)}}[b,a])$, $h_4 = [b,a]$, $h'_1 = ((h_1)({}^{b^{h_2}}((a^{[-1]})^{h_3}, b^{[-1]^{h_4}})^{-1}))$ and $h'_2 = h_2((a^{[-1]})^{h_3}, (b^{[-1]})^{h_4})^{-1})$.

Now from Lemma **??** and the normality of B, $\left[b'', \left[b', \left[a^{h'_1}, (a^{[-1]})^{h_3}\right]\right]\right] = \left[b''', \left[a^{h'_1}, (a^{[-1]})^{h_3}\right]\right] \in [B, A]$. Thus $\left[\widehat{a,b}\right] \in [A,B] \cap [B,A]$. Since A, B are normal in an ultra-group M, $[A,B]$ and $[B,A]$ are normal subultra-groups of M.

Lemma 3.10. *Let N be a normal subultra-group of an ultra-group M such that every $n \in N$ commute with every $m \in M$ and for every $h \in H$, $n^h = n$. Then $\left[N, m^h\right] = [N, m]$.*

Proof. First for each $n \in N$ and $m_1, m_2 \in M$, $h \in H$ we have

$$[n, [m_1, m_2]] = [[n, m_1], m_2] = [[m_1, n], m_2] = \left[m_1^{(n,m_2)}, [n, m_2]\right].$$

On the other hand $[[m_1, m_2], n] = \left[m_1^{(m_2,n)}, [m_2, n]\right] = \left[m_1^{(m_2,n)}, [n, m_2]\right]$, thus $m_1^{(m_2,n)} = m_1^{(n,m_2)}$. So $m_1^{(n,m_2)(m_2,n)^{-1}} = m_1$. Now we can write

$$[n, [m_1, m_2]] = [[m_1, m_2], n] = \left[m_1^{(m_2,n)}, [m_2, n]\right] =$$
$$\left[\left[m_1^{(m_2,n)(n,m_2)^{-1}}, n\right], m_2\right] = [[m_1, n], m_2]. \qquad (i)$$

Also

$$[n, [m_1, m_2]] = [m_1, [n_1, m_2]] = \left[\left[m_1^{(n_1,m_2)^{-1}}, n_1\right], m_2\right]. \qquad (ii)$$

From $(i), (ii)$ we can deduced $[m_1, n] = \left[m_1^{(n_1,m_2)^{-1}}, n_1\right]$. Thus $m_1 = \left[\left[m_1^{(n_1,m_2)^{-1}}, n_1\right]^{n^{(-1)}}, n^{[-1]}\right]$. So $m_1 = \left[m_1^h, n'\right]$ where $n' = \left[n_1^{n^{(-1)}}, n^{[-1]}\right]$ and $h = (n_1, m_2)^{-1}({}^{n_1}n^{(-1)})(n^{n^{(-1)}, n^{[-1]}})$. Therefore there exist $n' \in N$ such that $m_1 = \left[m^h, n'\right] = \left[n', m^h\right]$ for every $n \in N$. So $m_1 \in \left[N, m_1^h\right]$ and $\left[N, m_1^h\right] = [N, m_1]$. ☐

Theorem 3.11. *Let N, K be normal subultra-groups of ultra-group M and $N < K$. Then $\frac{K}{N} < Z(\frac{M}{N})$ if and only if $\left[\widehat{K, M}\right] < N$.*

Proof. First we note that:
(1). For every $k \in K, h \in H$ we have $\left[(k^h)^{k^{(-1)}}, k^{[-1]}\right] \in N$. Thus $[N,k] = \left[N, k^h\right]$. Therefore for every $[N,k] \in \dfrac{K}{N}$, $[N,k]^h = \left[N^{k^h}, k^h\right] = [N,k]$.
(2). For every $k \in K$, $m \in M$ if $\left[\widehat{K,M}\right] < N$, then $\left[[k,m]^{[m,k]^{(-1)}}, [m,k]^{[-1]}\right] \in N$ and $[N,[k,m]] = [N,[m,k]]$. Hence $[[N,k],[N,m]] = [[N,m],[N,k]]$. So, $[N,k]$ commute with every $[N,m] \in \dfrac{M}{N}$.
(3). K is normal in M, thus $\dfrac{K}{N}$ is normal in $\dfrac{M}{N}$ since

$$\begin{aligned}[][[N,k],[[N,m_1],[N,m_2]]] &= [[N,k],[N,[m_1,m_2]]] \\ &= [N,[k,[m_1,m_2]]] \\ &= [N,[m_1,[k,m_2]]] \\ &= [[N,m_1],[[N,k],[N,m_2]]] .\end{aligned}$$

Now, by Lemma 3.10, for $\dfrac{N}{K}$ and for every $m_1, m_2 \in M$, we have

$$\begin{aligned}[][[N,k],[[N,m_1],[N,m_2]]] &= [[[N,m_1],[N,m_2]],[N,k]] \\ &= \left[[N,m_1]^{([N,m_2],[N,k])}, [[N,m_2],[N,k]]\right] \\ &= [[N,m_1],[[N,k],[N,m_2]]] .\end{aligned}$$

Thus $[N,k] \in Z(\dfrac{M}{N})$ for every $k \in K$.
Conversely, let $[N,k] < Z(\dfrac{M}{N})$. Hence for every $k \in K, m \in M$

$$\begin{aligned}[][[N,k],[[N,m],[N,e]]] &= [[N,m],[[N,k],[N,e]]] \\ [[N,k],[[N,m],N]] &= [[N,m],[[N,k],N]] \\ [[N,k],[N,m]] &= [[N,m],[N,k]] \\ [N,[k,m]] &= [N,[m,k]] .\end{aligned}$$

Thus for every $k \in K, m \in M$ we have, $\left[[k,m]^{[m,k]^{(-1)}}, [m,k]^{[-1]}\right] \in N$. So $\left[\widehat{K,M}\right] < N$. □

Now we give an equivalent characterization of nilpotent ultra-groups, namely by descending central series.

Let M be an ultra-group and

$$\gamma_1(M) = M, \quad \gamma_2(M) = \left[\widehat{\gamma_1(M), M}\right], \quad \gamma_i(M) = \left[\widehat{\gamma_{i-1}(M), M}\right].$$

Then the chain of normal subultra-groups $M = \gamma_1(M) > \gamma_2(M) > \ldots$ is called descending central series of M, that have the following properties: $\gamma_i(M) \triangleleft M$ for every i, and by the above lemma, $\frac{\gamma_i(M)}{\gamma_{i+1}(M)} < Z(\frac{M}{\gamma_{i+1}(M)})$ since $\left[\widehat{\gamma_i(M), M}\right] = \gamma_{i+1}(M)$.

Definition 3.12. A series $\{e\} = M_0 < M_1 < \ldots < M_n = M$ of an ultra-group M is called *central series* if for each i, $M_i \triangleleft M$ and $\frac{M_{i+1}}{M_i} < Z(\frac{M}{M_i})$.

Lemma 3.13. *If $\{e\} = M_0 < M_1 < \ldots < M_n = M$ is a central series of an ultra-group M, then*

(i) $\gamma_i(M) < M_{n-i+1}$,

(ii) $M_i < Z_i(M)$.

Proof. The proof is straightforward by induction on i. □

Theorem 3.14. *Let M be an ultra-group. Then M is nilpotent if and only if $\gamma_{n+1}(M) = \{e\}$ for some integer $n \geqslant 0$.*

Proof. Assume that there is an integer $n \geqslant 0$ such that $\gamma_{n+1}(M) = \{e\}$. Consider the series

$$M = \gamma_1(M) > \gamma_2(M) > \ldots > \gamma_n(M) > \gamma_{n+1}(M) = \{e\}.$$

In this series, $\frac{\gamma_i(M)}{\gamma_{i+1}(M)} < Z(\frac{M}{\gamma_{i+1}(M)})$ and $\gamma_{n+1-i}(M) < Z_i(M)$ for all $i = 0, 1, \ldots, n$ (Lemma 3.13). Therefore $M = \gamma_1(M) < Z_n(M)$, so M is nilpotent.

Conversely, if M is nilpotent then there exists $n \geqslant 1$ such that $Z_n(M) = M$. Therefore we have a series of normal subultra-groups

$$\{e\} = Z_0(M) < Z_1(M) < Z_2(M) < \ldots < Z_n(M) = M.$$

Then it follows that $\gamma_i(M) < Z_{n+1-i}(M)$ for all i and $\gamma_{n+1}(M) < Z_0(M)$. So $\gamma_{n+1}(M)$=$\{e\}$. □

Theorem 3.15. *Every subultra-group of nilpotent ultra-group is nilpotent.*

Proof. Let M be an ultra-group and K be a subultra-group of M. Since M is nilpotent, so there exists an integer $n \geqslant 0$ such that $\gamma_{n+1}(M) = \{e\}$. Now we will show (by induction on i) that $\gamma_i(K) < \gamma_i(M)$. For $i = 1$, it is clear. Suppose that $\gamma_i(K) < \gamma_i(M)$. Then $\gamma_{i+1}(K) = \left[\widehat{\gamma_i(K), K}\right] < \left[\widehat{\gamma_i(M), M}\right] = \gamma_{i+1}(M)$. Hence $\gamma_{i+1}(K) < \gamma_{i+1}(M) = \{e\}$. □

Let M be an ultra-group of a subgroup H over the group G. We define the i-th derived subultra-group of M inductively as follows:

$M^{(1)} = M' = \left[\widehat{M, M}\right]$,

$M^{i+1} = M^{(i)'} = \left[\widehat{M^{(i)}, M^{(i)}}\right]$.

So we obtain the sequence of subultra-groups of M such that each one is normal sn the previous.

The series

$$M^{(0)} = M > M^1 > M^2 > \dots$$

is called *derived*.

Lemma 3.16. *Let M be an arbitrary ultra-group and $M = M_0 > M_1 > M_2 > \dots$ be the solvable series. Then for every i, $M^{(i)} \subseteq M_{(i)}$.*

Proof. By induction on i. If $i = 0$, then $M^{(0)} = M \subseteq M_{(0)} = M$. Suppose $M^{(i)} \subseteq M_{(i)}$. We show that this is true for $i + 1$. By $M^{(i)} \subseteq M_{(i)}$ we have $\left[\widehat{M^{(i)}, M^{(i)}}\right] \subseteq \left[\widehat{M_{(i)}, M_{(i)}}\right]$. So $M^{i+1} \subseteq M_i'$. According to the conditions of the solvable series $\frac{M_i}{M_{i+1}}$ is abelian. Thus by Theorem 3.8, $M_i' \subseteq M_{i+1}$. Therefore $M^{(i+1)} \subseteq M_{(i+1)}$. □

Theorem 3.17. *Let M be an ultra-group of subgroup H over the group G. The ultra-group M is solvable if and only if there exists $n \geqslant 0$ such that $M^{(n)} = \{e\}$.*

Proof. If M is a solvable ultra-group, then it has solvable series $M = M_0 > M_1 > M_2 > \dots > M_n = \{e\}$. Now by the above lemma for $i = n$ we have $M^{(n)} > M_n = \{e\}$. Conversely, let there exists $n \geqslant 0$ such that $M^{(n)} = \{e\}$. In this case the derived series has the conditions of a solvable series, that means $M^{i+1} \lhd M^i$ and $\frac{M_i}{M_{i+1}}$ is abelian. So it is a solvable series for an ultra-group M. □

We conclude this paper by presenting the following lemma, which demonstrates the relationship between nilpotent and solvable ultra-groups.

Lemma 3.18. *Every nilpotent ultra-group is solvable.*

Proof. If an ultra-group M is nilpotent then the upper central series of M, $\{e\} < Z_1(M) < Z_2(M) < \dots < Z_n(M) = M$ is a normal series. All quotients of the upper central series are abelian since $\frac{Z_i(M)}{Z_{i-1}(M)} = Z(\frac{M}{Z_{i-1}(M)})$ and $Z(\frac{M}{Z_{i-1}(M)})$ is abelian. □

References

[1] **J. Adamek, H. Herrlich and G. Strecker**, *Abstract and Concrete Categories*, John Willy and Sons, New York (1990).

[2] **S. Burris and H.P. Sankappanavar**, *A Course in Universal Algebra*, Springer-Verlag Graduate Texts in Mathematics, (1981).

[3] **S.H. Dalalyan**, *On hypergroups, prenormal subgroups and the simplest groups*, Conf. dedicated to 90th-anniversary of M.M. Jrbashyan, Yerevan, (Russian),(2008), $12-14$.

[4] **T.W. Hungerford**, *Algebra*, Springer, New York, (2003).

[5] **M. Kilp, U. Knauer and A. Mikhalev**, *Monoids, Acts and Categories*, Walter De Gruyter, Berlin, New York, (2000).

[6] **A.G. Kurosh**, *The Theory of Groups*, Amer. Math. Soc., (1960).

[7] **Gh. Moghaddasi, B. Tolue and P. Zolfaghari**, *On the structure of the ultra-group over a finite group*, U.P.B. Sci. Bull., Series A, **78** (2016), $173-187$.

[8] **Gh. Moghaddasi and P. Zolfaghari**, *The quotient ultra-groups*, Italian J. Pure Appl. Math., **36** (2016), $211-218$.

[9] **B. Tolue, Gh. Moghaddasi and P. Zolfaghari**, *On the category of ultra-groups*, acettepe J. Math. Statistics, **46** (2017), $437-447$.

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