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On primary ordered semigroups

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Abstract. In this paper, left primary, right primary, primary and semiprimary ideals of ordered semigroups are introduced. Moreover, we introduce an ordered semigroups in which every ideal is primary and every ideal is semiprimary which is a generalization of primary and semiprimary semigroups.

1. Introduction and preliminaries

A primary semigroup was introduced and studied by M. Satyanarayana in [10] and some results from [10] were extended to semiprimary semigroups by H. Lal [8]. Their study was restricted to commutative semigroups. The concepts of primary and semiprimary semigroups pass to noncommutative semigroups by A. Anjaneyulu [1, 2]. In [2], a class of semigroups knows as pseudo symmetric semigroups, which includes the classes of commutative, narmal, idempotent, duo semigroups was introduced. In this paper, the notions of primary and semiprimary semigroups extended to ordered semigroups. We introduce left primary, right primary, primary and semiprimary ideals of ordered semigroups and also a class of ordered semigroups, namely pseudo symmetric ordered semigroups, which includes the classes of commutative, narmal, idempotent, duo ordered semigroups. Moreover, we study the connection between prime and semiprime ideals of an ordered semigroups.

We recall some certain definitions and results used throughout this paper. A semigroup $(S, \cdot)$ together with a partial oder $\leqslant$ that is *compatible* with the semigroup operation, meaning that for any x, y, z in S, $x \leqslant y$ implies $zx \leqslant zy$ and $xz \leqslant yz$, is called a *partially ordered semigroup*, or simply an *ordered semigroup* [4]. Under the trivial relation, $x \leqslant y$ if and only if $x = y$, it is observed that every semigroup is an ordered semigroup.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For two nonempty subsets A, B of S, we write AB for the set of all elements xy in S where $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements x in S such that $x \leqslant a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

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In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [5] that the following hold: (1) $A \subseteq (A]$; (2) $A \subseteq B \Rightarrow (A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $(A \cup B] = (A] \cup (B]$; (5) $((A]] = (A]$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a *left* (respectively, *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) $A = (A]$, that is, for any x in A and y in S, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of two ideals of S is an ideal of S.

An element a of an ordered semigroup $(S, \cdot, \leqslant)$, the *principal left* (respectively, *right, two-sided*) *ideal* generated by a is of the form $L(a) = (a \cup Sa]$ (respectively, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$).

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A left ideal A of S is said to be *proper* if $A \subset S$. A proper right and two-sided ideals are defined similarly. If S does not contain proper ideals then we call S *simple*. A proper ideal A of S is said to be *maximal* if for any ideal B of S, if $A \subset B \subseteq S$, then $B = S$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal I of S is said to be *prime* if for any ideals A, B of S, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal I of S is said to be *completely prime* if for any $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. An ideal I of S is said to be *semiprime* if for any ideal A of S, $A^2 \subseteq I$ implies $A \subseteq I$. An ideal I of S is said to be *completely semiprime* if for any $a \in S$, $a^n \in I$ for any positive integer n implies $a \in I$ [11].

An ideal A of an ordered semigroup $(S, \cdot, \leqslant)$, the intersection of all prime ideals of S containing A, will be denoted by $Q^*(A)$ and the intersection of all completely prime ideals of S containing A, will be denoted by $P^*(A)$.

A subset A of an ordered semigroup $(S, \cdot, \leqslant)$, the radical of A, will be denoted by $\sqrt{A}$ defined by

$$\sqrt{A} = \{x \in S \mid x^n \in A \text{ for some positive integer } n\ \} \text{ [3].}$$

An element a of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *semisimple element* in S if $a \in (SaSaS]$. And S is said to be *semisimple* if every element of S is semisimple [11].

An element a of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *left regular* (respectively, *right regular*, *regular*, *intra-regular*) if there exist x, y in S such that $a \leqslant xa^2$ (respectively, $a \leqslant a^2x$, $a \leqslant axa$, $a \leqslant xa^2y$) [11]. It is observed that left regular elements, right regular elements, regular elements, and intra-regular elements are all semisimple.

A subset M of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *m-system* of S, if for any $a, b \in M$, there exists $x \in S$ such that $(axb] \cap M \neq \emptyset$. A subset N of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *n-system* of S, if for any $a \in N$, there exists $x \in S$ such that $(axa] \cap N \neq \emptyset$ [7].

An ordered semigroup $(S, \cdot, \leqslant)$ is said to be a *left(right) duo* if every left(right) ideal of S is a two-sided ideal of S. An ordered semigroup S is said to be a *duo* if it is both a left duo and a right duo. An ordered semigroup S is said to be *normal* if $(xS] = (Sx]$ for all $x \in S$.

An element a of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *ordered idempotent* if $a \leqslant a^2$. We call an ordered semigroup S *idempotent ordered semigroup* if every element of S is an ordered idempotent [6]. The set of all ordered idempotents of an ordered semigroup S denoted by $E(S)$.

An element e of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *identity element* of S if $ex = x = xe$ for any $x \in S$. The *zero element* of S, defined by Birkhoff, is an element 0 of S such that $0 \leqslant x$ and $0x = 0 = x0$ for all $x \in S$.

2. Pseudo symmetric ordered semigroups

In this section, we introduce a class of ordered semigroups, namely pseudo symmetric ordered semigroups, which includes the classes of commutative, narmal, idempotent, duo ordered semigroups.

Definition 2.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal A of S is said to be *pseudo symmetric* if $xy \in A$ for some $x, y \in S$ implies $(xsy] \subseteq A$ for all $s \in S$.

Definition 2.2. An ordered semigroup $(S, \cdot, \leqslant)$ is said to be *pseudo symmetric* if every ideal of S is pseudo symmetric.

Example 2.3. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	b	b

$$\leq\, = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}.$$

The ideals of S are: $\{a\}$, $\{a, b\}$ and S. As is easily seen, $\{a\}$, $\{a, b\}$ and S, are pseudo symmetric. So, it is pseudo symmetric ordered semigroup.

Remark 1. Every commutative and normal ordered semigroup is a pseudo symmetric ordered semigroup.

Proposition 2.4. *Every duo ordered semigroup is a pseudo symmetric ordered semigroup.*

Proof. Let $(S, \cdot, \leqslant)$ be a duo ordered semigroup and A an ideal of S such that $xy \in A$ for some $x, y \in S$. Since S is duo, $L(a) = R(a)$ for all $a \in S$. Let $s \in S$. We have $xs \in (xS \cup x] = (Sx \cup x]$. Thus $xs \in (Sx]$ or $xs \in (x]$. And each of the cases implies $(xsy] \subseteq A$. Thus S is a pseudo symmetric. □

Proposition 2.5. *Every idempotent ordered semigroup is a pseudo symmetric ordered semigroup.*

Proof. Let $(S, \cdot, \leqslant)$ be an idempotent ordered semigroup and A an ideal of S such that $xy \in A$ for some $x, y \in S$. Since S is an idempotent ordered semigroup, we have $yx \leqslant yxyx = y(xy)x \in A$ and also $xsy \leqslant xsyxsy \in A$ for all $s \in S$. Thus S is a pseudo symmetric. □

Proposition 2.6. *Let $(S, \cdot, \leqslant)$ be a pseudo symmetric ordered semigroup and A an ideal of S. Then A is prime if and only if A is completely prime.*

Proof. Assume that A is prime. Let $ab \in A$ for any $a, b \in S$. Since S is pseudo symmetric, $(asb] \subseteq A$ for all $s \in S$. It follows that $(aSb] \subseteq A$. Thus $I(a)I(b) \subseteq A$. Since A is prime, we have $I(a) \subseteq A$ or $I(b) \subseteq A$. Thus $a \in A$ or $b \in A$, which shows that A is completely prime. The converse statement is clear. □

Lemma 2.7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and A an ideal of S. Then $Q^*(A) \subseteq \sqrt{A}$.*

Proof. Let $x \in Q^*(A)$. If $x^n \notin A$ for all positive integer n. By Lemma 2.4 in [11], then there exists a prime ideal P of S containing A such that $x^n \notin P$ for all positive integer n. Thus $x \notin Q^*(A)$. This is a contradiction. Thus $Q^*(A) \subseteq \sqrt{A}$. □

Theorem 2.8. *Let $(S, \cdot, \leqslant)$ be a pseudo symmetric ordered semigroup and A an ideal of S. Then $Q^*(A) = \sqrt{A}$.*

Proof. We have $Q^*(A) \subseteq \sqrt{A}$ by Lemma 2.7. If $x \notin Q^*(A)$. Then there exists a prime ideal P of S containing A such that $x \notin P$. We have P is a completely prime ideal by Proposition 2.6. Thus $x^n \notin P$ for all positive integer n. It follows that $x^n \notin A$ for all positive integer n. Thus $x \notin \sqrt{A}$ and so $\sqrt{A} \subseteq Q^*(A)$. Hence $Q^*(A) = \sqrt{A}$. □

3. Prime and semiprime ideals of ordered semigroups

In this section, we study the relation between prime and semiprime ideals of an ordered semigroups.

Lemma 3.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and A an ideal of S. Then A is prime if and only if for any $a, b \in S$, $(aSb] \subseteq A$ implies $a \in A$ or $b \in A$.*

Proof. Assume that A is prime. Let $(aSb] \subseteq A$ for any $a, b \in S$. Thus $I(a)I(b) \subseteq A$. Since A is prime, we have $a \in I(a) \subseteq A$ or $b \in I(b) \subseteq A$. Conversely, assume that for any $a, b \in S$, $(aSb] \subseteq A$ implies $a \in A$ or $b \in A$. Let B, C be ideals of S such that $BC \subseteq A$. If $B \nsubseteq A$ and $C \nsubseteq A$, then there exists $b \in B \setminus A$ and $c \in C \setminus A$. Thus $(bSc] \subseteq A$. It follows that $b \in A$ or $c \in A$. This is a contradiction. Thus $B \subseteq A$ or $C \subseteq A$. □

Similarly, we prove the following:

Lemma 3.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and A an ideal of S. Then A is semiprime if and only if for any $a \in S$, $(aSa] \subseteq A$ implies $a \in A$.*

Proposition 3.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and A an ideal of S. Then A is prime if and only if either $S \setminus A = \emptyset$ or the set $S \setminus A$ is an m-system.*

Proof. Assume that A is prime. If $S \setminus A \neq \emptyset$. Let $a, b \in S \setminus A$. Since A is a prime, we have $(aSb] \nsubseteq A$ by Lemma 3.1. Then there exists $y \in S$ such that $ayb \notin A$. Thus $ayb \in S \setminus A$ and so $S \setminus A$ is an m-system. Conversely, assume that either $S \setminus A = \emptyset$ or the set $S \setminus A$ is an m-system. Let $a, b \in A$ such that $(aSb] \subseteq A$. If $a, b \notin A$. Since $S \setminus A$ is an m-system, then there exists $x \in S$ and $c \in S \setminus A$ such that $c \leqslant axb \in (aSb] \subseteq A$. This is a contradiction. Thus $a \in A$ or $b \in A$. □

Similarly, we prove the following:

Proposition 3.4. *Let* $(S,\cdot,\leqslant)$ *be an ordered semigroup and* A *an ideal of* S*. Then* A *is semiprime if and only if either* $S\setminus A=\emptyset$ *or the set* $S\setminus A$ *is an* n*-system.*

Proposition 3.5. *Any semiprime ideal of an ordered semigroup* $(S,\cdot,\leqslant)$ *is an intersection of prime ideals of* S*.*

Proof. Let A be a semiprime ideal of S. If $x\notin A$, choose elements $x_1,x_2,x_3,...$ inductively as follows: $x_1=x$. Since $(x_1Sx_1]=(xSx]\nsubseteq A$, take $x_2\in S$ such that $x_2\in(x_1Sx_1]$ and $x_2\notin A$. Since $(x_2Sx_2]\nsubseteq A$, we have $x_3\in S$ such that $x_3\in(x_2Sx_2]$, $x_3\notin A,\cdots,x_{i+1}\in(x_iSx_i]$, $x_{i+1}\notin A,\cdots$. We set $B=\{x_1,x_2,x_3,\cdots\}$. Let $x_i,x_j\in B$ and $i\leqslant j$. Then $x_{j+1}\in(x_iSx_j]$, $x_{j+1}\in(x_jSx_i]$ and $x_{j+1}\in B$. Thus B is an m-system. Let $T=\{Q\mid Q$ is an m-system of S, $x\in Q$ and $Q\cap A=\emptyset\ \}$. Then $T\neq\emptyset$. By Zorn's Lemma, there exists a maximal element in T, namely M. Let $H=\{J\mid J$ is an ideal of S, $A\subseteq J$ and $J\cap M=\emptyset\ \}$. Then $H\neq\emptyset$. By Zorn's Lemma, there exists a maximal element in H, namely I. Let $a,b\in S\setminus I$, then $(I(a)\cup I)\cap M\neq\emptyset$ and $(I(b)\cup I)\cap M\neq\emptyset$. Thus there exists $m_1,m_2\in M$ such that $m_1\leqslant s_1as_2$, $m_2\leqslant s_3bs_4$, where $s_1,s_2,s_3,s_4\in S$. Since M is an m-system, then there exists $m\in M$ such that $m\leqslant m_1zm_2$ for some $z\in S$. We have $m\leqslant s_1as_2zs_3bs_4$ and so $s_1as_2zs_3bs_4\notin I$. It follows that $as_2zs_3b\notin I$. Thus $as_2zs_3b\in S\setminus I$ and so $S\setminus I$ is an m-system. We have I is prime ideal of S containing A by Proposition 3.3. Since $x\notin I$, $x\notin Q^*(A)$. Thus $Q^*(A)\subseteq A$ and so $Q^*(A)=A$. □

Proposition 3.6. *Let* $(S,\cdot,\leqslant)$ *be an ordered semigroup and* A *an ideal of* S*. Then* A *is semiprime if and only if* $Q^*(A)=A$*.*

Proof. If A is semiprime, then $Q^*(A)=A$ by Proposition 3.5. The converse statement is obvious. □

It is easy to see the following:

Lemma 3.7. *Let* $(S,\cdot,\leqslant)$ *be an ordered semigroup and* A *an ideal of* S*. Then* A *is completely prime if and only if* $S\setminus A$ *is a subsemigroups of* S*.*

Proposition 3.8. *Any completely semiprime ideal of an ordered semigroup* $(S,\cdot,\leqslant)$ *is an intersection of completely prime ideals of* S*.*

Proof. Let A be completely semiprime ideal of S. If $x\notin A$, then $x^n\notin A$ for all positive integer n. Let $B=\{x,x^2,x^3,\cdots\}$. Then B is an m-system and $A\cap B=\emptyset$. Let $T=\{Q\mid Q$ is an m-system of S, $x\in Q$ and $Q\cap A=\emptyset\ \}$. Then $T\neq\emptyset$. By Zorn's Lemma, there exists a maximal element in T, namely M. Let $H=\{J\mid J$ is an ideal of S, $A\subseteq J$ and $J\cap M=\emptyset\ \}$. Then $H\neq\emptyset$. By Zorn's Lemma, there exists a maximal element in H, namely I. By the same method given in Proposition 3.6, we have $S\setminus I=M$. Let $<M>$ be a subsemigroup of S generated by M. Then $<M>$ is an m-system. If $<M>\cap A\neq\emptyset$, then there exists $m_1,m_2,m_3,\cdots,m_n\in M$ such that $m_1m_2m_3\cdots m_n\in A$. Since M is an m-system, there exists $m\in M$ and $x_1,x_2,x_3,\cdots,x_{n-1}\in S$ such that $m\leqslant m_1x_1m_2x_2m_3\cdots m_{n-1}x_{n-1}m_n$. Since A is a completely semiprime, $ab\in A$ implies $ba\in A$. It follows that $m_1x_1m_2x_2m_3\cdots m_{n-1}x_{n-1}m_n\in A$. Thus $m\in A$. This is a contradiction. By the maximality of M, we have $<M>=M$. Thus I is a completely prime ideal of S containing A by Lemma 3.7. Since $x\notin I$, $x\notin P^*(A)$. Thus $P^*(A)\subseteq A$ and so $P^*(A)=A$. □

Corollary 3.9. *Any completely semiprime ideal of an ordered semigroup* $(S, \cdot, \leqslant)$ *is an intersection of prime ideals of* S.

Proposition 3.10. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup and* A *an ideal of* S. *Then* A *is completely semiprime if and only if* $P^*(A) = A$.

Proof. If A is completely semiprime, then $P^*(A) = A$ by Proposition 3.8. The converse statement is obvious. □

Lemma 3.11. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. The following statements are equivalent:*

(1) S *is semisimple.*

(2) $(A^2] = A$ *for any ideal* A *of* S.

(3) $A \cap B = (AB]$ *for any ideal* A, B *of* S.

(4) $I(a) \cap I(b) = (I(a)I(b)]$ *for any* $a, b \in S$.

(5) $(I(a)^2] = I(a)$ *for any* $a \in S$.

Proof. The implications (3) ⇒ (4) and (4) ⇒ (5) are obvious and we will prove (1) ⇒ (2) ⇒ (3) and (5) ⇒ (1). (1) ⇒ (2). Let A be an ideal of S and $x \in A$. Then $x \leqslant s_1xs_2xs_3$ for some $s_1, s_2, s_3 \in S$. We have $s_1xs_2 \in A$ and $xs_3 \in A$. Then $x \leqslant s_1xs_2xs_3 \in A^2$ and so $x \in (A^2]$. Thus $(A^2] = A$. (2) ⇒ (3). Let A and B be an ideals of S. Clearly $(AB] \subseteq A \cap B$. Since $A \cap B$ is an ideal, $A \cap B = ((A\cap B)(A\cap B)] \subseteq (AB]$. Thus $A \cap B = (AB]$. (5) ⇒ (1). Let $a \in S$. Then $I(a)^3 = I(a)I(a)I(a) \subseteq SI(a)S \subseteq (SaS]$. We have

$$a \in I(a) = (I(a)^2] \subseteq (I(a)^5] = (I(a)^3I(a)I(a)] \subseteq ((SaS]I(a)S] \subseteq (SaSaS].$$

Thus S is semisimple. □

Proposition 3.12. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. Then* S *is semisimple if and only if every ideal of* S *is semiprime.*

Proof. Assume that S is semisimple. Let I and A be an ideals of S such that $A^2 \subseteq I$. We have $A = (A^2] \subseteq I$ by Lemma 3.11. Thus I is semiprime. Converesly, assume that every ideal of S is semiprime. Let A be an ideal of S. Since $A^2 \subseteq (A^2]$, $A \subseteq (A^2]$. Clearly $(A^2] \subseteq A$. Thus $A = (A^2]$, which shows that S is semisimple by Lemma 3.11. □

4. Primary ordered semigroups

In this section, we introduce left primary, right primary, primary and semiprimary ideals of ordered semigroups and an ordered semigroups in which every ideal is primary and every ideal is semiprimary.

Definition 4.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal I of S is said to be *left(right) primary* if

(i) If A, B are ideals of S such that $AB \subseteq I$ and $B \nsubseteq I (A \nsubseteq I)$ implies $A \subseteq Q^*(I) (B \subseteq Q^*(I))$.

(ii) $Q^*(I)$ is a prime ideal.

An ideal I of S is said to be *primary* if it is both the left and right primary ideal.

Remark 2. An ideal I of S satisfies condition (i) of Definition 4.1 if and only if for every $x, y \in S$ such that $I(x)I(y) \subseteq I$ and $y \notin I(x \notin I)$, then $x \in Q^*(I)(y \in Q^*(I))$.

We have the example to show that left primary, right primary and primary ideals are different.

Example 4.2. Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	a	c

$$\leq = \{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}.$$

The ideals of S are: $\{a\}$, $\{a, b\}$ and S. It is evident that the ideal $\{a\}$ is right primary but not left primary.

Definition 4.3. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An ideal I of S is said to be *semiprimary* if $Q^*(I)$ is a prime ideal.

It is clear that every left(right) primary ideal is a semiprimary ideal.

Definition 4.4. An ordered semigroup $(S, \cdot, \leqslant)$ is said to be *(left, right, semi)primary* if every ideal of S is (left, right, semi)primary.

Theorem 4.5. *Let $(S, \cdot, \leqslant)$ be a pseudo symmetric ordered semigroup and A an ideal of S. Then A is left(right) primary if and only if for $x, y \in S$ such that $xy \in A$ and $y \notin A(x \notin A)$, then $x \in Q^*(A)(y \in Q^*(A))$.*

Proof. Assume that A a left primary. Let $x, y \in S$ such that $xy \in A$ and $y \notin A$. Since S is pseudo symmetric, we have $(xsy] \subseteq A$ for all $s \in S$. Thus $(xSy] \subseteq A$. It follows that $I(x)I(y) \subseteq A$. Since A is left primary and $I(y) \nsubseteq A$, we have $x \in I(x) \subseteq Q^*(A)$. Conversely, let $x, y \in S$ such that $I(x)I(y) \subseteq A$ and $y \notin A$. Then $xy \in A$ and so $x \in Q^*(A)$. Let $ab \in Q^*(A)$ for any $a, b \in S$ and $b \notin Q^*(A)$. Then $(ab)^n \in A$ for some positive integer n by Theorem 2.8. Let k be the least positive integer such that $(ab)^k \in A$. If $k = 1$, then $ab \in A$. Thus $a \in Q^*(A)$, which shows that $Q^*(A)$ is completely prime. It follows that $Q^*(A)$ is prime. If $k > 1$, then $ab(ab)^{k-1} = (ab)^k \in A$. If $b(ab)^{k-1} \in A$. Since $(ab)^{k-1} \notin A$, we have $b \in Q^*(A)$. This is a contradiction. Thus $b(ab)^{k-1} \notin A$ and so $a \in Q^*(A)$. It follows that $Q^*(A)$ is prime. Thus A is a left primary. □

It is easy to see the following lemma:

Lemma 4.6. *Let A and B be an ideals of an ordered semigroup $(S, \cdot, \leqslant)$. Then*

(1) *If $A \subseteq B$ then $Q^*(A) \subseteq Q^*(B)$;*

(2) *$Q^*(Q^*(A)) = Q^*(A)$;*

(3) $Q^*(A \cap B) = Q^*(A) \cap Q^*(B)$.

Theorem 4.7. *An ordered semigroup $(S, \cdot, \leqslant)$ is a left(right) primary if and only if every ideal in S satisfies condition (i) in Definition* 4.1.

Proof. Assume that every every ideal in S satisfies condition (i) in Definition 4.1. Let I be an ideal of S such that $AB \subseteq Q^*(I)$ for any ideals A, B of S. If $B \nsubseteq Q^*(I)$, then $A \subseteq Q^*(Q^*(I)) = Q^*(I)$. Thus $Q^*(I)$ is prime and so I is a left primary. The converse statement is clear. □

Proposition 4.8. *Let A be an ideal of a pseudo symmetric semiprimary ordered semigroup $(S, \cdot, \leqslant)$. Then A is completely semiprime if and only if A is completely prime.*

Proof. Assume that A is completely semiprime. Let $ab \in A$ for any $a, b \in S$. Since S is a pseudo symmetric, we have $Q^*(A)$ is completely prime by Proposition 2.6. Thus $a \in Q^*(A)$ or $b \in Q^*(A)$. If $a, b \notin A$. Since A is completely semiprime, $a^n, b^n \notin A$ for all positive integer n. Thus $a, b \notin Q^*(A)$ by Theorem 2.8. This is contradiction. Thus A is completely prime. The converse statement is obvious. □

Proposition 4.9. *Let $(S, \cdot, \leqslant)$ be a pseudo symmetric ordered semigroup. Then S is semiprimary if and only if every ideal A of S satisfies the condition: If $xy \in A$ for any $x, y \in S$, then $x \in Q^*(A)$ or $y \in Q^*(A)$.*

Proof. Assume that S is semiprimary. Let A be an ideal of S such that $xy \in A$ for any $x, y \in S$. Since S is a pseudo symmetric, we have $Q^*(A)$ is completely prime. Thus $x \in Q^*(A)$ or $y \in Q^*(A)$. Conversely, let A be an ideal of S and $xy \in Q^*(A)$ for any $x, y \in S$. Then $x \in Q^*(Q^*(A)) = Q^*(A)$ or $y \in Q^*(Q^*(A)) = Q^*(A)$, which shows that $Q^*(A)$ is completely prime. Thus $Q^*(A)$ is prime. Hence S is semiprimary. □

Lemma 4.10. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then a maximal ideal M of S is prime if and only if $M = Q^*(M)$.*

Proof. If a maximal ideal M is prime, then $M = Q^*(M)$ is clear. Conversely, assume that $M = Q^*(M)$. Since M is a maximal ideal, we have M is prime. □

Proposition 4.11. *Let A be an ideal of an ordered semigroup $(S, \cdot, \leqslant)$. If $Q^*(A)$ is a maximal ideal of S, then A is a semiprimary ideal.*

Proof. If $Q^*(A)$ is a maximal ideal of S, then $Q^*(A)$ is prime by Lemma 4.10. Thus A is a semiprimary ideal. □

Lemma 4.12. *Let A be an ideal of an ordered semigroup $(S, \cdot, \leqslant)$ with identity. If $Q^*(A) = M$, where M is the unique maximal ideal of S, then A is a primary ideal.*

Proof. Let $x, y \in S$ such that $I(x)I(y) \subseteq A$ and $y \notin A$. If $x \notin Q^*(A) = M$. Then $I(x) \nsubseteq M$. Since each proper ideal of S is contained in M, we have $I(x) = S$. Thus $y = ey \in I(x)I(y) \subseteq A$. This is contradiction. Thus $x \in Q^*(A)$. We have $Q^*(A)$ is prime by Lemma 4.10. Thus A is a left primary ideal. Similarly, we have A is a right primary ideal. Hence A is a primary ideal. □

Theorem 4.13. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with identity. If every(nonzero, assume this if S has 0) proper prime ideals are maximal, then S is a primary.*

Proof. If S is not a simple, then S has a unique maximal ideal M, which is the union of all proper ideals of S. By hypothesis M is the only proper (nonzero) prime ideal of S. If A is a proper (nonzero) ideal, then $Q^*(A) = M$. Thus A is primary by Lemma 4.12. If S has 0. If $I(0)$ is a prime ideal, then $I(0)$ is primary. If $I(0)$ is not prime, then $Q^*(I(0)) = M$. Thus $I(0)$ is primary by Lemma 4.12. Hence S is primary. □

Proposition 4.14. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If A is a semiprime ideal in S, then the following conditions are equivalent:*

(1) *A is a prime.*

(2) *A is a primary.*

(3) *A is a left primary.*

(4) *A is a right primary.*

(5) *A is a semiprimary.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious. Since A is a semiprime, we have $Q^*(A) = A$ by Proposition 3.6. Thus (1) and (5) are equivalent. □

Theorem 4.15. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then S is semiprimary if and only if the set of all prime ideals of S forms a chain under the set inclusion.*

Proof. Let A and B be any prime ideals of S. Thus $A \cap B = Q^*(A \cap B)$. Since S is a semiprimary, $A \cap B$ is prime. If $A \nsubseteq B$ and $B \nsubseteq A$, then there exists elements $a, b \in S$ such that $a \in A \setminus B$ and $b \in B \setminus A$. Thus $I(a)I(b) \subseteq A \cap B$ and $a, b \notin A \cap B$. This is contradiction. Hence either $A \subseteq B$ or $B \subseteq A$. Conversely, let A be any ideal of S. If the set of all prime ideals of S forms a chain under the set inclusion, then $Q^*(A)$ is a prime, which shows that A is a semiprimary ideal. Thus S is a semiprimary. □

Theorem 4.16. *Let $(S, \cdot, \leqslant)$ be a duo semiprimary ordered semigroup. Then S has the following properties:*

(1) *Set of all prime ideals of S forms a chain under the set inclusion.*

(2) *For any $e, f \in E(S)$, either $e \leqslant xf$ and $e \leqslant fy$ or $f \leqslant xe$ and $f \leqslant ey$ for some $x, y \in S$.*

Proof. (1) This follow by Theorem 4.15. (2) Let $e, f \in E(S)$. Since S is semiprimary, we have $Q^*(I(e))$ and $Q^*(I(f))$ are prime. Thus $Q^*(I(e)) \subseteq Q^*(I(f))$ or $Q^*(I(f)) \subseteq Q^*(I(e))$ by (1). If $Q^*(I(e)) \subseteq Q^*(I(f))$. Then $e^n \in I(f)$ for some positive integer n by Lemma 2.7. It follows that $e \in I(f)$. Since S is a duo ordered semigroup, we have $I(f) = (Sf] = (fS]$. Thus $e \leqslant xf$ and $e \leqslant fy$ for some $x, y \in S$. Similarly, if $Q^*(I(f)) \subseteq Q^*(I(e))$ then $f \leqslant xe$ and $f \leqslant ey$ for some $x, y \in S$. □

Theorem 4.17. *Let $(S, \cdot, \leqslant)$ be a regular pseudo symmetric ordered semigroup. The following statements are equivalent:*

(1) *Every ideal of S is prime.*

(2) *S is a primary ordered semigroup.*

(3) *S is a left primary ordered semigroup.*

(4) *S is a right primary ordered semigroup.*

(5) *S is a semiprimary ordered semigroup.*

(6) *The set of all prime ideals of S forms a chain under the set inclusion.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious. (5) $\Rightarrow$ (1) Let A be an ideal of S and $x^2 \in A$ for any $x \in S$. Since S is regular pseudo symmetric, we have $x \in (xSx] \subseteq A$, which shows that A is completely semiprime. It follows that A is prime by Proposition 4.8 and Proposition 2.6. We have (5) and (6) are equivalent by Theorem 4.15. □

Following result is obvious its proof is omitted.

Lemma 4.18. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. The following statements are equivalent:*

(1) *Set of all the principal ideals of S forms a chain under the set inclusion.*

(2) *Set of all the ideals of S forms a chain under the set inclusion.*

Theorem 4.19. *Let $(S, \cdot, \leqslant)$ be a semisimple ordered semigroup. The following statements are equivalent:*

(1) *Every ideal of S is prime.*

(2) *S is a primary ordered semigroup.*

(3) *S is a left primary ordered semigroup.*

(4) *S is a right primary ordered semigroup.*

(5) *S is a semiprimary ordered semigroup.*

(6) *The set of all prime ideals of S forms a chain under the set inclusion.*

(7) *The set of all principal ideals of S forms a chain under the set inclusion.*

(8) *The set of all the ideals of S forms a chain under the set inclusion.*

Proof. Let A be an ideal of S. Since S is semisimple, we have A is a semiprime by Proposition 3.12. Thus (1) to (5) are equivalent by Proposition 4.14. We have (5) and (6) are equivalent by Theorem 4.15. The implication (8) $\Rightarrow$ (6) is obvious. (6) $\Rightarrow$ (7). Let $I(a)$ and $I(b)$ be a principal ideals of S. We have $Q^*(I(a)) \subseteq Q^*(I(b))$ or $Q^*(I(b)) \subseteq Q^*(I(a))$. Since S is a semisimple, $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. We have (7) and (8) are equivalent by Lemma 4.18. This complete the proof of the theorem. □

Theorem 4.20. *Let $(S, \cdot, \leqslant)$ be a duo semisimple ordered semigroup. The following statements are equivalent:*

(1) *Every ideal of S is prime.*

(2) *S is a primary ordered semigroup.*

(3) *S is a left primary ordered semigroup.*

(4) *S is a right primary ordered semigroup.*

(5) *S is a semiprimary ordered semigroup.*

(6) *The set of all prime ideals of S forms a chain under the set inclusion.*

(7) *The set of all principal ideals of S forms a chain under the set inclusion.*

(8) *The set of all the ideals of S forms a chain under the set inclusion.*

(9) *For any* $e, f \in E(S)$, *either* $e \leqslant xf$ *and* $e \leqslant fy$ *or* $f \leqslant xe$ *and* $f \leqslant ey$ *for some* $x, y \in S$.

Proof. We have (1) to (8) are equivalent by Theorem 4.19. (5) $\Rightarrow$ (9). By Theorem 4.16. (9) $\Rightarrow$ (7). Let $I(a)$ and $I(b)$ be a principal ideals of S. Since S is duo semisimple , we have S is regular. Thus $a \leqslant axa$ and $b \leqslant byb$ for some $x, y \in S$. It follows that $ax, by \in E(S)$. Then either $ax \leqslant sby$ and $ax \leqslant byt$ or $by \leqslant sax$ and $by \leqslant axt$ for some $s, t \in S$ by (9). If $ax \leqslant sby$ and $ax \leqslant byt$. We have $a \leqslant axa \leqslant axaxa \leqslant sbybyta \in (SbS] \subseteq I(b)$. Thus $I(a) \subseteq I(b)$. Similarly, if $by \leqslant sax$ and $by \leqslant axt$ then $I(b) \subseteq I(a)$. This complete the proof. □

Corollary 4.21. *Let* $(S, \cdot, \leqslant)$ *be a duo regular ordered semigroup. The following statements are equivalent:*

(1) *Every ideal of S is prime.*

(2) *S is a primary ordered semigroup.*

(3) *S is a left primary ordered semigroup.*

(4) *S is a right primary ordered semigroup.*

(5) *S is a semiprimary ordered semigroup.*

(6) *The set of all prime ideals of S forms a chain under the set inclusion.*

(7) *The set of all principal ideals of S forms a chain under the set inclusion.*

(8) *The set of all the ideals of S forms a chain under the set inclusion.*

(9) *For any* $e, f \in E(S)$, *either* $e \leqslant xf$ *and* $e \leqslant fy$ *or* $f \leqslant xe$ *and* $f \leqslant ey$ *for some* $x, y \in S$.

Corollary 4.22. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. Then every ideal of S is prime if and only if S is a semisimple (semi)primary.*

Proof. Assume that every ideal of S is prime. Let $x \in S$. We have $I(x)I(x) \subseteq (I(x)^2]$. Since $(I(x)^2]$ is an ideal of S, $I(x) \subseteq (I(x)^2]$ and so $I(x) = (I(x)^2]$. Thus S is a semisimple (semi)primary by Lemma 3.11 and Theorem 4.19. Conversely, if S is a semisimple (semi)primary, then every ideal of S is prime by Theorem 4.19. □

Corollary 4.23. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup. Then every ideal of S is prime if and only if S is a semisimple and the set of all the ideals of S forms a chain under the set inclusion.*

Corollary 4.24. *Let* $(S, \cdot, \leqslant)$ *be a duo ordered semigroup. The following statements are equivalent:*

(1) *Every ideal of S is prime.*

(2) *S is regular semiprimary.*

(3) *S is regular and for any $e, f \in E(S)$, either $e \leqslant xf$ and $e \leqslant fy$ or $f \leqslant xe$ and $f \leqslant ey$ for some $x, y \in S$.*

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