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Generalized essential ideals in R-groups

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Abstract. In this paper, we consider an R-group where R is a zero-symmetric right nearring. We define generalized essential ideal of an R-group and prove several properties. Further, we extend this notion to obtain a one-one correspondence between s-essential ideals of R-group and those of $M_n(R)$-group R^n.

1. Preliminaries

The concept of uniform dimension in modules over rings is a generalization of the dimension of a vector space over a field. A module in which every non-zero submodule is essential is called uniform. Uniform submodules play a significant role to establish various finite dimension conditions in modules over associative rings. Goldie [11] characterized equivalent conditions for a module to have finite uniform dimension. In Bhavanari [20], uniform dimension was generalized to modules over nearrings (also known as, R-groups) and proved a characterization for a R-group to have finite Goldie dimension (in short, $f.G.d.$). Goldie dimension aspects in modules over nearrings were extensively studied by [5, 7, 20]. In case of a module over a matrix nearring, the notions essential ideal, uniform ideal were defined in [6], and proved a characterization for a module over a matrix nearring to have a $f.G.d.$. In [10], the authors studied prime and semiprime aspects in connection with $f.G.d.$ in R-groups and matrix nearrings.

In section 2, we introduce generalized essential ideal in R-groups and prove some properties. In section 3, we extend the notion of generalized essential ideal to modules over matrix nearrings and obtain a one-one correspondence between s-essential ideals of an R-group (over itself) and those

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of $M_n(N)$-group R^n.

A (right) nearring $(R, +, \cdot)$ is an algebraic system (Pilz [18]), where R is an additive group (need not be abelian), and a multiplicative semigroup, satisfying only one distributive axioms (say, right): $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in R$. If R is a right nearring, then $0a = 0$ and $(-a)b = -ab$, for all $a,\ b \in R$, but in general, $a0 \neq 0$ for some $a \in R$. R is zero-symmetric (denoted as, $R = R_0$) if $a0 = 0$ for all $a \in R$. An additive group $(G, +)$ is called an R-group (or module over a nearring R), denoted by ${}_RG$ (or simply by G) if there exists a mapping $R \times G \to G$ (image $(n, g) \to ng$), satisfying: $(n + m)g = ng + mg$; $(nm)g = n(mg)$ for all $g \in G$ and $n,\ m \in R$. It is evident that every nearring is an R-group (over itself). Also, if R is a ring, then each (left) module over R is an R-group. Throughout, G denotes an R-group where R is a right nearring.

A subgroup $(H, +)$ of G with $RH \subseteq H$ is called an R-subgroup of G. A normal subgroup H of G is called an ideal if $n(g + h) - ng \in H$ for all $n \in R,\ h \in H,\ g \in G$. For any two R-groups G_1 and G_2, a map $f\colon G_1 \to G_2$ is called an R-homomorphism, $f(x + y) = f(x) + f(y)$ and $f(nx) = nf(x)$ hold for all $x, y \in G_1$ and $n \in R$. If f is one-one and onto, then f is an R-isomorphism.

In case of a zero symmetric nearring, for any ideals A and B of G, $A+B$ is an ideal of G ([18], Corollary 2.3). For each $g \in G$, Rg is an R-subgroup of G. The ideal (or R-subgroup) generated by an element $g \in G$ is denoted by $\langle g \rangle$.
An ideal H of an R-group G is essential (see, [20]), if for any ideal K of G, $H \cap K = (0)$ implies $K = (0)$. If every ideal $(0) \neq H$ of G is essential then we say G is uniform. An ideal (R-subgroup) S of G is said to be superfluous ideal (see, [2, 3]), if $S + K = G$ and K is an ideal of G, imply $K = G$ and G is called hollow if every proper ideal of G is superfluous in G. Generalizations of essential ideals, prime ideals, superfluous ideals in R-groups, matrix nearrings, and hyperstructures were extensively studied in [13, 14, 17, 19, 21, 22, 23, 24, 25].

For standard definitions and notations in nearrings, we refer to [8, 18].

2. Generalized essential ideals

Definition 2.1. Let K be an R-ideal (or R-subgroup) of G. K is said to be *s-essential* in G (denoted by $K \trianglelefteq_s G$) if for any superfluous R-ideal (or R-subgroup) L of G, $K \cap L = (0)$ implies $L = (0)$.

Note 2.2. Every essential R-ideal of G is s-essential in G.

Remark 2.3. Converse of Note 2.2 need not be true. Let $R = \mathbb{Z}$ and $G = \mathbb{Z}_6$. Then $K_1 = \{\bar{0}, \bar{3}\}$ and $K_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ are the R-ideals of G. Then K_2 is s-essential but not essential, since $K_2 \cap K_1 = (\bar{0})$. but $K_1 \neq (\bar{0})$.

Example 2.4. Consider the nearring with addition and multiplication tables listed in K(135) and K(139) of p.418 of Pilz [18]. Let $G = D_8 = \langle \{a, b \mid 4a = 2b = 0,\ a+b = b-a\} \rangle = \{a, 2a, 3a, 4a = 0, b, a+b, 2a+b, 3a+b\}$, where a is the rotation in an anti-clockwise direction about the origin through $\frac{\pi}{2}$ radians and b is the reflection about the line of symmetry, and $G = R$. Then G is an R-group. Consider the operations:

$+$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
a	a	$2a$	$3a$	0	$a+b$	$2a+b$	$3a+b$	b
$2a$	$2a$	$3a$	0	a	$2a+b$	$3a+b$	b	$a+b$
$3a$	$3a$	0	a	$2a$	$3a+b$	b	$a+b$	$2a+b$
b	b	$3a+b$	$2a+b$	$a+b$	0	$3a$	$2a$	a
$a+b$	$a+b$	b	$3a+b$	$2a+b$	a	0	$3a$	$2a$
$2a+b$	$2a+b$	$a+b$	b	$3a+b$	$2a$	a	0	$3a$
$3a+b$	$3a+b$	$2a+b$	$a+b$	b	$3a$	$2a$	a	0

$*_1$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	a	b	$a+b$	$2a+b$	$3a+b$
b	0	b	$2a$	$2a+b$	b	$a+b$	$2a+b$	$3a+b$
$a+b$	0	$a+b$	0	$a+b$	0	0	0	0
$2a+b$	0	$2a+b$	$2a$	b	b	0	$2a+b$	$3a+b$
$3a+b$	0	$3a+b$	0	$3a+b$	0	0	0	0

The proper ideals are $I_1 = \{0, 2a\}$, $I_2 = \{0, a+b, 2a, 3a+b\}$, and R-subgroups are $J_1 = \{0, 2a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, a+b\}$, $J_4 = \{0, 2a+b\}$, $J_5 = \{0, 3a+b\}$, $J_6 = \{0, b, 2a, 2a+b\}$, $J_7 = \{0, 2a, a+b, 3a+b\}$. Then J_1 is s-essential but not essential, as $J_1 \cap J_3 = (0)$, whereas $J_3 \neq (0)$.

Proposition 2.5. *Let G be a unitary R-group and $(0) \neq K$ be an R-subgroup of G. Then $K \trianglelefteq_s G$ if and only if for each $0 \neq x \in G$, if $Rx \ll G$, then there exists an element $n \in R$ such that $0 \neq nx \in K$.*

Proof. Let $(0) \neq K$ be an R-subgroup of G such that $K \trianglelefteq_s G$. For each $0 \neq x \in G$, if $Rx \ll G$, then since $1 \in R$ and $x \neq 0$, we have $Rx \neq (0)$. Clearly, Rx is a R-subgroup of G. Since $K \trianglelefteq_s G$, we get $K \cap Rx \neq (0)$. Then there exists $0 \neq a \in K \cap Rx$. Since $a \in Rx$, there exists $n \in R$ such that $a = nx$. Therefore, $0 \neq nx \in K$. Conversely, suppose that L be an R-subgroup of G such that $(0) \neq L \ll G$. Then $0 \neq x \in L \subseteq G$. To show $Rx \ll G$, let T be an R-subgroup of G such that $Rx + T = G$. Now $Rx \subseteq RL \subseteq L$. Thus, $G = Rx + T \subseteq L + T$. So $L + T = G$. Now $L \ll G$ implies $T = G$. Therefore, $Rx \ll G$. Then by hypothesis, there exists an element $n \in R$ such that $0 \neq nx \in K$. Hence $0 \neq nx \in K \cap L$, and so $K \cap L \neq (0)$. Therefore, $K \trianglelefteq_s G$. □

Proposition 2.6. *Let K, L, T be R-ideals of G with $K \subseteq T$. If $K \trianglelefteq_s G$, then $K \trianglelefteq_s T$ and $T \trianglelefteq_s G$.*

Proof. Suppose that K be an R-ideal of G with $K \cap P = (0)$, where $P \ll T$. To show $P \ll G$, let M be an R-ideal of G such that $P + M = G$. Then $(P + M) \cap T = G \cap T$. Now by modular law, $P + (M \cap T) = T$. Since $P \ll T$, we get $M \cap T = T$. This implies $M \subseteq T$. Thus, $G = P + M \subseteq T = T$. Therefore, $T = G$. Hence $P \ll G$. Since $K \trianglelefteq_s G$, we have $P = (0)$. Thus $K \trianglelefteq_s T$. Now to show $T \trianglelefteq_s G$, let $Q \ll G$ such that $T \cap Q = (0)$. Since $K \subseteq T$, we have $K \cap Q \subseteq T \cap Q = (0)$. Then by hypothesis, $Q = (0)$. Therefore $T \trianglelefteq_s G$. □

Remark 2.7. *The converse of Proposition 2.6 need not be true. Let $R = \mathbb{Z}$ and $G = \mathbb{Z}_{36}$. $K = 6\mathbb{Z}_{36}$ and $L = 18\mathbb{Z}_{36}$ are R-ideals of G. Now $L \trianglelefteq_s K$ and $K \trianglelefteq_s G$. But $L \ntrianglelefteq_s G$, since $L \cap 12\mathbb{Z}_{36} = (0)$, but $12\mathbb{Z}_{36} \neq (0)$.*

Proposition 2.8. *Let K and L be R-ideals of G. Then $K \cap L \trianglelefteq_s G$ if and only if $K \trianglelefteq_s G$ and $L \trianglelefteq_s G$.*

Proof. Let $K \cap L \trianglelefteq_s G$. To show $K \trianglelefteq_s G$, let $P \ll G$ such that $K \cap P = (0)$. Now, $(K \cap L) \cap P \subseteq K \cap P = (0)$. Since $K \cap L \trianglelefteq_s G$, we have $P = (0)$. Thus $K \trianglelefteq_s G$. Similarly, $L \trianglelefteq_s G$. Conversely, suppose that $K \trianglelefteq_s G$ and $L \trianglelefteq_s G$. Let $P \ll G$ such that $(K \cap L) \cap P = (0)$. Then $K \cap (L \cap P) = (0)$. Now we show that $K \cap P \ll G$. Let T be a R-ideal of G such that $(K \cap P) + T = G$. Since $K \cap P \subseteq P$, we have $G = (K \cap P) + T \subseteq P + T$. Now $P \ll G$, implies $T = G$. Thus $K \cap P \ll G$. Now, $L \trianglelefteq_s G$ and $K \cap P \ll G$, implies $K \cap P = (0)$. Also $K \trianglelefteq_s G$ and $P \ll G$ implies $P = (0)$. Therefore, $K \cap L \trianglelefteq_s G$. □

Proposition 2.9. *Let $f : G \to G'$ be an N-epimorphism. If $K \trianglelefteq_s G'$, then $f^{-1}(K) \trianglelefteq_s G$.*

Proof. Let $L \ll G$ such that $f^{-1}(K) \cap L = (0)$. To show that $K \cap f(L) = (0)$, let $x \in K \cap f(L)$. Then $x \in K$ and $x \in f(L)$. This implies $x = f(y)$, for some $y \in L$. Then $y = f^{-1}(x) \in f^{-1}(K)$ and $y \in L$. Thus $y \in f^{-1}(K) \cap L = (0)$, and so $y = 0$. Hence $x = f(0) = 0$. Therefore, $K \cap f(L) = (0)$. Now we show that $f(L) \ll G'$. Let T be an N-ideal of G' such that $f(L) + T = G'$. Then $L + f^{-1}(T) = f^{-1}(G') = G$. This implies $f^{-1}(T) = G$, and so $T = f(G) = G'$. Therefore, $f(L) \ll G'$. Now since $K \trianglelefteq_s G_2$ and $K \cap f(L) = (0)$, we get $f(L) = (0)$. Hence $L \subseteq f^{-1}(0) \subseteq f^{-1}(K) \cap L = (0)$. Therefore, $L = (0)$. □

Theorem 2.10. *Suppose that $K_1 \leq_R G_1 \leq_R G$, $K_2 \leq_R G_2 \leq_R G$, and $G = G_1 \oplus G_2$; then $K_1 \oplus K_2 \trianglelefteq_s G_1 \oplus G_2$ if and only if $K_1 \trianglelefteq_s G_1$ and $K_2 \trianglelefteq_s G_2$.*

Proof. Suppose that $K_1 \trianglelefteq_s G_1$. That is, $K_1 \cap L_1 = (0)$, for some $(0) \neq L_1 \ll G_1$. We show that $(K_1 + K_2) \cap L_1 = (0)$. Let $x \in (K_1 + K_2) \cap L_1$. Then $x = k_1 + k_2$ and $x = l_1$, where $k_1 \in K_1$, $k_2 \in K_2$. This implies $l_1 = k_1 + k_2$, and so $k_2 = -k_1 + l_1 \in G_1 \cap G_2 = (0)$. Therefore, $k_2 = (0)$. Hence $l_1 = k_1 \in K_1 \cap L_1 = (0)$. Therefore, $x = 0$. This shows that $(K_1 + K_2) \cap L_1 = (0)$. Now to show $L_1 \ll G_1 + G_2$, let $T \trianglelefteq_R G_1 + G_2$ such that $L_1 + T = G_1 + G_2$. Then $(L_1 + T) \cap G_1 = (G_1 + G_2) \cap G_1$. Now by modular law, since $L_1 \subseteq G_1$, we get $L_1 + (T_1 \cap G_1) = G_1$. Since $L_1 \ll G_1$ and $T \cap G_1 \trianglelefteq_R G_1$, we have $T \cap G_1 = G_1$, and so $G_1 \subseteq T$. Thus, $G_1 + G_2 = L_1 + T \subseteq G_1 + T = T$. Therefore, $T = G_1 + G_2$ shows that

$$L_1 \ll G_1 + G_2 \cdots \qquad (*).$$

Now $K_1 \oplus K_2 \trianglelefteq_s G_1 \oplus G_2$ implies $L = (0)$, a contradiction. Therefore $K_1 \trianglelefteq_s G_1$. In a similar way, it can be proved that $K_2 \trianglelefteq_s G_2$. Conversely, suppose that $K_i \trianglelefteq_s G_i$ and $0 \neq g_i \in G_i$ $(i = 1, 2)$. Then by Proposition 2.5 and by $(*)$ we have $Rg_i \ll G_1 + G_2$. Then by Proposition 2.5, there exists $r_1 \in R$ such that $0 \neq r_1 g_1 \in K_1$. If $r_1 g_2 \in K_2$, then $0 \neq r_1 g_1 + r_1 g_2 \in K_1 \oplus K_2$. If $r_1 g_2 \notin K_2$, then again by Proposition 2.5, there exists an $r_2 \in R$ with $0 \neq r_2 r_1 g_2 \in K_2$, and we have $0 \neq r_2 r_1 g_1 + r_2 r_1 g_2 \in K_1 \oplus K_2$. Then $K_1 \oplus K_2 \trianglelefteq_s G_1 \oplus G_2$. □

3. Generalized essential ideals in $M_n(R)$-group R^n

For a zero-symmetric right nearring R with 1, let R^n will be the direct sum of n copies of $(R, +)$. The elements of R^n are column vectors and written as $(r_1, \cdots, r_n)$. The symbols i_j and π_j respectively, denote the i^{th} coordinate injective and j^{th} coordinate projective maps.
For an element $a \in R$, $i_i(a) = (0, \cdots, \underbrace{a}_{i^{th}}, \cdots, 0)$, and $\pi_j(a_1, \cdots, a_n) = a_j$, for any $(a_1, \cdots, a_n) \in R^n$. The nearring of $n \times n$ matrices over R, denoted by $M_n(R)$, is defined to be the subnearring of $M(R^n)$, generated by the set of functions $\{f_{ij}^a : R^n \to R^n \mid a \in R, 1 \leq i, j \leq n\}$ where $f_{ij}^a(k_1, \cdots, k_n) := (l_1, l_2, \cdots, l_n)$ with $l_i = ak_j$ and $l_p = 0$ if $p \neq i$. Clearly, $f_{ij}^a = i_i f^a \pi_j$, where $f^a(x) = ax$, for all $a, x \in R$. If R happens to be a ring, then f_{ij}^a corresponds to the $n \times n$-matrix with a in position (i, j) and zeros elsewhere.

Notation 3.1. ([6], Notation 1.1)
For any ideal $\mathcal{A}$ of $M_n(R)$-group R^n, we write

$$\mathcal{A}_{**} = \{a \in R : a = \pi_j A, \text{ for some } A \in \mathcal{A}, 1 \leq j \leq n\}, \text{ an ideal of } {}_R R.$$

We denote $M_n(R)$ for a matrix nearring, R^n for an $M_n(R)$-group R^n. We refer to Meldrum & Van der Walt [16] for preliminary results on matrix nearrings.

Theorem 3.2. (Theorem 1.4 of [6]) *Suppose $A \subseteq R$.*

1. *If A^n is an ideal of ${}_{M_n(R)}R^n$, then $A = (A^n)_{\star\star}$.*
2. *If A is an ideal of ${}_R R$ if and only if A^n is an ideal of ${}_{M_n(R)}R^n$.*
3. *If A is an ideal of ${}_R R$, then $A = (A^n)_{\star\star}$.*

Lemma 3.3. (Lemma 1.5 of [6])

1. *If $\mathcal{I}$ is an ideal of ${}_{M_n(R)}R^n$, then $(\mathcal{I}_{\star\star})^n = I$.*
2. *Every ideal $\mathcal{I}$ of ${}_{M_n(R)}R^n$ is of the form K^n for some ideal K of ${}_R R$.*

Remark 3.4. (Remark 1.6 of [6]) Suppose I, J are ideals of ${}_R R$. Then

(i) $(I \cap J)^n = I^n \cap J^n$;

(ii) $I \cap J = (0)$ if and only if $(I \cap J)^n = (\bar{\mathbf{0}})$ if and only if $I^n \cap J^n = (\bar{\mathbf{0}})$.

Lemma 3.5. *If I and J are ideals of R, then $(I+J)^n = I^n + J^n$.*

Proof. Clearly, $I \subseteq I+J$ and $I \subseteq I+J$ which implies $I^n \subseteq (I+J)^n$ and $J^n \subseteq (I+J)^n$ and so $I^n + J^n \subseteq (I+J)^n$. To prove the other part, let $(x_1, x_2, \cdots, x_n) \in (I+J)^n$. Then $x_i \in I+J$ for every $1 \leq i \leq n$ which implies $x_i = a_i + b_i$, where $a_i \in I$ and $b_i \in J$.
Now,

$$\begin{aligned}(x_1, x_2, \cdots, x_n) &= (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n) \\ &= (a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) \\ &\in I^n + J^n\end{aligned}$$

Therefore, $(I+J)^n \subseteq I^n + J^n$. Hence, $(I+J)^n = I^n + J^n$. □

Lemma 3.6. *$I + J = G$ if and only if $(I+J)^n = G^n$ if and only if $I^n + J^n = G^n$.*

Lemma 3.7. (Note 1.7(iii) of [6]) *Let A be an ideal of ${}_RR$. Then $A \leq_e {}_RR$ if and only if $A^n \leq_e {}_{M_n(R)}R^n$.*

Definition 3.8. An ideal $\mathcal{A}$ of $M_n(R)$-group R^n is said to be *superfluous* if for any ideal $\mathcal{K}$ of R^n, $\mathcal{A} + \mathcal{K} = R^n$ implies $\mathcal{K} = R^n$.

Definition 3.9. An ideal $\mathcal{K}$ of $M_n(R)$-group R^n is said to be *s-essential* if for any ideal $\mathcal{A}$ of R^n, $\mathcal{K} \cap \mathcal{A} = (\bar{\mathbf{0}})$ and $\mathcal{A} \ll R^n$ implies $\mathcal{K} = (\bar{\mathbf{0}})$.

Lemma 3.10. *Let K be an ideal of ${}_RR$. If $K \trianglelefteq_s {}_RR$, then $K^n \trianglelefteq_s {}_{M_n(R)}R^n$.*

Proof. Let $K \trianglelefteq_s {}_RR$. To show $K^n \trianglelefteq_s {}_{M_n(R)}R^n$, let $\mathcal{L}$ be an ideal of ${}_{M_n(R)}R^n$ such that $K^n \cap \mathcal{L} = (\bar{\mathbf{0}})$ and $\mathcal{L} \ll {}_{M_n(R)}R^n$. Now to show $\mathcal{L}_{\star\star} \ll {}_RR$, let $B \trianglelefteq {}_RR$ such that $\mathcal{L}_{\star\star} + B = R$. By Lemma 3.6, we have $(\mathcal{L}_{\star\star} + B)^n = R^n$. By Lemma 3.5, we have $(\mathcal{L}_{\star\star})^n + B^n = R^n$. Now by Lemma 3.3, we get $\mathcal{L} = (\mathcal{L}_{\star\star})^n$, which implies $\mathcal{L} + B^n = R^n$. Since $B^n \trianglelefteq {}_{M_n(R)}R^n$ and $\mathcal{L} \ll {}_{M_n(R)}R^n$, we have $B^n = R^n$. Let $n \in R$. Then $(n, 0, \cdots, 0) \in R^n = B^n$. Therefore, $n \in (B^n)_{\star\star} = B$ (by Theorem 3.2(3)). Therefore, $B = R$, and so $\mathcal{L}_{\star\star} \ll {}_RR$. So $K^n \cap \mathcal{L} = (\bar{\mathbf{0}})$ implies $K^n \cap (\mathcal{L}_{\star\star})^n = (\bar{\mathbf{0}})$, and by Remark 3.4 (ii), $K \cap (\mathcal{L}_{\star\star}) = (0)$. Now since $K \trianglelefteq_s R$, we get $\mathcal{L}_{\star\star} = (0)$. Thus $\mathcal{L} = (\mathcal{L}_{\star\star})^n = (\bar{\mathbf{0}})$. This shows that $K^n \trianglelefteq_s {}_{M_n(R)}R^n$. □

Lemma 3.11. *Let $\mathcal{A}$ be an ideal of ${}_{M_n(R)}R$. If $\mathcal{A} \trianglelefteq_s {}_{M_n(R)}R^n$, then $\mathcal{A}_{\star\star} \trianglelefteq_s {}_RR$.*

Proof. Let $\mathcal{A} \trianglelefteq_s {}_{M_n(R)}R^n$. To show $\mathcal{A}_{\star\star} \trianglelefteq_s {}_R R$, let $B \ll {}_R R$ such that $\mathcal{A}_{\star\star} \cap B = (0)$. Then by Remark 3.4, we have $(\mathcal{A}_{\star\star})^n \cap B^n = (\bar{\mathbf{0}})$ and by Lemma 3.3, we have $\mathcal{A} = (\mathcal{A}_{\star\star})^n$, and so $\mathcal{A} \cap B^n = (0)$. Now to show $B^n \ll {}_{M_n(R)}R^n$, let $\mathcal{L} \trianglelefteq {}_{M_n(R)}R^n$ such that $B^n + \mathcal{L} = R^n$. To show $\mathcal{L} = R^n$. Since $\mathcal{L} \trianglelefteq {}_{M_n(R)}R^n$, by Lemma 3.3, we have $\mathcal{L} = (\mathcal{L}_{\star\star})^n$, which implies $B^n + (\mathcal{L}_{\star\star})^n = R^n$. Now using Lemma 3.5, we get $(B + \mathcal{L}_{\star\star})^n = R^n$. Therefore, by Lemma 3.6, $B + \mathcal{L}_{\star\star} = R$, and since $B \ll {}_R R$, we get $\mathcal{L}_{\star\star} = R$. Hence, $\mathcal{L} = (\mathcal{L}_{\star\star})^n = R^n$. This shows that $B^n \ll {}_{M_n(R)}R^n$. Now $\mathcal{A} \trianglelefteq_s {}_{M_n(R)}R^n$ implies $B^n = (\bar{\mathbf{0}})$. Thus $B = (0)$. This shows that $\mathcal{A}_{\star\star} \trianglelefteq_s {}_R R$. □

Theorem 3.12. *There is a one-one correspondence between the set of s-essential ideals of ${}_R R$ and those of $M_n(R)$-group R^n.*

Proof. Let $P = \{A \trianglelefteq {}_R R : A \trianglelefteq_s {}_R R\}$. $Q = \{\mathcal{A} \trianglelefteq {}_{M_n(R)}R^n : \mathcal{A} \trianglelefteq_s {}_{M_n(R)}R^n\}$. Define $\Phi : P \to Q$ by $\Phi(A) = A^n$. Then by Lemma 3.10, $A^n \trianglelefteq_s {}_{M_n(R)}R^n$. Define $\Psi : Q \to P$ by $\Psi(\mathcal{A}) = \mathcal{A}_{\star\star}$. By Lemma 3.11, $\mathcal{A}_{\star\star} \trianglelefteq_s {}_R R$. Now $(\Psi \circ \Phi)(A) = \Psi(\Phi(A)) = \Psi(A^n) = (A^n)_{\star\star} = A$. $(\Phi \circ \Psi)(\mathcal{A}) = \Phi(\Psi(\mathcal{A})) = \Phi(\mathcal{A}_{\star\star}) = (\mathcal{A}_{\star\star})^n = \mathcal{A}$. Therefore, $(\Psi \circ \Phi) = Id_P$ and $(\Phi \circ \Psi) = Id_Q$. □

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