Weakly quasi invo-clean rings

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Abstract. We introduce the notion of weakly quasi invo-clean rings where every element $r$ can be written as $r = v + e$ or $r = v - e$, where $v \in Qinv(R)$ and $e \in Id(R)$. We study various properties of weakly quasi invo-clean elements and weakly quasi invo-clean rings. We prove that the ring $R = \prod_{i \in I} R_i$, where all rings $R_i$ are weakly quasi invo-clean, is weakly quasi invo-clean if and only if all factors but one are quasi invo-clean.

1. Introduction

Let $R$ be an associative ring with identity. An element $v$ of $R$ is said to be an involution if $v^2 = 1$ and a quasi-involution if either $v$ or $1 - v$ is an involution [6]. Let $U(R)$, $Id(R)$, $Nil(R)$, $Z(R)$, $Inv(R)$ and $Qinv(R)$ will denote respectively the set of units, the set of idempotents, the set of nilpotents, the set of centrals, the set of involutions and the set of quasi-involutions of $R$.

The ring $R$ is said to be clean if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$ [1, 8]. The ring $R$ is said to be invo-clean if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ [2, 4, 7]. In [2, Corollary 2.16], it is shown that, if $R$ is an invo-clean ring, then $J(R)$ is nil with index of nilpotence not exceeding 3. In [4, Theorem 2.2], it is proved that, if $R$ is an invo-clean ring, then $eRe$ is also an invo-clean ring for any idempotent $e$ of $R$. In addition, for all $n \in \mathbb{N}$, if $M_n(R)$ is invo-clean, then so is $R$.

The ring $R$ is said to be weakly invo-clean if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$ [3]. In [3, Theorem 4.18], it is shown that, a ring $R$ is weakly invo-clean if, and only if, $R \cong R' \times R''$, where $R'$ is a weakly invo-clean ring which is isomorphic to

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either \( \mathbb{Z}_3 \) or \( \mathbb{Z}_5 \) or can be embedded in a direct product of copies of \( \mathbb{Z}_3 \) and a single copy of \( \mathbb{Z}_5 \), and \( R' \) is either \( \{0\} \) or a nil-clean ring of characteristic at most 8 for which \( z^2 = 2z \) for all \( z \in J(R) \). In particular, any weakly invo-clean ring is clean.

The ring \( R \) is said to be quasi invo-clean if for each \( r \in R \) there exist \( v \in Qinv(R) \) and \( e \in Id(R) \) such that \( r = v + e \). If, in addition \( ve = ev \), \( R \) is said to be strongly quasi invo-clean [5]. In [5, Theorem 2.4], it is proved that, a ring \( R \) is quasi invo-clean if, and only if, \( R \cong R_1 \times R_2 \times R_3 \), where \( R_1 = \{0\} \) or \( R_1 \) is an invo-clean ring of characteristic not exceeding 8 which is nil-clean, \( R_2 = \{0\} \) or \( R_2 \) is a subdirect product of a family of copies of \( \mathbb{Z}_3 \), and \( R_3 = \{0\} \) or \( R_3 \cong \mathbb{Z}_5 \).

In this paper, we introduce the notion of a weakly quasi invo-clean ring as a new generalization of a weakly invo-clean ring and a quasi invo-clean ring. Let \( R \) be a ring. Then an element \( r \in R \) is called weakly quasi invo-clean if there exist \( v \in Qinv(R) \) and \( e \in Id(R) \) such that \( r = v + e \) or \( r = v - e \). A ring \( R \) is called weakly quasi invo-clean if every element of \( R \) is weakly quasi invo-clean. We study various properties of weakly quasi invo-clean elements and weakly quasi invo-clean rings. We show that, every homomorphically image of a weakly quasi invo-clean ring is weakly quasi invo-clean (Lemm 2.10). We prove that, if \( R \) is a weakly quasi invo-clean ring with the strong property and \( 4 = 0 \), then \( R \) is strongly quasi invo-clean (Lemma 2.14).

Finally, we show that the ring \( R = \prod_{i \in I} R_i \), where all rings \( R_i \) are weakly quasi invo-clean, is weakly quasi invo-clean ring if and only if all factors but one are quasi invo-clean (Theorem 2.19).

### 2. Main results

In conjunction with [2], [3] and [5], we start our work in this section with the following basic notion.

**Definition 2.1.** An element \( r \in R \) is said to be an *invo-clean element* if there exist \( v \in Inv(R) \) and \( e \in Id(R) \) such that \( r = v + e \). A ring \( R \) is said to be invo-clean if each element in \( R \) is invo-clean [2].

Simple examples of invo-clean rings that could be plainly verified are these: \( \mathbb{Z}_2 \), \( \mathbb{Z}_3 \) and \( \mathbb{Z}_4 \). Oppositely, \( \mathbb{Z}_5 \) is not invo-clean but however they are clean being finite [2].
Definition 2.2. Let \( R \) be a ring. Then an element \( r \in R \) is said to be \textit{weakly invo-clean} if there exist \( v \in \text{Inv}(R) \) and \( e \in \text{Id}(R) \) such that \( r = v + e \) or \( r = v - e \). A ring \( R \) is said to be weakly invo-clean if every element of \( R \) is weakly invo-clean [3].

Definition 2.3. An element \( v \in R \) is said to be a \textit{quasi-involution element} if \( v^2 = 1 \) or \((1-v)^2 = 1\) [5]. \( Qinv(R) \) denotes the set of all quasi-involutions in \( R \).

Definition 2.4. An element in \( R \) is said to be \textit{quasi invo-clean} if it can be written as the sum of an idempotent and a quasi-involution element. A ring \( R \) is said to be quasi invo-clean if each element in \( R \) is quasi invo-clean [5].

It is evident that invo-clean rings are both weakly invo-clean and quasi invo-clean as this implication is extremely non-reversible by looking quickly at the field \( \mathbb{Z}_5 \).

In the following, we define the weakly quasi invo-clean rings, then we study some of the basic properties of weakly quasi invo-clean rings. Moreover, we give some necessarily examples.

Definition 2.5. Let \( R \) be a ring. Then an element \( r \in R \) is called \textit{weakly quasi invo-clean} if there exist \( v \in Qinv(R) \) and \( e \in Id(R) \) such that \( r = v + e \) or \( r = v - e \). A ring \( R \) is called weakly quasi invo-clean if every element of \( R \) is weakly quasi invo-clean.

Every invo-clean or weakly invo-clean or quasi invo-clean ring is weakly quasi invo-clean. The following example shows that every weakly quasi invo-clean ring is neither weakly invo-clean nor quasi invo-clean nor invo-clean ring, in general.

Example 2.6.

(i) Let \( R = \mathbb{Z}_5 \). Then \( Inv(\mathbb{Z}_5) = \{0, 1, 2, 4\} \), \( Qinv(\mathbb{Z}_5) = \{0, 1, 2, 4\} \) and \( Id(\mathbb{Z}_5) = \{0, 1\} \). Hence \( \mathbb{Z}_5 \) is a weakly quasi invo-clean ring. Since the element 3 of \( \mathbb{Z}_5 \) cannot be expressed as sum of an idempotent and an involution, \( \mathbb{Z}_5 \) is not invo-clean.

(ii) Let \( R = \mathbb{Z}_5 \times \mathbb{Z}_5 \). Then \( R \) is not a weakly invo-clean and not quasi invo-clean ring, by [3, Example 4.16]. Since \( Qinv(\mathbb{Z}_5) = \{0, 1, 2, 4\} \) and \( Id(\mathbb{Z}_5) = \{0, 1\} \), \( R \) is a weakly quasi invo-clean ring.
(iii) Let \( R = \mathbb{Z}_7 \). Then \( \text{Qinv}(\mathbb{Z}_7) = \{0, 1, 2, 6\} \) and \( \text{Id}(\mathbb{Z}_7) = \{0, 1\} \). Since the element 4 of \( \mathbb{Z}_7 \) cannot be expressed as sum or difference of an idempotent and an quasi involution, \( \mathbb{Z}_7 \) is not a (weakly) quasi invo-clean ring.

(iv) Let \( R = \mathbb{Z}_8 \). Then \( \text{Qinv}(\mathbb{Z}_8) = \{0, 1, 2, 5, 6, 7\} \) and \( \text{Id}(\mathbb{Z}_8) = \{0, 1\} \). Hence \( \mathbb{Z}_8 \) is a weakly quasi invo-clean ring. Since the element 4 of \( \mathbb{Z}_8 \) cannot be expressed as sum of an idempotent and an quasi involution, \( \mathbb{Z}_8 \) is not quasi invo-clean.

**Proposition 2.7.** Let \( R \) be a ring and \( r \in R \). Then \( r \) is weakly quasi invo-clean if and only if \( r \) or \( r + 1 \) is quasi invo-clean.

**Proof.** Suppose that \( r \) is weakly quasi invo-clean. Hence \( r = v + e \) or \( r = v - e \) for some \( v \in \text{Qinv}(R) \) and \( e \in \text{Id}(R) \). If \( r = v + e \), then \( r \) is quasi invo-clean. If \( r = v - e \), then \( r + 1 = v - e + 1 = v + (1 - e) \). Conversely, if \( r \) is quasi invo-clean, then it is clear that \( r \) is weakly quasi invo-clean. If \( r + 1 \) is quasi invo-clean, then \( r + 1 = v + e \), where \( v \in \text{Qinv}(R) \) and \( e \in \text{Id}(R) \), and so \( r = v - (1 - e) \). Therefore \( r \) is weakly quasi invo-clean.

**Proposition 2.8.** Let \( R \) be a ring and \( r \in R \). Then \( r \) is weakly quasi invo-clean if and only if \( 1 - r \) or \( 1 + r \) is quasi invo-clean.

**Proof.** Suppose that \( r \) is weakly quasi invo-clean. Hence \( r = v + e \) or \( r = v - e \) for some \( v \in \text{Qinv}(R) \) and \( e \in \text{Id}(R) \). Hence \( 1 - r = 1 - v - e = -v + (1 - e) \) or \( 1 + r = v + (1 - e) \). Conversely, if \( 1 - r \) or \( 1 + r \) is quasi invo-clean, then \( 1 - r = v + e \) or \( 1 + r = v - e \) for some \( v \in \text{Qinv}(R) \) and \( e \in \text{Id}(R) \). Hence \( r = -v + (1 - e) \) or \( r = v - (1 - e) \). Therefore \( r \) is quasi invo-clean.

**Proposition 2.9.** Let \( R \) be a ring and \( r \in R \). Then \( r \) is weakly quasi invo-clean if and only if \( r = v + e \), where \( v \in \text{Qinv}(R) \) or \( 1 + v \in \text{Qinv}(R) \).

**Proof.** Suppose that \( r \) is weakly quasi invo-clean. Hence \( r = v + e \) or \( r = v - e \) for some \( v \in \text{Qinv}(R) \) and \( e \in \text{Id}(R) \). If \( r = v + e \), then \( v \in \text{Qinv}(R) \). If \( r = v - e \), then \( r = (v - 1) + (1 - e) \), where \( 1 + (v - 1) = v \in \text{Qinv}(R) \). Conversely, is clear.

**Lemma 2.10.** Every homomorphic image of a weakly quasi invo-clean ring is weakly quasi invo-clean.
\textbf{Proof.} Since homomorphic images of quasi involutions and idempotents are again quasi involutions and idempotents, respectively, the assertion holds. \hfill \Box

\textbf{Lemma 2.11.} Let \( R \) be a weakly quasi invo-clean ring and \( 3, 7 \in U(R) \). Then \( 120 = 0 \). In particular, \( 30 \in \text{Nil}(R) \).

\textbf{Proof.} Suppose that \( R \) is weakly quasi invo-clean. Hence \( 5 = v + e \) or \( 5 = v - e \) for some \( v \in Qinv(R) \) and \( e \in Id(R) \). If \( 5 = v + e \) and \( v^2 = 1 \), then \( e = 5 - v \), and so \( (5 - v)^2 = 5 - v \). Hence \( 9v = 21 \). Since \( 3 \in U(R) \), \( 3v = 7 \). Then \( 9v^2 = 49 \), and so \( 40 = 0 \). So \( 3 \cdot 40 = 120 = 0 \). If \( 5 = v + e \) and \( (1 - v)^2 = 1 \), then \( e = 5 - v \), and so \( (5 - v)^2 = 5 - v \). Hence \( -13 = 7(1 - v) \), and so \( 169 = 49 \). Then \( 120 = 0 \). If \( 5 = v - e \) and \( v^2 = 1 \), then \( 31 = 11v \), and so \( 961 = 121 \). Hence \( 840 = 0 \). Since \( 7 \in U(R) \), \( 120 = 0 \). If \( 5 = v - e \) and \( (1 - v)^2 = 1 \), then \( -21 = 9(1 - v) \). Since \( 3 \in U(R) \), \( -7 = 3(1 - v) \). Hence \( 49 = 9 \), and so \( 40 = 0 \). Then \( 3 \cdot 40 = 120 = 0 \). Therefore in the every case \( 120 = 0 \). Since \( 30^3 = 120 \cdot 225 = 0, 30 \in \text{Nil}(R) \). \hfill \Box

\textbf{Corollary 2.12.} Let \( R \) be a weakly quasi invo-clean ring and \( 3, 7 \in U(R) \). Then the following statements hold.

(i) \( 5 \in U(R) \) if and only if \( 6 \in \text{Nil}(R) \).

(ii) \( 6 \in U(R) \) if and only if \( 5 \in \text{Nil}(R) \).

\textbf{Proof.} Since \( 1 + \text{Nil}(R) \subseteq U(R) \) and by Lemma 2.11, \( 30 \in \text{Nil}(R) \), the assertion holds. \hfill \Box

\textbf{Lemma 2.13.} Let \( R \) be a weakly quasi invo-clean ring. If \( R \) is strongly indecomposable and \( 4 = 0 \), then \( R \) is quasi invo-clean.

\textbf{Proof.} Suppose that \( r \in R \). Hence \( r = v \) or \( r = v + 1 \) or \( r = v - 1 \) for some \( v \in Qinv(R) \). If \( r = v \) or \( r = v + 1 \), then \( r = v + 0 \) or \( r = v + 1 \), where \( v \in Qinv(R) \) and \( 0, 1 \in Id(R) \). If \( r = v - 1 \) and \( v^2 = 1 \), then \( r = (v - 2) + 1 \), where \( v - 2 \in Qinv(R) \) and \( 1 \in Id(R) \). If \( r = v - 1 \) and \( (1 - v)^2 = 1 \), then \( r = -(1 - v) + 0 \), where \( -(1 - v) \in Qinv(R) \) and \( 0 \in Id(R) \). Then even \( R \) is quasi invo-clean. \hfill \Box

\textbf{Lemma 2.14.} Let \( R \) be a weakly quasi invo-clean ring with the strong property and \( 4 = 0 \), then \( R \) is strongly quasi invo-clean.
Proof. Suppose that \( r \in R \). Hence \( r = v + e \) or \( r = v - e \) with \( ev = ve \) for some \( v \in Qinv(R) \) and \( e \in Id(R) \). If \( r = v + e \), then the assertion holds. If \( r = v - e \) and \( v^2 = 1 \), then \( r = (v - 2e) + e \) and \((v - 2e)e = e(v - 2e)\), where \( v - 2e \in Qinv(R) \) and \( e \in Id(R) \). If \( r = v - e \) and \((1 - v)^2 = 1 \), then \( r = -(1 - v) + (1 - e) \) and \((v - 1)(1 - e) = (1 - e)(v - 1)\), where \(-(1 - v) \in Qinv(R) \) and \(1 - e \in Id(R) \). Then even \( R \) is strongly quasi invo-clean. 

\( \Box \)

**Proposition 2.15.** Let \( R \) be a weakly quasi invo-clean ring and \( 4 = 0 \). Then \( Z(R) \) is quasi invo-clean.

Proof. Suppose that \( R \) is weakly quasi invo-clean and \( z \in Z(R) \). Hence \( z = v + e \) or \( z = v - e \) for some \( v \in Qinv(R) \) and \( e \in Id(R) \). If \( z = v - e \) and \( v^2 = 1 \), then \((z + e)^2 = 1 \), and so \( z^2 + 2ze = 1 - e \). Since \( 4 = 0 \) and \((z^2 + 2ze)^2 = 1 - e \), \( z^4 = 1 - e \). Hence \( e = 1 - z^4 \in Z(R) \). Therefore \( v \in Z(R) \), and so \( z = (v - 2e) + e \), where \((v - 2e)^2 = 1 \) and \( e^2 = 1 \). If \( z = v - e \) and \((1 - v)^2 = 1 \), then \((1 - (z + e))^2 = 1 \), and so \( e = 2z - z^2 - 2ze \). Since \( 4 = 0 \), \( e = z^4 \in Z(R) \) and \( 1 - e \in Z(R) \). Therefore \( v \in Z(R) \), and so \( z = (v - 1) + (1 - e) \), where \((v - 1)^2 = 1 \) and \((1 - e)^2 = 1 \). Similarly, if \( z = v + e \) for some \( v \in Qinv(R) \) and \( e \in Id(R) \), then \( e \in Z(R) \), and so \( v \in Z(R) \). Therefore even \( Z(R) \) is quasi invo-clean. 

\( \Box \)

**Lemma 2.16.** Let \( R \) be a weakly quasi invo-clean ring. If \( R \) is indecomposable and \( 2 \in U(R) \), then \( R \cong Z_3 \) or \( R \cong Z_5 \).

Proof. Assume that \( R \) is a weakly quasi invo-clean ring and \( Id(R) = \{0, 1\} \). Assume that \( r \in R \). Hence \( r = v \) or \( r = v + 1 \) or \( r = v - 1 \) for some \( v \in Qinv(R) \). If \( v^2 = 1 \), then \((1 - v)^2 \in Id(R) = \{0, 1\} \). Hence \( v = 1 \) or \( v = -1 \). Then \( R = \{0, -1, 1, -2, 2\} \). Since \( 2 \in U(R) \), \( 3 = 0 \) or \( 5 = 0 \). Then \( R \cong Z_3 \) or \( R \cong Z_5 \). If \((1 - v)^2 = 1 \), then \((2 - v)^2 \in Id(R) = \{0, 1\} \). Hence \( v = 0 \) or \( v = 2 \). Then \( R = \{0, -1, 1, 2, 3\} \). Since \( 2 \in U(R) \), \( 3 = 0 \) or \( 5 = 0 \). Then \( R \cong Z_3 \) or \( R \cong Z_5 \). 

\( \Box \)

**Corollary 2.17.** Let \( R \) be a weakly quasi invo-clean ring. If \( R \) is indecomposable and \( 3 \in Nil(R) \), then \( R \cong Z_4 \).

Proof. Since \( 1 + Nil(R) \subseteq U(R) \), \( 2 \in U(R) \). Hence \( R \) is a field of three elements, by Lemma 2.16. 

\( \Box \)
Let \( R \) be a ring and \( R M_R \) be an \( R-R \)-bimodule which is a ring possibly without a unity in which \((mn)r = m(nr), (mr)n = m(rn)\) and \((rm)n = r(mn)\) hold for all \( m, n \in M \) and \( r \in R \). The ideal extension of \( R \) by \( M \) is defined to be the additive abelian group \( I(R, M) = R \oplus M \) with multiplication \((r, m)(s, n) = (rs, rn + ms + mn)\).

**Lemma 2.18.** Let \( R \) be a weakly quasi invo-clean ring and \( R M_R \) be an \( R-R \)-bimodule such that for any \( m \in M \) and \( v \in \text{Qinv}(R) \), \( vm + mv + m^2 = 1 \) and \( 4 - 4m - v = 1 \). Then the ideal-extension \( I(R, M) \) of \( R \) by \( M \) is weakly quasi invo-clean.

**Proof.** Suppose that \((r, m) \in I(R, M)\). Hence \( r = v + e \) or \( r = v - e \) for some \( e \in Id(R) \) and \( v \in \text{Qinv}(R) \). Then \((r, m) = (v, m) + (e, 0)\) or \((r, m) = (v, m) - (e, 0)\). It is clear that \((e, 0) \in Id(I(R, M))\). Assume that \( v^2 = 1 \). Hence \((v, m)^2 = (v^2, vm + mv + m^2) = (1, 1)\), and so \((v, m) \in \text{Qinv}(I(R, M))\). If \((1 - v)^2 = 1, ((1, 1) - (v, m))^2 = ((1 - v)^2, 4 - 4m - v) = (1, 1)\), and so \((v, m) \in \text{Qinv}(I(R, M))\). Therefore \( Id(I(R, M))\) is weakly quasi invo-clean.

**Theorem 2.19.** Let \( R = \prod_{i \in I} R_i \), where all rings \( R_i \) are weakly quasi invo-clean. Then \( R \) is weakly quasi invo-clean ring if and only if all factors but one are quasi invo-clean.

**Proof.** Suppose that \( R' = (R_1, R_2, \cdots, R_n) \) is a direct factor of \( R \), where \( n \geq 1 \) and \(|I| \geq n\). Assume that \( R' \) is weakly quasi invo-clean. If \( R_1 \) and \( R_2 \) are not quasi invo-clean, then there exist \( r_1 \in R_1 \) and \( r_2 \in R_2 \) such that \( r_1 \in \text{Qinv}(R_1) + Id(R_1) \) and \( r_2 \in \text{Qinv}(R_2) - Id(R_2) \) but \( r_1 \notin \text{Qinv}(R_1) - Id(R_1) \) and \( r_2 \notin \text{Qinv}(R_2) + Id(R_2) \). Then \( r = (r_1, r_2, 0, \cdots, 0) \notin \text{Qinv}(R) \leq Id(R)\), a contradiction. Conversely, Assume that \( r = (r_1, r_2, \cdots) \in R \). Suppose that \( R_1 \) is weakly quasi invo-clean whereas \( R_i \) is quasi invo-clean for every \( i \neq 1 \). Since \( r_1 \in R_1, r_1 = v_1 - e_1 \) or \( r_1 = v_1 + e_1 \) for some \( v \in \text{Qinv}(R) \) and \( e \in Id(R) \). Since \( R_i \) is quasi invo-clean for every \( i \neq 1, r_i = v_i + e_i \) for some \( v \in \text{Qinv}(R_i) \) and \( e \in Id(R_i) \) for every \( i \neq 1 \). Suppose that \( r_1 = v_1 - e_1 \). Since \( R_i \) is quasi invo-clean for every \( i \neq 1, 1 + r_i = v_i + f_i, \) and so \( r_i = v_i - (1 - f_i) = v_i + e_i \) for every \( i \neq 1 \). Then \( r = (r_1, r_2, \cdots) = (v_1, v_2, \cdots) - (e_1, e_2, \cdots) \). Therefore \( r \) is weakly quasi invo-clean. If \( r_1 = v_1 + e_1 \), then the assertion holds.

The following example shows that the condition all factors but one are quasi invo-clean is essential.
Example 2.20. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_8$. Hence $Id(\mathbb{Z}_8) = \{0, 1\}$ and $Qinv(\mathbb{Z}_8) = \{0, 1, 2, 5, 6, 7\}$. Then $\mathbb{Z}_8$ is weakly quasi invo-clean. Since the element 4 of $\mathbb{Z}_8$ cannot be expressed as sum of an idempotent and an quasi involution, $\mathbb{Z}_8$ is not quasi invo-clean. Since the element $(3, 4)$ of $R$ cannot be expressed as sum or difference of an idempotent and an quasi involution, $R$ is not weakly quasi invo-clean.

Corollary 2.21. Let $R$ be a ring and $n \geq 2$. Then $R^n$ is weakly quasi invo-clean if and only if $R^n$ is quasi invo-clean if and only if $R$ is quasi invo-clean.

Proof. It follows from Theorem 2.19. \qed

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References


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