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Weakly quasi invo-clean rings

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Abstract. We introduce the notion of weakly quasi invo-clean rings where every element r can be written as $r = v + e$ or $r = v - e$, where $v \in Qinv(R)$ and $e \in Id(R)$. We study various properties of weakly quasi invo-clean elements and weakly quasi invo-clean rings. We prove that the ring $R = \prod_{i \in I} R_i$, where all rings R_i are weakly quasi invo-clean, is weakly quasi invo-clean ring if and only if all factors but one are quasi invo-clean.

1. Introduction

Let R be an associative ring with identity. An element v of R is said to be an involution if $v^2 = 1$ and a quasi-involution if either v or $1 - v$ is an involution [6]. Let $U(R)$, $Id(R)$, $Nil(R)$, $Z(R)$, $Inv(R)$ and $Qinv(R)$ will denote respectively the set of units, the set of idempotents, the set of nilpotents, the set of centrals, the set of involutions and the set of quasi-involutions of R.

The ring R is said to be clean if each $r \in R$ can be expressed as $r = u+e$, where $u \in U(R)$ and $e \in Id(R)$ [1, 8]. The ring R is said to be invo-clean if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ [2, 4, 7]. In [2, Corollary 2.16], it is shown that, if R is an invo-clean ring, then $J(R)$ is nil with index of nilpotence not exceeding 3. In [4, Theorem 2.2], it is proved that, if R is an invo-clean ring, then eRe is also an invo-clean ring for any idempotent e of R. In addition, for all $n \in \mathbb{N}$, if $M_n(R)$ is invo-clean, then so is R.

The ring R is said to be weakly invo-clean if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$ [3]. In [3, Theorem 4.18], it is shown that, a ring R is weakly invo-clean if, and only if, $R \cong R' \times R''$, where R' is a weakly invo-clean ring which is isomorphic to

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either $\mathbb{Z}_3$ or $\mathbb{Z}_5$ or can be embedded in a direct product of copies of $\mathbb{Z}_3$ and a single copy of $\mathbb{Z}_5$, and R'' is either $\{0\}$ or a nil-clean ring of characteristic at most 8 for which $z^2 = 2z$ for all $z \in J(R)$. In particular, any weakly invo-clean ring is clean.

The ring R is said to be quasi invo-clean if for each $r \in R$ there exist $v \in Qinv(R)$ and $e \in Id(R)$ such that $r = v + e$. If, in addition $ve = ev$, R is said to be strongly quasi invo-clean [5]. In [5, Theorem 2.4], it is proved that, a ring R is quasi invo-clean if, and only if, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = \{0\}$ or R_1 is an invo-clean ring of characteristic not exceeding 8 which is nil-clean, $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of $\mathbb{Z}_3$, and $R_3 = \{0\}$ or $R_3 \cong \mathbb{Z}_5$.

In this paper, we introduce the notion of a weakly quasi invo-clean ring as a new generalization of a weakly invo-clean ring and a quasi invo-clean ring. Let R be a ring. Then an element $r \in R$ is called weakly quasi invo-clean if there exist $v \in Qinv(R)$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$. A ring R is called weakly quasi invo-clean if every element of R is weakly quasi invo-clean. We study various properties of weakly quasi invo-clean elements and weakly quasi invo-clean rings. We show that, every homomorphic image of a weakly quasi invo-clean ring is weakly quasi invo-clean (Lemm 2.10). We prove that, if R is a weakly quasi invo-clean ring with the strong property and $4 = 0$, then R is strongly quasi invo-clean (Lemma 2.14).

Finally, we show that the ring $R = \prod_{i \in I} R_i$, where all rings R_i are weakly quasi invo-clean, is weakly quasi invo-clean ring if and only if all factors but one are quasi invo-clean (Theorem 2.19).

2. Main results

In conjunction with [2], [3] and [5], we start our work in this section with the following basic notion.

Definition 2.1. An element $r \in R$ is said to be an *invo-clean element* if there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$. A ring R is said to be invo-clean if each element in R is invo-clean [2].

Simple examples of invo-clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$. Oppositely, $\mathbb{Z}_5$ is not invo-clean but however they are clean being finite [2].

Definition 2.2. Let R be a ring. Then an element $r \in R$ is said to be *weakly invo-clean* if there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$. A ring R is said to be weakly invo-clean if every element of R is weakly invo-clean [3].

Definition 2.3. An element $v \in R$ is said to be a *quasi-involution element* if $v^2 = 1$ or $(1-v)^2 = 1$ [5]. $Qinv(R)$ denotes the set of all quasi-involutions in R.

Definition 2.4. An element in R is said to be *quasi invo-clean* if it can be written as the sum of an idempotent and a quasi-involution element. A ring R is said to be quasi invo-clean if each element in R is quasi invo-clean [5].

It is evident that invo-clean rings are both weakly invo-clean and quasi invo-clean as this implication is extremely non-reversible by looking quickly at the field $\mathbb{Z}_5$.

In the following, we define the weakly quasi invo-clean rings, then we study some of the basic properties of weakly quasi invo-clean rings. Moreover, we give some necessarily examples.

Definition 2.5. Let R be a ring. Then an element $r \in R$ is called *weakly quasi invo-clean* if there exist $v \in Qinv(R)$ and $e \in Id(R)$ such that $r = v+e$ or $r = v - e$. A ring R is called weakly quasi invo-clean if every element of R is weakly quasi invo-clean.

Every invo-clean or weakly invo-clean or quasi invo-clean ring is weakly quasi invo-clean. The following example shows that every weakly quasi invo-clean ring is neither weakly invo-clean nor quasi invo-clean nor invo-clean ring, in general.

Example 2.6.

(i) Let $R = \mathbb{Z}_5$. Then $Inv(\mathbb{Z}_5) = \{0, 1, 2, 4\}$, $Qinv(\mathbb{Z}_5) = \{0, 1, 2, 4\}$ and $Id(\mathbb{Z}_5) = \{0, 1\}$. Hence $\mathbb{Z}_5$ is a weakly quasi invo-clean ring. Since the element 3 of $\mathbb{Z}_5$ cannot be expressed as sum of an idempotent and an involution, $\mathbb{Z}_5$ is not invo-clean.

(ii) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then R is not a weakly invo-clean and not quasi invo-clean ring, by [3, Example 4.16]. Since $Qinv(\mathbb{Z}_5) = \{0, 1, 2, 4\}$ and $Id(\mathbb{Z}_5) = \{0, 1\}$, R is a weakly quasi invo-clean ring.

(iii) Let $R = \mathbb{Z}_7$. Then $Qinv(\mathbb{Z}_7) = \{0, 1, 2, 6\}$ and $Id(\mathbb{Z}_7) = \{0, 1\}$. Since the element 4 of $\mathbb{Z}_7$ cannot be expressed as sum or difference of an idempotent and an quasi involution, $\mathbb{Z}_7$ is not a (weakly) quasi invo-clean ring.

(iv) Let $R = \mathbb{Z}_8$. Then $Qinv(\mathbb{Z}_8) = \{0, 1, 2, 5, 6, 7\}$ and $Id(\mathbb{Z}_8) = \{0, 1\}$. Hence $\mathbb{Z}_8$ is a weakly quasi invo-clean ring. Since the element 4 of $\mathbb{Z}_8$ cannot be expressed as sum of an idempotent and an quasi involution, $\mathbb{Z}_8$ is not quasi invo-clean.

Proposition 2.7. *Let R be a ring and $r \in R$. Then r is weakly quasi invo-clean if and only if r or $r + 1$ is quasi invo-clean.*

Proof. Suppose that r is weakly quasi invo-clean. Hence $r = v + e$ or $r = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. If $r = v + e$, then r is quasi invo-clean. If $r = v - e$, then $r + 1 = v - e + 1 = v + (1 - e)$. Conversely, if r is quasi invo-clean, then it is clear that r is weakly quasi invo-clean. If $r + 1$ is quasi invo-clean, then $r + 1 = v + e$, where $v \in Qinv(R)$ and $e \in Id(R)$, and so $r = v - (1 - e)$. Therefore r is weakly quasi invo-clean. □

Proposition 2.8. *Let R be a ring and $r \in R$. Then r is weakly quasi invo-clean if and only if $1 - r$ or $1 + r$ is quasi invo-clean.*

Proof. Suppose that r is weakly quasi invo-clean. Hence $r = v + e$ or $r = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. Hence $1 - r = 1 - v - e = -v + (1 - e)$ or $1 + r = v + (1 - e)$. Conversely, if $1 - r$ or $1 + r$ is quasi invo-clean, then $1 - r = v + e$ or $1 + r = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. Hence $r = -v + (1 - e)$ or $r = v - (1 - e)$. Therefore r is quasi invo-clean. □

Proposition 2.9. *Let R be a ring and $r \in R$. Then r is weakly quasi invo-clean if and only if $r = v + e$, where $v \in Qinv(R)$ or $1 + v \in Qinv(R)$.*

Proof. Suppose that r is weakly quasi invo-clean. Hence $r = v + e$ or $r = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. If $r = v + e$, then $v \in Qinv(R)$. If $r = v - e$, then $r = (v - 1) + (1 - e)$, where $1 + (v - 1) = v \in Qinv(R)$. Conversely, is clear. □

Lemma 2.10. *Every homomorphic image of a weakly quasi invo-clean ring is weakly quasi invo-clean.*

Proof. Since homomorphic images of quasi involutions and idempotents are again quasi involutions and idempotents, respectively, the assertion holds. □

Lemma 2.11. *Let R be a weakly quasi invo-clean ring and $3, 7 \in U(R)$. Then $120 = 0$. In particular, $30 \in Nil(R)$.*

Proof. Suppose that R is weakly quasi invo-clean. Hence $5 = v + e$ or $5 = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. If $5 = v + e$ and $v^2 = 1$, then $e = 5 - v$, and so $(5 - v)^2 = 5 - v$. Hence $9v = 21$. Since $3 \in U(R)$, $3v = 7$. Then $9v^2 = 49$, and so $40 = 0$. So $3 \cdot 40 = 120 = 0$. If $5 = v + e$ and $(1 - v)^2 = 1$, then $e = 5 - v$, and so $(5 - v)^2 = 5 - v$. Hence $-13 = 7(1 - v)$, and so $169 = 49$. Then $120 = 0$. If $5 = v - e$ and $v^2 = 1$, then $31 = 11v$, and so $961 = 121$. Hence $840 = 0$. Since $7 \in U(R)$, $120 = 0$. If $5 = v - e$ and $(1 - v)^2 = 1$, then $-21 = 9(1 - v)$. Since $3 \in U(R)$, $-7 = 3(1 - v)$. Hence $49 = 9$, and so $40 = 0$. Then $3 \cdot 40 = 120 = 0$. Therefor in the every case $120 = 0$. Since $30^3 = 120 \cdot 225 = 0$, $30 \in Nil(R)$. □

Corollary 2.12. *Let R be a weakly quasi invo-clean ring and $3, 7 \in U(R)$. Then the following statements hold.*

- *(i)* $5 \in U(R)$ *if and only if* $6 \in Nil(R)$.
- *(ii)* $6 \in U(R)$ *if and only if* $5 \in Nil(R)$.

Proof. Since $1 + Nil(R) \subseteq U(R)$ and by Lemma 2.11, $30 \in Nil(R)$, the assertion holds. □

Lemma 2.13. *Let R be a weakly quasi invo-clean ring. If R is strongly indecomposable and $4 = 0$, then R is quasi invo-clean.*

Proof. Suppose that $r \in R$. Hence $r = v$ or $r = v + 1$ or $r = v - 1$ for some $v \in Qinv(R)$. If $r = v$ or $r = v + 1$, then $r = v + 0$ or $r = v + 1$, where $v \in Qinv(R)$ and $0, 1 \in Id(R)$. If $r = v - 1$ and $v^2 = 1$, then $r = (v - 2) + 1$, where $v - 2 \in Qinv(R)$ and $1 \in Id(R)$. If $r = v - 1$ and $(1 - v)^2 = 1$, then $r = -(1 - v) + 0$, where $-(1 - v) \in Qinv(R)$ and $0 \in Id(R)$. Then even R is quasi invo-clean. □

Lemma 2.14. *Let R be a weakly quasi invo-clean ring with the strong property and $4 = 0$, then R is strongly quasi invo-clean.*

Proof. Suppose that $r \in R$. Hence $r = v + e$ or $r = v - e$ with $ev = ve$ for some $v \in Qinv(R)$ and $e \in Id(R)$. If $r = v + e$, then the assertion holds. If $r = v - e$ and $v^2 = 1$, then $r = (v - 2e) + e$ and $(v - 2e)e = e(v - 2e)$, where $v - 2e \in Qinv(R)$ and $e \in Id(R)$. If $r = v - e$ and $(1 - v)^2 = 1$, then $r = -(1 - v) + (1 - e)$ and $(v - 1)(1 - e) = (1 - e)(v - 1)$, where $-(1 - v) \in Qinv(R)$ and $1 - e \in Id(R)$. Then even R is strongly quasi invo-clean. □

Proposition 2.15. *Let R be a weakly quasi invo-clean ring and $4 = 0$. Then $Z(R)$ is quasi invo-clean.*

Proof. Suppose that R is weakly quasi invo-clean and $z \in Z(R)$. Hence $z = v + e$ or $z = v - e$ for some $v \in Qinv(R)$ and $e \in Id(R)$. If $z = v - e$ and $v^2 = 1$, then $(z + e)^2 = 1$, and so $z^2 + 2ze = 1 - e$. Since $4 = 0$ and $(z^2 + 2ze)^2 = 1 - e$, $z^4 = 1 - e$. Hence $e = 1 - z^4 \in Z(R)$. Therefore $v \in Z(R)$, and so $z = (v - 2e) + e$, where $(v - 2e)^2 = 1$ and $e^2 = 1$. If $z = v - e$ and $(1 - v)^2 = 1$, then $(1 - (z + e))^2 = 1$, and so $e = 2z - z^2 - 2ze$. Since $4 = 0$, $e = z^4 \in Z(R)$ and $1 - e \in Z(R)$. Therefore $v \in Z(R)$, and so $z = (v - 1) + (1 - e)$, where $(v - 1)^2 = 1$ and $(1 - e)^2 = 1$. Similarly, if $z = v + e$ for some $v \in Qinv(R)$ and $e \in Id(R)$, then $e \in Z(R)$, and so $v \in Z(R)$. Therefore even $Z(R)$ is quasi invo-clean. □

Lemma 2.16. *Let R be a weakly quasi invo-clean ring. If R is indecomposable and $2 \in U(R)$, then $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_5$.*

Proof. Assume that R is a weakly quasi invo-clean ring and $Id(R) = \{0, 1\}$. Assume that $r \in R$. Hence $r = v$ or $r = v + 1$ or $r = v - 1$ for some $v \in Qinv(R)$. If $v^2 = 1$, then $(\frac{(1 - v)}{2}) \in Id(R) = \{0, 1\}$. Hence $v = 1$ or $v = -1$. Then $R = \{0, -1, 1, -2, 2\}$. Since $2 \in U(R)$, $3 = 0$ or $5 = 0$. Then $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_5$. If $(1 - v)^2 = 1$, then $(\frac{(2 - v)}{2}) \in Id(R) = \{0, 1\}$. Hence $v = 0$ or $v = 2$. Then $R = \{0, -1, 1, 2, 3\}$. Since $2 \in U(R)$, $3 = 0$ or $5 = 0$. Then $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_5$. □

Corollary 2.17. *Let R be a weakly quasi invo-clean ring. If R is indecomposable and $3 \in Nil(R)$, then $R \cong \mathbb{Z}_3$.*

Proof. Since $1 + Nil(R) \subseteq U(R)$, $2 \in U(R)$. Hence R is a field of three elements, by Lemma 2.16. □

Let R be a ring and ${}_RM_R$ be an R-R-bimodule which is a ring possibly without a unity in which $(mn)r = m(nr)$, $(mr)n = m(rn)$ and $(rm)n = r(mn)$ hold for all $m, n \in M$ and $r \in R$. The ideal extension of R by M is defined to be the additive abelian group $I(R, M) = R \oplus M$ with multiplication $(r, m)(s, n) = (rs, rn + ms + mn)$.

Lemma 2.18. *Let R be a weakly quasi invo-clean ring and ${}_RM_R$ be an R-R-bimodule such that for any $m \in M$ and $v \in Qinv(R)$, $vm+mv+m^2 = 1$ and $4-4m-v = 1$. Then the ideal-extension $I(R, M)$ of R by M is weakly quasi invo-clean.*

Proof. Suppose that $(r, m) \in I(R, M)$. Hence $r = v + e$ or $r = v - e$ for some $e \in Id(R)$ and $v \in Qinv(R)$. Then $(r, m) = (v, m) + (e, 0)$ or $(r, m) = (v, m) - (e, 0)$. It is clear that $(e, 0) \in Id(I(R, M))$. Assume that $v^2 = 1$. Hence $(v, m)^2 = (v^2, vm + mv + m^2) = (1, 1)$, and so $(v, m) \in Qinv(I(R, M))$. If $(1-v)^2 = 1$, $((1, 1)-(v, m))^2 = ((1-v)^2, 4-4m-v) = (1, 1)$, and so $(v, m) \in Qinv(I(R, M))$. Therefore $Id(I(R, M)$ is weakly quasi invo-clean. □

Theorem 2.19. *Let $R = \prod_{i\in I} R_i$, where all rings R_i are weakly quasi invo-clean. Then R is weakly quasi invo-clean ring if and only if all factors but one are quasi invo-clean.*

Proof. Suppose that $R' = (R_1, R_2, \cdots, R_n)$ is a direct factor of R, where $n \geq 1$ and $|I| \geq n$. Assume that R' is weakly quasi invo-clean. If R_1 and R_2 are not quasi invo-clean, then there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r_1 \in Qinv(R_1) + Id(R_1)$ and $r_2 \in r_2 \in Qinv(R_2) - Id(R_2)$ but $r_1 \notin Qinv(R_1) - Id(R_1)$ and $r_2 \notin r_2 \in Qinv(R_2) + Id(R_2)$. Then $r = (r_1, r_2, 0, \cdots, 0) \notin Qinv(R) \pm Id(R)$, a contradiction. Conversely, Assume that $r = (r_1, r_2, \cdots) \in R$. Suppose that R_1 is weakly quasi invo-clean whereas R_i is quasi invo-clean for every $i \neq 1$. Since $r_1 \in R_1$, $r_1 = v_1 - e_1$ or $r_1 = v_1 + e_1$ for some $v \in Qinv(R)$ and $e \in Id(R)$. Since R_i is quasi invo-clean for every $i \neq 1$, $r_i = v_i + e_i$ for some $v \in Qinv(R_i)$ and $e \in Id(R_i)$ for every $i \neq 1$. Suppose that $r_1 = v_1 - e_1$. Since R_i is quasi invo-clean for every $i \neq 1$, $1 + r_i = v_i + f_i$, and so $r_i = v_i - (1 - f_i) = v_i + e_i$ for every $i \neq 1$. Then $r = (r_1, r_2, \cdots) = (v_1, v_2, \cdots) - (e_1, e_2, \cdots)$. Therefore r is weakly quasi invo-clean. If $r_1 = v_1 + e_1$, then the assertion holds. □

The following example shows that the condition all factors but one are quasi invo-clean is essential.

Example 2.20. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_8$. Hence $Id(\mathbb{Z}_8) = \{0, 1\}$ and $Qinv(\mathbb{Z}_8) = \{0, 1, 2, 5, 6, 7\}$. Then $\mathbb{Z}_8$ is weakly quasi invo-clean. Since the element 4 of $\mathbb{Z}_8$ cannot be expressed as sum of an idempotent and an quasi involution, $\mathbb{Z}_8$ is not quasi invo-clean. Since the element $(3, 4)$ of R cannot be expressed as sum or difference of an idempotent and an quasi involution, R is not weakly quasi invo-clean.

Corollary 2.21. *Let R be a ring and $n \geq 2$. Then R^n is weakly quasi invo-clean if and only if R^n is quasi invo-clean if and only if R is quasi invo-clean.*

Proof. It follows from Theorem 2.19. □

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