On the weight of finite groups

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Abstract. For a finite group $G$, let $W(G)$ denotes the set of the orders of the elements of $G$. In this paper we study $|W(G)|$ and show that the cyclic group of order $n$ has the maximum value of $|W(G)|$ among all groups of the same order. Furthermore we study this notion in nilpotent and non-nilpotent groups and state some inequality for it. Among the result we show that the minimum value of $|W(G)|$ is power of 2 or it pertains to a non-nilpotent group.

1. Introduction

Let $G$ be a finite group. The connection between structure and the set of the orders of the elements of $G$, has been studied in several works. In 1932, Levi and Waerden [4] showed that under some conditions the groups with weight 2 are nilpotent of class at most 3. Later in 1937, Neumann [6] proved that if $W(G) = \{1, 2, 3\}$, then $G$ is an elementary abelian-by-prime order group. Sanov [9] showed that, when $W(G) \subseteq \{1, 2, 3, 4\}$ $G$ is a locally finite group. Novikov and Adjan [7] in 1968 answered negatively to the following question. Does the finiteness of $W(G)$ imply $G$ to be locally finite? In the same line of research Gupta et. al, [3] proved if $W(G) \subseteq \{1, 2, 3, 4, 5\}$ and $W(G) \neq \{1, 5\}$, then $G$ is locally finite. In 2007, D. V. Lytkina [5] showed that for the group $G$, with $W(G) = \{1, 2, 3, 4\}$, either $G$ is an extension of an elementary abelian 3-group by a cyclic or a quaternion group, or it is an extension of a nilpotent 2-group of class 2 by a subgroup of $S_3$. The sum of element orders in finite groups is studied by Amiri, Jafarian Amiri and Isaacs [1]. We denote by $|W(G)|$, the number of element orders of $G$. The group $G$ is $m$-weight group, if $|W(G)| = m$. It is easy to see that if $G$ is trivial, then $|W(G)| = 1$. If $G$ be a non-trivial group then, the weight of $G$ is at least 2. In the following lemma, we state a result about 2-weight group.

2010 Mathematics Subject Classification: Primary 20D15; Secondary 20K01.
Keywords: Weight of group, finite p-group, non-nilpotency property.
Lemma 1.1. Let $G$ be a group, then $G$ is a 2-weight group if and only if $\exp(G) = p$.

Proof. First assumethat, $G$ is a 2-weight group. If $\exp(G) = p$ has two
distinct prime divisors $p$ and $q$, then $\{1, p, q\} \subseteq W(G)$, so $\exp(G)$ must be
a $p$-number for some prime $p$. Now, if $\exp(G) = p^n$, for some $n \geq 2$, then,
$\{1, p, p^2\} \subseteq W(G)$. The converse is trivial. 

2. Preliminary results

This section contains some basic properties on the weight of a finite group.
The following proposition shows the relation of the weight of a direct prod-
uct of a finite number of finite groups with the weights of its factors.

Proposition 2.1. Let $H$ and $K$ be two arbitrary finite groups, then
$$|W(H \times K)| \leq |W(H)| \times |W(K)|,$$
and the equality holds if $(\exp(H), \exp(K)) = 1$.

Proof. Let $m \in W(H \times K)$ then, there exists $(h, k) \in H \times K$, such that
$m = o(h, k) = [o(h), o(k)] = \frac{o(h)}{g_1} \times \frac{o(k)}{g_2} = rs$. Since $[o(h), o(k)]$ is the
least common multiple of $o(h)$ and $o(k)$, and $g_1g_2 = gcd(o(h), o(k))$, on the
other hand $r = \frac{o(h)}{g_1}, s = \frac{o(k)}{g_2}$. So we have $r \in W(H)$ and $s \in W(K)$.
Hence $|W(H \times K)| \leq |W(H)| \times |W(K)|$. Now, if $(\exp(H), \exp(K)) = 1$
and $(r, s) \in W(H) \times W(K)$, then there exist $h \in H$ and $k \in K$ of orders $r$
and $s$, respectively. Therefore, $(h, k)$ is an element of $H \times K$ of order $rs$, so
the result holds. 

Now, using induction in order to prove the following corollary.

Corollary 2.2. Let $G_{i=1}^n$ be a family of finite groups. Then,$|W(\prod_{i=1}^n G_i)| \leq
\prod_{i=1}^n |W(G_i)|$. Furthermore, the equality holds if the exponent of distinct
direct factors are mutully coprime.

It is easy to see that the cyclic group of order $p^{m-1}$, $C_{p^{m-1}}$ is an $m$-
weight group, in which $p$ is an arbitrary prime number, so for every natural
number $n$, there exists a finite group (in fact a finite $p$-group) of weight $m$.

The following theorem gives an upper bound for the weight of a finite
group in terms of its order.
Theorem 2.3. Let $G$ be a finite group of order $n$, then $|W(G)| \leq |W(C_n)|$ and the equality holds if and only if $G \cong C_n$.

Proof. Since the order of each element of $G$ is a divisor of $n$ and $|W(C_n)| = d(n)$, in which $d(n)$ is the number of natural divisors of $n$, it is trivial, such that $|W(G)| \leq |W(C_n)|$. Now, if $|W(G)| = |W(C_n)|$, then $n \in W(G)$ and hence $G \cong C_n$. □

3. Nilpotent groups

In this section, we state some facts on $W(G)$, when $G$ is a nilpotent group. The following proposition gives the upper and lower bound for $W(G)$, when $G$ is a finite nilpotent group.

Proposition 3.1. Let $\mathcal{N}$ be class of nilpotent groups of order $n$, then for each $G \in \mathcal{N}$ we have $2^{\pi(n)} \leq |W(G)| \leq d(n)$, and equality in the first inequality holds if and only if all Sylow subgroups of $G$ has prime exponent.

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $d(n) = (\alpha_1 + 1) \cdots (\alpha_k + 1)$. Let $G$ be a nilpotent group of order $n$, so $G \cong \prod_{i=1}^k S_i$, in which $S_i$ is the Sylow $p_i$-subgroup of $G$ of order $p_i^{\alpha_i}$ ($1 \leq i \leq k$). Now, by Proposition 2.1, we have $|W(G)| = |\prod_{i=1}^k W(S_i)|$. Applying, Theorem 2.3, thus $2^{\pi(n)} \leq |W(S_i)| \leq \alpha_i + 1$, for all $i, 1 \leq i \leq k$. So $2^{\pi(n)} \leq |W(G)| \leq \prod_{i=1}^k (\alpha_i + 1) = d(n)$. Hence, $|W(G)| = 2^{\pi(n)}$ if and only if $\alpha_i = 1$, for all $i, 1 \leq i \leq k$ which is equal to $\exp(S_i) = p_i$, for all $i, 1 \leq i \leq k$.

As an immediate result we have.

Corollary 3.2. Let $G$ be a finite group of order $n$, if $|W(G)| < 2^{\pi(n)}$ then $G$ is non-nilpotent.

Theorem 3.3. Let $G$ be a group of prime weight then $G$ is nilpotent if and only if $G$ is a $p$-group.

Proof. Since $G$ is a nilpotent group we have $G = P_1 \times \cdots \times P_k$ so $W(G) = W(P_1) \cdots W(P_k)$ this implies $k = 1$ hence $G$ is a $p$-group □

Immediate consequence of Theorem 3.3, we get the following corollary.
Corollary 3.4. In the class of all finite groups of prime weight, each group is either a $p$-group or non-nilpotent.

Proposition 3.5. (See [8, Theorem 1]) Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, in which $p_i$'s are distinct prime numbers. Then, every finite group of order $n$ is a nilpotent group if and only if $p_i \nmid p_j^{\beta_j} - 1$, for each $j$, $0 < \beta_j \leq \alpha_j$ and $i \neq j$.

In above proposition such these numbers are called nilpotent numbers.

Now in order to prove our main result, we need the following results.

Lemma 3.6. Every finite nilpotent group of order $n$ is cyclic if and only if $n$ is square free.

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be decomposition of $n$ into prime factors and $G$ be a nilpotent group of order $n$. By Proposition 3.1, we have $2^k \leq \lvert W(G) \rvert \leq \lvert W(C_n) \rvert$, since every nilpotent group of order $n$ is cyclic, so both inequalities are in fact equality and hence $\alpha_i = 1$, for all $i$, $1 \leq i \leq k$. Conversely, let $G$ be a nilpotent group of order $n = p_1 \cdots p_k$. Applying, Proposition 3.1 again, so we have $\lvert W(G) \rvert = 2^k = d(n) = \lvert W(C_n) \rvert$, it implies that $G \cong C_n$.

Using, the above lemma we can prove the following theorem.

Theorem 3.7. Every finite group of order $n$ is cyclic if and only if $n = p_1 \cdots p_k$, in which $p_1 < \cdots < p_k$ and $p_i \nmid p_{i+s} - 1$, where $1 \leq i \leq k - 1$ and $1 \leq s \leq k - i$.

Proof. If every finite group of order $n$ is cyclic, then by Lemma 3.6 and Proposition 3.5, the result holds. If $n = p_1 \cdots p_k$, in which $p_1 < \cdots < p_k$ and $p_i \nmid p_{i+s} - 1$, where $1 \leq i \leq k - 1$ and $1 \leq s \leq k - i$, then every group of order $n$ is nilpotent, so we have $\lvert W(G) \rvert = 2^k = d(n) = \lvert W(C_n) \rvert$ and hence $G \cong C_n$.

4. Non-nilpotent groups

This section is devoted to some results on non-nilpotent groups.

Let $\mathcal{K}_{(n)}$ denote the class of all groups of order $n$.

Definition 4.1. We say that $\mathcal{K}_{(n)}$ has non-nilpotency property if there exists a non-nilpotent group $T$ in $\mathcal{K}_{(n)}$, such that $\min \{ \lvert W(G) \rvert \mid G \in \mathcal{K}_{(n)} \} = \lvert W(T) \rvert$. 
Theorem 4.2. If $K_n$ has non-nilpotency property, then $K_n^{(nm)}$, has also non-nilpotency property, for any natural number $m$, such that $(n,m) = 1$.

Proof. Let $H$ be a nilpotent group of order $nl$, since $(n,l) = 1$ and $H$ is nilpotent, there exist normal subgroups $N$ and $L$ of $H$, such that $|L| = l$, $|N| = n$ and $H = N \times L$. Now, as $N \in K_n$ and $K_n$ has non-nilpotency property, so there is a non-nilpotent group $T$ in $K_n$ such that

$$|W(T)| = \min\{|W(G)| \mid G \in K_n\}$$

so

$$|W(T)| \leq |W(N)|.$$ 

If $E = T \times L$, then $E$ is also a non-nilpotent group, and clearly $|T| = |N| = n$ and $|L| = l$. Now, we have


So, as $E$ is a non-nilpotent group, and $H$ is nilpotent group in $K_{nl}$ and $|W(E)| \leq |W(H)|$, then $K_{nl}$ has non-nilpotency property. \square

Example 4.3. It is easy to see that $K_{(6)}$ has the non-nilpotency property, so $K_{(30)}$ has the non-nilpotency property, we know that

$$K_{(30)} = \{C_{30}, C_3 \times D_{10}, C_5 \times D_6, D_{30}\}$$

and

$$\omega(C_{30}) = 8, \omega(C_3 \times D_{10}) = 6, \omega(C_5 \times D_6) = 6 \text{ and } \omega(D_{30}) = 5.$$ 

Therefore, the minimum weight occurs at the non-nilpotent group $D_{30}$.

In the following lemma, we construct non-nilpotent groups with small enough weights.

Lemma 4.4. Let $p$ and $q$ be two distinct prime numbers and $\alpha \in \text{Aut}(C_q^r)$ be of order $p$. If $\{a_1, \ldots, a_m\}$ be the standard generating set for $C_p^m$, then the semidirect product $C_p^m \rtimes C_q^r$, by the homomorphism $\mu : C_p^m \rightarrow \text{Aut}(C_q^r)$, such that $\mu(a_i) = \alpha$, for each $i, i = 1, \ldots, m$, is a non-nilpotent group with weight at most 4.

Proof. Let $b \neq 0$ and $(0, b) \in C_p^m \rtimes C_q^r$. Clearly $(0, b)^q = (0, b^q) = (0, 0)$ and hence $o(0, b) = q$. So, if $a \neq 0$ and $(a, 0) \in C_p^m \rtimes C_q^r$, we have $(a, 0)^p = (a^p, 0) = (0, 0)$, it implies that $o(a, 0) = p$. 

Now, assume that \( a \neq 0 \) and \( b \neq 0 \), as \((a,b)^{pq} = (0,0)\) and \( o(a,b) \leq pq \), it follows that 
\[
W(C_p^m \ltimes C_q^n) \subseteq \{1, p, q, pq\},
\]
therefore \( C_p^m \ltimes C_q^n \) is a non-nilpotent group with maximum weight 4.

We use the following useful result in the next theorem.

**Proposition 4.5.** (See [2]) For a finite \( p \)-group \( G \), \( \text{Aut}(G) \cong \text{Gl}(n,p) \) if and only if \( G \) is an elementary abelian \( p \)-group of order \( p^n \).

**Theorem 4.6.** The class of \( K_{(n)} \) has non-nilpotency property, for any non-nilpotent natural number \( n \).

**Proof.** As \( n \) is not a nilpotent number according to Proposition 3.5, there exist distinct and prime divisors \( p \) and \( q \) of \( n \) such that
\[
p \mid q^i - 1
\]
Now, we consider \( n = p^m q^r k \) that \((pq,k) = 1\). By Proposition 4.5, we have
\[
|\text{Aut}(C_q^r)| = (q^r - 1)(q^r - q)\cdots(q^r - q^{r-1})
\]
As
\[
p \mid q^i - 1,
\]
thus
\[
p \mid (q^i - 1)q^{r-i} = q^r - q^{r-i}.
\]
Therefore, \( p \mid |\text{Aut}(C_q^r)| \) and hence there exists \( \alpha \in \text{Aut}(C_q^r) \) with \( o(\alpha) = p \).

Now, if \( \{a_1, \ldots, a_m\} \) is standard generator set of \( C_p^m \), we consider homomorphism \( \mu \), such that
\[
\mu : C_p^m \rightarrow \text{Aut}(C_q^r)
\]
given by \( \mu(a_i) = \alpha \) for \( i = 1, \ldots, m \). We get semidirect product \( C_p^m \) and \( C_q^r \) by homomorphism \( \mu \). Then, \( C_p^m \ltimes C_q^r \) is a non-nilpotent group of order \( p^m q^r \). On the other hand by Lemma 4.4, we have
\[
|W(C_p^m \ltimes C_q^r)| \leq 4
\]
So, if \( G \) is a nilpotent group of order \( p^m q^r \), then we have
\[
|W(G)| \geq 2^2 = 4
\]
Thus, we conclude that \( K_{(p^m q^r)} \) has non-nilpotency property. Since \((pq,k) = 1\) and \( p^m q^r k = n \), by Theorem 4.2, \( K_{(n)} \) has non-nilpotency property. \( \Box \)
Theorem 4.7. Let \( n \) be an even number, such that \( n \) is not a power of 2, then \( \mathcal{K}_n \) has the non-nilpotency property.

Proof. Suppose that \( n = 2^{\alpha_1}p^{\alpha_2}q_3^{\alpha_3} \cdots q_r^{\alpha_r} \), for some \( r \geq 2 \). Since 2 is a divisor of \( |\text{Aut}(Z_p^{\alpha_2})| \), we have \( \omega\left(Z_2^{\alpha_1} \rtimes Z_p^{\alpha_2}\right) \subseteq \{1,2,p,2p\} \). Now, let \( G \) be a nilpotent group of order \( n \), thus \( \omega(G) \geq 2^r \), also we have

\[
\omega\left((Z_2^{\alpha_1} \rtimes Z_p^{\alpha_2}) \rtimes Z_3^{\alpha_3} \times \cdots \times Z_{q_r}^{\alpha_r}\right) \leq 4(2^{r-2}) = 2^r
\]

Therefore

\[
\omega\left((Z_2^{\alpha_1} \rtimes Z_p^{\alpha_2}) \rtimes Z_3^{\alpha_3} \times \cdots \times Z_{q_r}^{\alpha_r}\right) \leq \omega(G)
\]

and the results hold.

Example 4.8. \( \mathcal{K}_{(12)} \), \( \mathcal{K}_{(22)} \) and \( \mathcal{K}_{(30)} \) has the non-nilpotency property. We know that \( \mathcal{K}_{(12)} = \{A_4,D_{12},T,C_{12},C_3 \times C_2 \times C_2\} \) in which

\[
T = \langle a,b \mid a^4 = b^3 = 1; a^{-1}ba = b^{-1} \rangle.
\]

We have \( \omega(T) = \omega(D_{12}) = \omega(C_2 \times C_2 \times C_3) = 4 \) also \( \omega(A_4) = 3 \) and \( \omega(C_{12}) = 6 \).

\[
\mathcal{K}_{(22)} = \{C_{22},D_{22}\}, \omega(C_{22}) = 4 \text{ and } \omega(D_{22}) = 3.
\]

\[
\mathcal{K}_{(30)} = \{C_{30},C_3 \times D_{10},C_5 \times D_6,D_{30}\} \text{ (see Theorem 4.2).}
\]

Here, we can prove the main theorem.

Theorem 4.9. Let \( G \) be a finite group of order \( n \), then \( |W(G)| \leq |W(C_n)| \).

If \( \min\{|W(G)| \mid |G| = n\} = m \), then \( m = 2^{\pi(n)} \) or there is a non-nilpotent group \( T \) that \( |T| = n \) and \( |W(T)| = m \). In other words, the class of groups of order \( n \), cyclic group \( C_n \) has the most weight and if the least weight on the above groups equals \( m \), then \( m \) is a power of 2, such that the power equals to numbers of distinct prime factors of \( n \). Therefore \( m \) is the weight of a non-nilpotent group.

Proof. Let \( C_n \) be a cyclic group of order \( n \). If \( m \) is a divisor of \( n \), then \( m \in W(G) \) and it follows that

\[
\{m \in \mathbb{Z} \mid m > 0, m \mid n\} \subseteq W(C_n).
\]

Now, if \( G \) is a group of order \( n \) and \( m \in W(G) \), then \( m \mid n \) and hence \( W(G) \subseteq \{m \in \mathbb{Z} \mid m > 0, m \mid n\} \).

Thus, \( W(G) \subseteq W(C_n) \), and so we have

\[
|W(G)| \leq |W(C_n)|.
\]
For the finite group $G$ if $n$ is a nilpotent number, then
\[ |W(G)| \geq 2^{|\pi(n)|}, \]

If $n$ is not a nilpotent number, then $\mathcal{K}(n)$ has nonnilpotency property. So, there exists a nonnilpotent group $T$ in $\mathcal{K}(n)$, such that for every group $G$ in $\mathcal{K}(n)$, we have
\[ |W(T)| \leq |W(G)|. \]

Hence
\[ |W(T)| = \min \{|W(G)| \mid G \in \mathcal{K}(n)\}, \]

Therefore, the proof is completed \[\square\]

References


Received December 23, 2021

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