Generalized Green’s relations and $GV$-ordered semigroups

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Abstract In this paper an extensive study of the concepts of generalized Green’s relations and $GV$-semigroups without order to ordered semigroups have been given. Our approach allows one to see the nature of generalized Green’s relations in the class of $GV$-ordered semigroups. Moreover we show that an ordered semigroup $S$ is a $GV$-ordered semigroup if and only if $S$ is a complete semilattice of completely $\pi$-regular and Archimedean ordered semigroups.

1. Introduction and preliminaries

An ordered semigroup $S$ is a partially ordered set $(S, \leq)$ and at the same time a semigroup $(S, \cdot)$ such that for all $a, b$ and $c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup $S$ and $H \subseteq S$, denote the downward closure of $H$ by $\downarrow H = \{ t \in S : t \leq h, \text{ for some } h \in H \}$. Throughout this paper $S$ will stand for an ordered semigroup unless otherwise stated.

An ordered semigroup $S$ is said to be Archimedean if for every $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in (SbS)$. A nonempty subset $I$ of $S$ is said to be a left (resp. right) ideal of $S$, if $SI \subseteq I$ (resp. $IS \subseteq I$) and $I \subseteq I$. If $I$ is both a left and right ideal, then it is called an ideal of $S$. We call $S$ a (resp. left, right) simple ordered semigroup if it does not contain any proper (resp. left, right) ideal. We denote by $R(x), L(x), I(x)$ the right ideal, left ideal, ideal of $S$, respectively, generated by $x$ $(x \in S)$, where $R(x) = (x \cup xS), L(x) = (x \cup Sx), I(x) = (x \cup xS \cup Sx \cup SxS)$ for all $x \in S$. For an ordered semigroup $(S, \cdot, \leq)$, we denote $S^1 = S \cup \{1\}$, where 1 is a symbol, such that $1a = a$, $a1 = a$ for each $a \in S$ and $1 \cdot 1 = 1$. An ordered semigroup $S$ is said to be regular (resp. completely regular) ordered semigroup if for every $a \in S$, $a \in (aSa]$ (resp. $a \in (a^2Sa^2]$). An ordered semigroup $S$ is called $\pi$-regular (resp. completely $\pi$-regular) if for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \in (a^mSa^m]$ (resp. $a^m \in (a^{2m}Sa^{2m}]$). The set of all regular, completely regular and $\pi$-regular elements in an ordered semigroup $S$ are denoted by $\text{Reg}_\leq(S), \text{Gr}_\leq(S)$ and $\pi\text{Reg}_\leq(S)$ respectively.

The class of completely regular ordered semigroups is a subclass of the class of regular ordered semigroups. Galbiati and Veronesi [3] studied class of semigroups

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(without order), where these two notion coincides. These semigroups are named after them as GV-semigroups. In this paper we extend the notion of GV-semigroups to ordered semigroups.

In a semigroup (without a partial order) Green’s relations use to play a significant role to study regular semigroups. Following L. Marki, O. Steinfeld [10] and J.L. Galbiati, M.L. Veronesi [3] we generalized Green-Kehayopulu relations [7], to study \(\pi\)-regular, completely \(\pi\)-regular and GV-ordered semigroups.

Due to Kehayopulu [7] Green’s relations on a regular ordered semigroup given as follows: \(aLb\) if \(L(a) = L(b)\), \(aRb\) if \(R(a) = R(b)\), \(aJb\) if \(I(a) = I(b)\), \(H = L \cap R\).

These four relations \(L, R, J\), and \(H\) are equivalence relations.

A congruence \(\rho\) on \(S\) is called a semilattice congruence if for every \(a, b \in S\), \(a \rho a^2\) and \(ab \rho ba\). By a complete semilattice congruence we mean a semilattice congruence \(\sigma\) on \(S\) such that for \(a, b \in S\), \(a \sigma b\) implies that \(a \sigma ab\). If \(\sigma\) is a semilattice congruence on \(S\), then \((x)_\alpha\) is a subsemigroup for any \(x \in S\).

An ordered semigroup \(S\) is called a complete semilattice of subsemigroups of type \(\tau\) if there exists a complete semilattice congruence \(\rho\) such that each \(\rho\)-congruence class \((x)_\alpha\) is a type \(\tau\) subsemigroup of \(S\). Equivalently [9], there exist a semilattice \(Y\) and a family of subsemigroups \(\{S_\alpha\}_{\alpha \in Y}\) of type \(\tau\) of \(S\) such that:

1. \(S_\alpha \cap S_\beta = \phi\) for any \(\alpha, \beta \in Y\) with \(\alpha \neq \beta\),
2. \(S = \bigcup_{\alpha \in Y} S_\alpha\),
3. \(S_\alpha S_\beta \subseteq S_{\alpha \beta}\) for any \(\alpha, \beta \in Y\),
4. \(S_\beta \cap (S_\alpha) \neq \phi\) implies \(\beta \preceq \alpha\), where \(\preceq\) is the order of the semilattice \(Y\) defined by \(\preceq := \{(\alpha, \beta) \mid \beta = \alpha \beta (\beta \alpha)\}\).

An element \(e \in S\) is called an ordered idempotent [5] if \(e \preceq e^2\). We denote the set of all ordered idempotents of an ordered semigroup \(S\) by \(E_\preceq(S)\). An element \(b \in S\) is inverse of \(a \in S\) if \(a \preceq aba\) and \(b \preceq bab\). We denote the set of all ordered inverses of an element \(a\) of an ordered semigroup \(S\) by \(V_\preceq(a)\).

The zero of an ordered semigroup \((S, \cdot, \preceq)\) is an element of \(S\), usually denoted by 0, such that \(0 \preceq x\) and \(0.x = x.0 = 0\) for all \(x \in S\). An ordered semigroup \(S\) with 0 is called nil if for every \(a \in S\) there is \(n \in \mathbb{N}\) such that \(a^n = 0\).

Cao and Xu [4] defined a nil-extension of an ordered semigroup as follows:

Let \(I\) be an ideal of an ordered semigroup \(S\). Then \((S/I, \cdot, \preceq)\) is called the Rees factor ordered semigroup of \(S\) modulo \(I\), and \(S\) is called an ideal extension of \(I\) by the ordered semigroup \(S/I\). Moreover \(S\) is said to be a nil-extension of \(I\) if \((S/I, \cdot, \preceq)\) is a nil ordered semigroup.
2. Main results

Let $S$ be a $\pi$-regular ordered semigroup. Following Galbiati and Veronesi [3], let us define the relations $\mathcal{L}^*, \mathcal{R}^*, \mathcal{J}^*, \mathcal{H}^*$ by: For $a, b \in S$,

- $a\mathcal{L}^* b$ if and only if $a^m L b^n$
- $a\mathcal{R}^* b$ if and only if $a^m R b^n$
- $a\mathcal{J}^* b$ if and only if $a^m J b^n$
- $a\mathcal{H}^* b$ if and only if $a^m H b^n$

where $m, n$ are the smallest positive integers such that $a^m, b^n \in \text{Reg}_S(S)$. These four relations are equivalence relations on $S$.

We denote $\mathcal{L}^*(a), \mathcal{R}^*(a), \mathcal{H}^*(a)$, and $\mathcal{J}^*(a)$ respectively the $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, and $\mathcal{J}^*$-classes containing an element $a$ of $S$.

Lemma 3.1. Let $S$ be a $\pi$-regular ordered semigroup. Every $\mathcal{L}^*$ ($\mathcal{R}^*$, $\mathcal{J}^*$)-class contains at least one ordered idempotent.

Proof. Let $L$ be a $\mathcal{L}^*$-class and $a \in L$. Let $m$ be the smallest positive integer such that $a^m \leq a^m xa^m$, for some $x \in S$. This implies $xa^m \leq (xa^m)^2$. Therefore $xa^m \in E_S(S)$. We have to show that $a\mathcal{L}^* xa^m$. Let $y = xa^m$. Now $a^m \leq a^m xa^m \leq a^m xa^m xa^m \leq a^m (xa^m)^2$, so that for every $r \in \mathbb{N}$, $a \leq a^m (xa^m)^r$. Let $a^m \leq a^m (xa^m)^{r_1}$, where $r_1$ is the smallest positive integer such that $(xa^m)^{r_1} \in \text{Reg}_S(S)$. Now $y^{r_1} = xa^m \ldots xa^m = (xa^m \ldots x)a^m = pa^m$ where $p = xa^m \ldots x \in S$. Therefore $a\mathcal{L}^*(a)$. This implies $xa^m \in L$. Therefore $L$ contains an ordered idempotent. \hfill $\Box$

Proposition 3.2. Let $S$ be a $\pi$-regular ordered semigroup and $a, b \in S$. Then the following statements hold in $S$:

1. $a\mathcal{L}^* b$ if and only if there exists $a' \in V_S(a^p)$ and $b' \in V_S(b^q)$ such that $a' a^p \mathcal{L}^* b' b'^q$ where $p, q$ are the smallest positive integers such that $a'^p, b'^q \in \text{Reg}_S(S)$.

2. $a\mathcal{R}^* b$ if and only if there exists $a' \in V_S(a^p)$ and $b' \in V_S(b^q)$ such that $a'^p a^p \mathcal{R}^* b' b'^q$ where $p, q$ are the smallest positive integers such that $a'^p, b'^q \in \text{Reg}_S(S)$.

3. $a\mathcal{H}^* b$ if and only if there exists $a'' \in V_S(a^m)$ and $b'' \in V_S(b^n)$ such that $a'' a^m \mathcal{L}^* b'' b''$ and $a^n a'' \mathcal{R}^* b'' b''$ where $m, n$ are the smallest positive integers such that $a^m, b^n \in \text{Reg}_S(S)$.

Proof. We proof only the last condition. Two first conditions follows similarly.

(3): Let $a\mathcal{H}^* b$. Then $a^n a^n \mathcal{H}^* b^n$ where $m, n$ are the smallest positive integer such that $a^m, b^n \in \text{Reg}_S(S)$. Since $a^m \in \text{Reg}_S(S)$ there exists $a' \in S$ such that $a^m \leq a^m a' a^m$. Clearly $a^m a', a'a^m \in E_S(S)$. Let $e = a' a^m$ and $f = a^m a'$. Then $e\mathcal{J}^* a^m$ and $f\mathcal{R}^* a^m$. So that $e\mathcal{J}^* a\mathcal{L}^* b$ and $f\mathcal{R}^* a\mathcal{R}^* b$. Since $b^n \in \text{Reg}_S(S)$ then
there exists $b' \in S$ such that $b^n \leq b' b' b$. Let $e_1 = b' b'$ and $f_1 = b' b'$. Then $e_1, f_1 \in E_\sigma(S)$. Clearly $e_1 L^* b L^* a$ and $f_1 R^* b R^* a$. Since $e L^* a$ we have $e \leq x_1 a^m$ for some $x_1 \in S^1$. Also $a^{m}e \leq a^{m}e x_1 a^{m} \leq a^{m}e x_1 f a^{m} \leq a^{m}a''a^{m}$ and $a'' = e x_1 f \leq e (e x_1 f) \leq e(e) a'' \leq e(x_1 a^{m})a'' \leq (e x_1 f) a^{m}a'' = d'' a''a''$. Therefore $a'' \in V_\sigma(a^{m})$. Therefore $e L^* a''a''$.

Also $e_1 L^* b$ gives $e_1 \leq x_2 b$ for some $x_2 \in S^1$. Also $b^n \leq b^n e_1$ and $b^n \leq f_1 b^n$. Take $b'' = e_1 x_2 f_1$. Now $b'' \leq b'' e_1 \leq b'' x_2 b^n \leq (b'' e_1) x_2 b^n \leq b'' e_1 x_2 (f_1 b^n) \leq b'' (e_1 x_2 f_1) b^n \leq b'' b^n b^n$ and $b'' = e_1 x_2 f_1 \leq e_1 (e_1 x_2 f_1) \leq e_1 e_1 b'' \leq e_1 (e_1 x_2 f_1) b'' \leq e_1 x_2 f_1 b'b'' \leq b'' b'b''$. Therefore $b'' \in V_\sigma(b^n)$. Therefore $e_1 L b'' b^n$. Thus a $a'' a'' \in L b'' L_1 b'' b^n$. Similarly $a'' a'' \in R b'' b''$. Hence the proof.

Conversely assume that the given conditions hold in $S$. Since $a'' a'' \in R b'' b''$ and $a'' a'' \in L b'' b''$ for some $a'' \in V_\sigma(a^{m})$, $b'' \in V_\sigma(b^n)$, then there are $x, y, z, w \in S^1$ such that $a'' a'' \leq (b'' b'')x$, $b'' b'' \leq (a'' a'')y$, $a'' a'' \leq z(b'' b'')$ and $b'' b'' \leq w(a'' a'')$. Since $a'' \in V_\sigma(a^{m})$, we have $a'' \leq a'' a'' a'' \leq (b'' b'') a'' a'' \leq b'' u$, where $u = b'' x a'' \in S$. Again $a'' \leq a'' a'' a'' \leq a'' (z b'' b'') \leq w_1 b''$ where $w_1 = a'' b'' \in S$. Similarly taking $b'' \in V_\sigma(b^n)$ it can shown that $b'' \leq a'' w_2$ and $b'' \leq w_2 a''$ for some $w_2, w_3 \in S$. Therefore $a'' \in H b''$ and hence $a H b$.

We now generalize the concept of GV-semigroups (without order) to ordered semigroups. Some interesting interplays between GV-ordered semigroups and generalized Green’s relations have been given here.

**Definition 3.3.** An ordered semigroup $S$ is said to be a **GV-ordered semigroup** if $S$ is $\pi$-regular and $Reg_\sigma(S) = Gr_\sigma(S)$.

**Example 3.4.** The set $S = \{a, b, c, d\}$ with respect to the multiplication ‘.’ and the order ‘$\leq$’ defined below forms a GV-ordered semigroup.

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$\leq_s = \{(a, a), (a, b), (b, b), (c, b), (c, c), (d, b), (d, d)\}$.

For an $e \in E_\sigma(S)$, Bhuniya and Hansda [1] introduced the set

$$G_e = \{a \in S : a \leq ea, a \leq ae \text{ and } e \leq za, e \leq az \text{ for some } z \in S\}.$$  

They showed that $G_e$ is a t-simple subsemigroup in a completely regular ordered semigroup.

**Lemma 3.5.** Let $S$ be a GV-ordered semigroup. Then for every $a \in S$ there exists $e \in E_\sigma(S)$ and $z \in G_e$ such that $a^n \leq a^m e$, $a^m \leq ea^n$, $e \leq za^n$, and $e \leq a^m z$. 

Proof. Let $S$ be a GV-ordered semigroup, then $S$ is $\pi$-regular and $Reg_{\pi}(S) = Gr_{\pi}(S)$. Let $a \in S$. Then $a^n \in Reg_{\pi}(S) = Gr_{\pi}(S)$ for some $m \in \mathbb{N}$. Therefore $S$ is a completely $\pi$-regular ordered semigroup. Therefore by [[11], Lemma 3.7] the result follows. \hfill \square

**Theorem 3.6.** Let $S$ be a GV-ordered semigroup. Then $G_e \subseteq H^*(e) \subseteq J^*(e)$ for every $e \in E_{\pi}(S)$.

**Proof.** Let $S$ be a GV-ordered semigroup and $e \in E_{\pi}(S)$. Consider the subsemigroup $G_e$ and $a \in G_e$, $y \in V_{\pi}(a)$ in $G_e$. Therefore $a \leq ue, a \leq ev, e \leq za$ for some $u, v, w, z \in G_e$ and $a \leq aya$. Now $ya \leq (yu)e, ay \leq e(ay)$, $e \leq za \leq (za)ya, e \leq au \leq ay(au)$. Therefore we have $yaLe$ and $ayRe$. Hence $yaLee$ and $ayRee$. Therefore we have $aH^*e$, by Proposition 3.2. Hence $a \in H^*(e)$. Therefore $G_e \subseteq H^*(e)$.

Next, let $a \in H^*(e)$. Then $a^nHHe$ where $n$ is the smallest positive integer such that $a^n \in Reg_{\pi}(S)$. Therefore $H^*(e) \subseteq J^*(e)$. Hence the proof. \hfill \square

**Corollary 3.7.** Let $S$ be a GV-ordered semigroup. Then for every $a \in S$ there is $e \in E_{\pi}(S)$ such that $a^n \in G_e \subseteq H^*(e) \subseteq J^*(e)$ for some $m \in \mathbb{N}$.

**Proof.** This follows from Lemma 3.5 and Lemma 3.6. \hfill \square

**Corollary 3.8.** Let $S$ be a GV-ordered semigroup. Then for every $a \in S$ there exists $e \in E_{\pi}(S)$ such that $J^*(a) = J^*(e)$.

**Proof.** This follows from Corollary 3.7. \hfill \square

**Lemma 3.9.** Let $S$ be a GV-ordered semigroup. Then for all $a \in S$, $J^*(a) = J^*(a^2)$.

**Proof.** Let $S$ be a GV-ordered semigroup and $a \in S$. Let $m$ be the smallest positive integer such that $a^n \in Reg_{\pi}(S) = Gr_{\pi}(S)$. Then there is $x \in S$ such that $a^n \leq a^{2m}xa^{2m}$. Let $k$ be the smallest positive integer such that $a^{2k} \in Reg_{\pi}(S)$. Then $k \leq m$, as $a^{2m} \in Reg_{\pi}(S)$. Let $m = k + t$ for some $t \in \mathbb{N}$. So $a^{m} \leq a^{2m}xa^{2m} \leq a^{2m}xa^{2m}xa^{3m} \leq a^{2m}xa^{2m}a^{2t}xa^{3m}$. Also $a^{2k} \leq a^{4k}za^{4k}$ for some $z \in S$. This implies $a^{2k} \leq a^{4k}za^{4k} \leq a^{2k}a^{2k}za^{4k} \leq a^{2k}a^{2k}za^{4k}za^{4k} \leq \ldots \leq wa^{mk}u = wa^{mk-m}a^m = wa^{m(k-1)}a^m$ for some $w, u \in S$. Thus $a^{2k} \in (Sa^mS]$. Therefore $aJ^*a^2$. \hfill \square

**Corollary 3.10.** Let $S$ be a GV-ordered semigroup. Then for all $a \in S$, $J^*(a) = J^*(a^m)$ for all $m \in \mathbb{N}$.

**Lemma 3.11.** Let $S$ be a GV-ordered semigroup. Then for all $a, b \in S$, $J^*(ab) = J^*(ba)$. 

Proof. Let $S$ be a GV-ordered semigroup and $a, b \in S$. Let $m, t$ be the smallest positive integers such that $(ba)^t, (ab)^m \in \text{Reg}_< (S)$. Now $(Sba)^t S = (Sba)^{2t} S$ as $S$ is a GV-ordered semigroup. Also $(Sba)^{2t} S \subseteq (S(ab)^m S$. If $t \geq m$, then $(Sba)^t S \subseteq (S(ab)^t S \subseteq (S(ab)^{m} S$. If $t < m$, then $(ba)^t \leq (ba)^{2t} x (ba)^{2t} \leq (ba)^{2t} x (ba)^{2t} x (ba)^{2t}$. Proceed on we get $(ba)^r \in (Sba)^{2t} S$ for all $r \in \mathbb{N}$. In particular $r = m + 1$. $(Sba)^t S \subseteq (Sba)^{m+1} S \subseteq (Sba)^{(m+1)} t (m+1) (ba)^{m+1} S \subseteq (Sba)^{(m+1)} t (m+1) (ba)^{m+1} S \subseteq (Sba)^{m+1} S \subseteq (Sba)^m S$. Similarly we can prove that $(Sba)^m S \subseteq (Sba)^t S$. Therefore $ab, ba$.

Lemma 3.12. Let $S$ be a GV-ordered semigroup. Then $aH^*a^n$ where $n$ is the smallest positive integer such that $a^n \in \text{Reg}_< (S)$.

Proof. Let $S$ be a GV-ordered semigroup and $a \in S$. Let $n$ be the smallest positive integer such that $a^n \in \text{Reg}_< (S) = Gr_<(S)$, as $S$ is a GV-ordered semigroup. Then there exists $a' \in S$ such that $a^n \leq a^{2m} a^{2n} \leq \ldots \leq a^{kn} x a^{kn}$, for some $x \in S$ and for all $k \in \mathbb{N}$. Let $r$ be the smallest positive integer such that $(a^n)^r \in \text{Reg}_< (S)$. Then there exists $a'' \in S$ such that $(a^n)^r \leq (a^n)^r (a^n)^r \leq y_1 a^n$, where $y_1 = (a^n)^r a'' a^{nr-n} \in S$. Similarly $(a^n)^r \leq a^n y_2$. Also we have $a^n \leq a^n y_3, a^n \leq y_3 a^n$ for some $y_3, y_4 \in S$. Therefore $a^n H (a^n)^r$, that is, $aH^*a^n$. □

In the following theorem the class of GV-ordered semigroups have been characterized by their subsemigroups which are both Archimedean and completely $\pi$-regular.

Theorem 3.13. Let $S$ be an ordered semigroup. Then the following conditions are equivalent:

1. $S$ is a GV-ordered semigroup,
2. $S$ is completely $\pi$-regular and every $H^*$-class of $S$ contains an ordered idempotent,
3. $S$ is a complete semilattice of completely $\pi$-regular and Archimedean ordered semigroups,
4. For all $a, b \in S$, there exist $n \in \mathbb{N}$ such that $(ab)^n \in ((ab)^{n+1} S a (ab)^{n+1})$,
5. For all $a, b \in S$, there exist $n \in \mathbb{N}$ such that $(ab)^n \in ((ab)^{n+1} b S (ab)^{n+1})$.

Proof. (1) $\Rightarrow$ (2): Let $S$ be a GV-ordered semigroup. Consider a $H^*$-class $H^*$ and $a \in H^*$. Let $m$ be the smallest positive integer such that $a^m \in \text{Reg}_< (S) = Gr_<(S)$. Then there exists $x \in S$ such that $a^m \leq a^{2m} x a^{2m}$. Let $e = a^{2m} x a^{2m} x a^{2m}$. Then $e = a^{2m} x a^{2m} x a^{2m} \leq a^{2m} x a^{2m} a^{2m} x a^{2m} \leq a^{2m} x a^{2m} x a^{2m}$, $a^{2m} x a^{2m} \leq e(a^{2m} x a^{2m} x a^{2m}) = e^2$. Thus $e \in E_<(S)$.

Now $a^m \leq a^{2m} x a^{2m} \leq a^{m}(a^{2m} x a^{2m} x a^{2m}) = a^m e$ and $a^m \leq a^{2m} x a^{2m} \leq (a^{2m} x a^{2m} x a^{2m}) a^m = e a^m$. Also $e = a^{2m} x a^{2m} x a^{2m} = (a^{2m} x a^{2m} x a^{2m}) a^m = y a^m$ and $e = a^m (a^{2m} x a^{2m} x a^{2m}) = a^m z$ for some $y = a^{2m} x a^{2m} x a^{2m}$ and $z =$
Therefore \( a^m x a^m e x a^m e \in S \). Therefore \( a^m \leq a^m e \leq a^m e a^m , a^m \leq e a^m \leq e a^m a^m \) for some \( n \in \mathbb{N} \). And \( e^n = e \ldots e(n \text{ times}) = (ya^n \ldots y)a^m , e^n = e \ldots e(n \text{ times}) \leq a^m (z \ldots a^m z) \). Therefore \( e^n H a^m \). Hence \( e H^* a \) and therefore \( e \in H^*(a) \). Therefore \( H^* \)-class contains an ordered idempotent. Since \( S \) is a GV-ordered semigroup, therefore it is completely \( \pi \)-regular.

(2) \( \implies \) (3): Let \( a \in S \). Consider an \( H^* \)-class \( H^*(a) \). Then there exists an ordered idempotent \( e \in H^*(a) \). Therefore \( e^a \leq x a^m , e^a \leq a^m y , a^m \leq e^a y \) for some \( x , y , u , v \in S^1 \) and \( m \) is the smallest positive integer such that \( a^m \in Reg_\varnothing(S) \). Since \( S \) is completely \( \pi \)-regular, \( (a^m)^k \leq ((a^m)^k y_1 (a^m)^k p) \) for all \( p \in \mathbb{N} \) and for some \( y_2 \in S , k \in \mathbb{N} \). Now \( e^a \leq e^a x a^m \leq x a a^m -1 \). So \( a \mid e \). Therefore \( e \leq x_1 a y_1 \) for some \( x_1 , y_1 \in S^1 \). \( e \leq e^2 \leq x_1 a y_1 e^a \leq x_1 a y_1 e^a e^a \leq x_1 a y_1 e^a a^m y \leq \ldots \leq x_1 a y_1 (a^m)^2 y_1 \) for some \( y_1 \in S \). Therefore we have \( e \leq x_1 a y_1 a^m a^m y a^m y \leq \ldots \leq x_1 a y_1 (a^m)^2 y_1 \). Therefore \( a^2 \mid e \). Thus \( S \) is a \( \pi \)-regular ordered semigroup and for all \( a \in S \), \( e \in E_\varnothing(S) \), \( a \mid e \) implies \( a^2 \mid e \). Therefore by [[2], Theorem 4.1], \( S \) is a complete semilattice \( Y \) of ordered semigroups \( \{S_\alpha \}_{\alpha \in Y} \). \( S_\alpha \) is a nil-extension of simple and \( \pi \)-regular ordered semigroups \( \{K_{\alpha} \}_{\alpha \in Y} \). Hence \( S \) is a complete semilattice \( Y \) of Archimedean and \( \pi \)-regular ordered semigroups \( \{S_\alpha \}_{\alpha \in Y} \) by [[4], Theorem 3.8]. Since \( S \) is completely \( \pi \)-regular, therefore \( S_\alpha \) is also completely \( \pi \)-regular by [[4], Theorem 2.4]. Hence \( S \) is a complete semilattice of completely \( \pi \)-regular and Archimedean ordered semigroup.

(3) \( \implies \) (1): Let \( S \) is a complete semilattice \( Y \) of completely \( \pi \)-regular and Archimedean ordered semigroups \( S_\alpha , S_\alpha \in Y \). Then \( S \) is \( \pi \)-regular. Let \( a \in Reg_\varnothing(S) \). Then \( a \in S_\alpha \) for some \( \alpha \in Y \). Now \( a \leq axa \) for some \( x \in S_\beta \). Therefore \( axa \in S_\alpha S_\beta S_\alpha \subseteq S_\alpha S_\beta \). Therefore \( a \in (S_\alpha \cap (S_\alpha S_\beta) \neq \phi \). Therefore \( \alpha \beta \cap (S_\alpha \cap (S_\alpha S_\beta) \neq \phi \). Therefore \( S_\alpha \subseteq S_\alpha S_\beta \). Again \( a \leq axa \leq a(xax)a \). Now \( y = xax \in S_\beta S_\alpha S_\beta \subseteq S_\alpha S_\beta S_\alpha = S_\alpha S_\beta \). Therefore \( a \in Reg_\varnothing(S_\alpha) \). Now let \( \sigma \) be the semilattice congruence. Then \( a \in (a_\varnothing) = ((a_\varnothing) x) = ((a_\varnothing) y) \subseteq LReg_\varnothing(a) \subseteq LReg_\varnothing(S) \). Similarly \( a \in RReg_\varnothing(S) \). Therefore \( a \in Gr_\varnothing(S) \). Hence \( S \) is a GV-ordered semigroup.

(3) \( \implies \) (4): Let \( S \) be a complete semilattice \( Y \) of completely \( \pi \)-regular and Archimedean ordered semigroups \( S_\alpha , a \in Y \). Now each \( S_\alpha \) is a nil-extension of simple and completely \( \pi \)-regular ordered semigroup \( K_{\alpha} , \alpha \in Y \). Let \( a , b \in S \). Then \( a \in S_\alpha , b \in S_{\beta} \) for some \( \alpha , \beta \in Y \). Therefore \( ab , ba \in S_{\beta} \). Hence \( (ab)^m , (ba)^m \in K_{\alpha} \) for some \( n , m \in \mathbb{N} \). Since \( K_{\alpha} \) is ideal, therefore \( (ab)^m , (ba)^m \in K_{\alpha} \). Again since \( K_{\alpha} \) is simple, \( (ab)^m \leq (ab)^{n+1}(ba)^m (ab)^{n+1} \in K_{\alpha} \). Therefore \( (ab)^m \in ((ab)^n S_\alpha (ab)^{n+1}). \)

(4) \( \implies \) (3): Clearly \( S \) is completely \( \pi \)-regular ordered semigroup by the given condition. Assume \( a , b \in S \). Then \( (ab)^n \in ((ab)^n S_\alpha (ab)^{n+1}) \), that is, \( (ab)^n \in (S_\alpha^2 S) \) for some \( n \in \mathbb{N} \). Therefore by [[12], Lemma 3.5], \( S \) is a complete semilattice of Archimedean ordered semigroup. Hence \( S \) is a complete semilattice of completely \( \pi \)-regular and Archimedean ordered semigroup.

(3) \( \iff \) (5): This is similar to the proof of (3) \( \iff \) (4).
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References


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