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On transiso-class graphs

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Abstract. In this paper, we have determined the number of isomorphism classes of transversals of subgroups of order 2 and 5 of $Alt(5)$. Further, we have introduced two new graphs $\Gamma_{tic}(G)$ and $\Gamma_{d,tic}(G)$ on a finite group G, where d is the order of a subgroup of G and studied some properties of these graphs.

1. Introduction

Let G be a finite group and H be a subgroup of G. We say that a subset S of G is a normalized right transversal (NRT) of H in G, if S is obtained by choosing one and only one element from each right coset of H in G and $1 \in S$. For $x,\ y \in S$, define $\{x \circ y\} = S \cap Hxy$. Then with respect to this binary operation, S is a right loop with identity 1, that is, a right-quasigroup with both-sided identity (see [12, Proposition 4.3.3]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [8, Theorem 3.4]).

Let S be an NRT of H in G. Let $\langle S\rangle$ be the subgroup of G generated by S and H_S be the subgroup $\langle S\rangle \cap H$. Then $H_S = \langle\{xy(x \circ y)^{-1} \mid x, y \in S\}\rangle$ and $H_S S = \langle S\rangle$ (see [8, Corollary 3.7]). Identifying S with the set $H \setminus G$ of all right cosets of H in G, we get a transitive permutation representation $\chi_S : G \to \mathrm{Sym}(S)$ defined by $\{\chi_S(g)(x)\} = S \cap Hxg, g \in G, x \in S$. The kernel $\ker\chi_S$ of this action is $\mathrm{Core}_G(H)$, the core of H in G. Let $G_S = \chi_S(H_S)$, the group torsion of the right loop S (see [8]). The group G_S depends only on the right loop structure $\circ$ on S and not on the subgroup H. Since χ_S is injective on S and if we identify S with $\chi_S(S)$, then $\chi_S(\langle S\rangle) = G_S S$ which also depends only on the right loop S and S is an NRT of G_S in $G_S S$. One can also verify that $\ker(\chi_S|_{H_S S} : H_S S \to G_S S) = \ker(\chi_S|_{H_S} : H_S \to G_S) = \mathrm{Core}_{H_S S}(H_S)$ and $\chi_S|_S =$ the identity map on S. Also, G_S is trivial if and only if $(S, \circ)$ is a group (see [8]).

We denote the set of all normalized right transversals (NRTs) of H in G by $\mathcal{T}(G,H)$. We say that S and $T \in \mathcal{T}(G,H)$ are isomorphic (denoted by $S \cong T$), if their induced right loop structures are isomorphic. Let $\mathcal{I}(G,H)$ denote the set of isomorphism classes of NRTs of H in G. It has been proved in [10] as well as in [7] that $|\mathcal{I}(G,H)| = 1$ if and only if $H \trianglelefteq G$. It has been shown in [4] that there is no pair (G,H) such that $|\mathcal{I}(G,H)| = 2$. It is easy to observe that if H is a non-normal subgroup of G of index 3, then $|\mathcal{I}(G,H)| = 3$. The converse of this statement is

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proved in [5]. Also, it has been proved in [6] that there is no pair (G, H) such that $|\mathcal{I}(G,H)| = 4$. The integers 5, 6 also realized in this way (see [6]). It is easy to observe that if H is a subgroup of order 3 of $Alt(4)$, then $|\mathcal{I}(G,H)| = 7$. Therefore it seems an interesting problem to know that which integer appears as $|\mathcal{I}(G,H)|$ for some pair $(G,\ H)$.

In the Section 2, we have determined $|\mathcal{I}(G,H)|$, where $G = Alt(5)$ and H be a non-normal subgroup of G of order 2 or 5. In the Section 3, we have defined two new graphs associated to the isomorphism classes of transversal of a subgroup in a finite group and studied some properties of these graphs.

2. Isomorphism classes of transversals in $Alt(5)$

Now, we state the following proposition whose proof is essentially the same proof of the Proposition 2.7 in [10].

Proposition 2.1. *Let G be a finite group and H be a corefree subgroup of G. Let $T \in \mathcal{T}(G,H)$ such that $\langle T \rangle = G$. Let $\mathcal{O} = \{L \in \mathcal{T}(G,H) | T \cong L\}$. Then $Aut_H(G)$ acts transitively on the set $\mathcal{O}$.*

Remark 2.2. If G is a finite group and H a subgroup of G such that $Core_G(H)$ is nontrivial, then the number $|\mathcal{I}(G,H)|$ may be different from the number of $Aut_H(G)$-orbits in $\mathcal{T}(G,H)$. For example, let $G = \langle x, y | x^6 = 1 = y^2, yxy^{-1} = x^{-1} \rangle \cong D_{12}$, the dihedral group of order 12 and $H = \{1, x^3, y, yx^3\}$, where 1 is the identity of G. Then H is non-normal in G and $[G : H] = 3$. Hence $|\mathcal{I}(G,H)| = 3$. However, NRTs $\{1, x, x^2\}$, $\{1, yx, x^2\}$, $\{1, x, yx^2\}$ and $\{1, yx, yx^2\}$ to H in G, lie in different $Aut_H(G)$-orbits (as the set of orders of group elements in any two NRTs are not same).

Lemma 2.3. *Let L be a subgroup of $G = Alt(5)$ of order 12. Then $L \cong Alt(4)$, the alternating group of degree 4.*

Proof. Up to isomorphism, there are only 5 groups of order 12 (see [1, Theorem 5.1]),

1. $\mathbb{Z}_{12}$;
2. $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;
3. D_{12}, the dihedral group of order 12;
4. $\langle x, y | x^4 = y^3 = 1, xy = y^2 x \rangle$;
5. $Alt(4)$.

Since G does not contain an element of order 12 or order 6 or order 4, hence it is not isomorphic to either of the groups in (1)-(4). Thus $L \cong Alt(4)$. □

Lemma 2.4. *Let K be a subgroup of $Sym(5)$ of order 20. Then K is isomorphic to the group $\langle x, y \mid x^5 = y^4 = 1, yxy^{-1} = x^2 \rangle$, which is the one dimensional affine group over $\mathbb{Z}_5$.*

Proof. Up to isomorphism, there are only five non-isomorphic groups of order 20 (see [3]),

1. $\mathbb{Z}_{20}$;
2. $\mathbb{Z}_{10} \times \mathbb{Z}_2$;
3. D_{20}, the dihedral group of order 20;
4. $M = \langle x, y \mid x^5 = y^4 = 1, yxy^{-1} = x^{-1} \rangle$;
5. $\langle x, y \mid x^5 = y^4 = 1, yxy^{-1} = x^2 \rangle$.

Since $Sym(5)$ does not contain an element of order 10, K cannot be isomorphic to the either of the groups $\mathbb{Z}_{20}$, $\mathbb{Z}_{10} \times \mathbb{Z}_2$, D_{20} and M. This implies that K is not isomorphic to either of the groups in (1) - (4) (we observe that $Z(M) = \langle y^2 \rangle$). Thus K is isomorphic to the group $\langle x, y \mid x^5 = y^4 = 1, yxy^{-1} = x^2 \rangle$. □

Remark 2.5. Let $G = Alt(5)$. Then $Aut(G) = Inn(Sym(5))$ (see [13, 2.17, p.299]). Since $Z(Sym(5)) = \{I\}$, we may identify $Aut(G)$ with $Sym(5)$ by identifying each $g \in Sym(5)$ with i_g, the inner automorphism of $Sym(5)$, determined by g ($x \mapsto gxg^{-1}$). Thus for a subgroup H of G, $Aut_H(G) = N_{Sym(5)}(H)$.

Proposition 2.6. *Let $G = Alt(5)$. Let H be a subgroup of G of order 5. Then $Aut_H(G)$ is isomorphic to $\langle x, y \mid x^5 = y^4 = 1, yxy^{-1} = x^2 \rangle$, the one dimensional affine group over $\mathbb{Z}_5$.*

Proof. Let H be a subgroup of G of order 5. Then by Remark 2.5, $Aut_H(G) = N_{Sym(5)}(H)$. Since there are 6 Sylow 5-subgroups in $Sym(5)$, $[Sym(5) : N_{Sym(5)}(H)] = 6$. This implies that $|N_{Sym(5)}(H)| = 20 = |Aut_H(G)|$. Now, the proposition follows from the Lemma 2.4. □

Proposition 2.7. *Let $G = Alt(5)$ and $H = \langle a = (12345) \rangle$. Let $S \in \mathcal{T}(G, H)$. Then $H \nsubseteq Stab_K(S)$, the stabilizer of S in K, where $K = N_{Sym(5)}(H)$ and the action of K is by conjugation.*

Proof. Let $S_0 = \{\alpha \in G : \alpha(5) = 5\}$. Then $S_0 \cong Alt(4)$ and $S_0 \in \mathcal{T}(G, H)$. Let $S_0 = \{I = a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}\}$, where $a_1 = (12)(34)$, $a_2 = (13)(24)$, $a_3 = (14)(23)$, $a_4 = (123)$, $a_5 = (132), a_6 = (124)$, $a_7 = (142), a_8 = (134)$, $a_9 = (143)$, $a_{10} = (234)$, $a_{11} = (243)$. Then there exists a unique map $\sigma : S_0 \to H$, with $\sigma(a_0) = a_0$ such that $S = S_\sigma = \{\sigma(a_i)a_i \mid 0 \leqslant i \leqslant 11\} \in \mathcal{T}(G, H)$. Assume that $Stab_K(S) \supseteq H$. Then

$$aSa^{-1} = S. \tag{1}$$

Now, $a\sigma(a_3)a_3a^{-1} = \sigma(a_3)aa_3a^{-1} = \sigma(a_3)a^2a_3$. Since $a\sigma(a_3)a_3a^{-1} \in S_\sigma$ $(= S)$, by (1), $\sigma(a_3)a^2a_3 \in S$. This gives $\sigma(a_3)a^2 = \sigma(a_3)$. This implies that $a^2 = I$, a contradiction. Thus $Stab_K(S) \not\supseteq H$. □

Corollary 2.8. *Let G, H, K and S be as in the Proposition 2.7. Then $Stab_K(S) \ncong D_{10}$, the dihedral group of order 10. Further, $Stab_K(S) \neq K$.*

Proof. We observe that K has only one subgroup L of order 10 isomorphic to the dihedral group D_{10}. Since L contains the subgroup H of K, by Proposition 2.7, $Stab_K(S) \neq L$. Since $H \subseteq K$, by Proposition 2.7 $Stab_K(S) \neq K$. □

Proposition 2.9. *Let $G = Alt(5)$ and $H = \langle(12345)\rangle$. Let $S \in \mathcal{T}(G,H)$ such that $\langle S\rangle = S$. Then $S = hS_0h^{-1}$, where $h \in H$ and $S_0 = \{\alpha \in G : \alpha(5) = 5\} \in \mathcal{T}(G,H)$.*

Proof. We observe that $S_0 = \langle(123),(124)\rangle \cong Alt(4)$. Let $S \in \mathcal{T}(G,H)$ such that $\langle S\rangle = S$. By Lemma 2.3, $S \cong S_0$. This implies that $S = \langle(abc),(def)\rangle$, where $a,b,c,d,e,f \in \{1,2,3,4,5\}$. Since $S \cong S_0$ and $|(123)(124)| = 2$, $|(abc)(def)| = 2$. This implies that $d = a$, $e = b$ and hence $S = \langle(abc),(abf)\rangle$, where a,b,c and f are distinct. Thus we have a permutation $\alpha \in Sym(5)$ with $\alpha(1) = a$, $\alpha(2) = b$, $\alpha(3) = c$, $\alpha(4) = f$ and $\alpha(5) = d_0$, where $d_0 \in \{1,2,3,4,5\} \setminus \{a,b,c,f\}$. Thus

$$\alpha S_0\alpha^{-1} = \big\langle\big(\alpha(1)\alpha(2)\alpha(3)\big),\big(\alpha(1)\alpha(2)\alpha(4)\big)\big\rangle = \big\langle(abc),(abf)\big\rangle = S. \tag{2}$$

Next, since $\alpha \in Sym(5)$, either $\alpha \in Alt(5)$ or $(12)\alpha \in Alt(5)$. First, assume that $\alpha \in Alt(5)$. Then there exists $h_1 \in H$ and $\beta_1 \in S_0$ such that $\alpha = h_1\beta_1$. Thus $h_1 = \alpha\beta_1{}^{-1} \in H$. Since $\beta_1 \in S_0$, by (2) $h_1S_0h_1^{-1} = \alpha\beta_1^{-1}S_0(\alpha\beta_1^{-1})^{-1} = S$.

Next, assume that $(12)\alpha \in Alt(5)$. Then there exists $h_2 \in H$ and $\beta_2 \in S_0$ such that $(12)\alpha = h_2\beta_2$. Thus $h_2 = (12)\alpha\beta_2^{-1}$. Now, since $\big((12)\alpha\big)(123)\big((12)\alpha\big)^{-1}$ $= \big(\alpha(2)\alpha(1)\alpha(3)\big)$ and $\big((12)\alpha\big)(124)\big((12)\alpha\big)^{-1} = \big(\alpha(2)\alpha(1)\alpha(4)\big)$, therefore

$$\big((12)\alpha\big)S_0\big((12)\alpha\big)^{-1} = \big\langle\big(\alpha(2)\alpha(1)\alpha(3)\big),\big(\alpha(2)\alpha(1)\alpha(4)\big)\big\rangle = \alpha S_0\alpha^{-1}. \tag{3}$$

Since $\beta_2 \in S_0$, by (3) $h_2S_0h_2^{-1} = S$. Thus in either cases, we have $S = hS_0h^{-1}$, for some $h \in H$. □

Remark 2.10. *Let G be a finite group. If H and K are subgroups of G such that $f(H) = K$ for some $f \in Aut(G)$, then $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$.*

Proposition 2.11. *Let $G = Alt(5)$, the alternating group of degree 5 and H be a subgroup of G of order 5. Then $|\mathcal{I}(G,H)| = 5^2 \cdot (13 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6 + 5^7)$.*

Proof. Since any two subgroups of order 5 of G are conjugate, by Remark 2.10, we may take $H = \langle a = (12345)\rangle$. Let $S_0 \in \mathcal{T}(G,H)$, where $S_0 = \{a_0 = I,\ a_1 = (12)(34),\ a_2 = (13)(24),\ a_3 = (14)(23),\ a_4 = (123),\ a_5 = (132), a_6 = (124),\ a_7 =$

$(142), a_8 = (134),\ a_9 = (143),\ a_{10} = (234),\ a_{11} = (243)\}$. Then $S_0 \cong Alt(4)$. We observe that for each $S \in \mathcal{T}(G,H)$, there exists a unique map $\sigma : S_0 \to H$ such that $\sigma(a_0) = a_0$ and $S = S_\sigma = \{\sigma(a_i)a_i : 0 \leqslant i \leqslant 11\}$. Let $S \in \mathcal{T}(G,H)$. Then $S = S_\sigma$ for a unique map $\sigma : S_0 \to H$ with $\sigma(a_0) = a_0$. Further, since $|H| = 5$, a prime number, either $\langle S\rangle = S$ or $\langle S\rangle = G$. Assume that $\langle S\rangle = S$. Then by Lemma 2.3, $S \cong S_0 \cong Alt(4)$. By Proposition 2.9 all non-generating NRTs of H in G are conjugate, all non-generating NRTs of H in G forms a single $Aut_H(G)$-orbit in $\mathcal{T}(G,H)$, where $Aut_H(G)$ is identified with the subgroup $K = N_{Sym(5)}(H)$ of $Sym(5)$ and the action of K on $\mathcal{T}(G,H)$ is by conjugation (see also Remark 2.5). If $\langle S\rangle = G$, then by Proposition 2.1, the isomorphism class of S on $\mathcal{T}(G,H)$ forms a single $Aut_H(G)$-orbit. Thus $\mathcal{I}(G,H)$ is precisely the orbits of K in $\mathcal{T}(G,H)$. Now, we describe the orbits of K in $\mathcal{T}(G,H)$. Since $H = \langle a = (12345)\rangle$, we have

$$N_{Sym(5)}(H) = K = \left\langle a, b = (1342) \mid a^5 = b^4 = 1, bab^{-1} = a^2 \right\rangle,$$

K is isomorphic to one dimensional affine group over $\mathbb{Z}_5$ (see Proposition 2.6). Further, by Proposition 2.7 and Corollary 2.8, $|Stab_K(S)| \in \{1,2,4\}$.

Assume that $|Stab_K(S)| = 4$. Since a subgroup of K of order 4 is a Sylow 2-subgroup of K, we may assume that $Stab_K(S) = \langle b = (1342)\rangle = K_1$. Since $bab^{-1} = a^2$, we obtain the following relations:

$$\left.\begin{array}{c}\sigma(a_0) \;=\; \sigma(a_1) \;=\; \sigma(a_2) \;=\; \sigma(a_3) \;=\; I\\ \sigma(a_6) = (\sigma(a_4))^2,\ \sigma(a_9) = (\sigma(a_4))^3,\ \sigma(a_{11}) = (\sigma(a_4))^4,\\ \sigma(a_7) = (\sigma(a_5))^2,\ \sigma(a_8) = (\sigma(a_5))^3,\ \sigma(a_{10}) = (\sigma(a_5))^4.\end{array}\right\} \tag{4}$$

Conversely, if $\sigma_1 : S_0 \to H$ is a map satisfying the relations (4), then $Stab_K(S_{\sigma_1}) = K_1$, for if $g \in K \setminus K_1$, then $a_3 \notin gS_{\sigma_1}g^{-1}$ (note that $a_3 \in S_{\sigma_1}$) and $K_1 \subseteq Stab_K(S_{\sigma_1})$. Let $\sigma_1 : S_0 \to H$ be a map satisfying (4). Then $S_{\sigma_1} = \{\sigma_1(a_i)a_i \mid 0 \leqslant i \leqslant 11\} \in \mathcal{T}(G,H)$ and $Stab_K(S_{\sigma_1}) = K_1$. Assume that $T \in \mathcal{T}(G,H)$ lies in the K-orbit of S_{σ_1}. Then there exists $g \in K$ such that $gS_{\sigma_1}g^{-1} = T$. This implies that $Stab_K(T) = gK_1g^{-1}$. Since $N_K(K_1) = K_1$, if $g \notin K_1$, then $Stab_K(T) \neq K_1$. Further, if $g \in K_1$, then $S_{\sigma_1} = gS_{\sigma_1}g^{-1} = T$. This implies that S_{σ_1} lies in the unique K-orbit of size 5. From the relations (4), we observe that a map $\sigma : S_0 \to H$ satisfying (4) can be completely determined by assigning values of $\sigma(a_4)$ and $\sigma(a_5)$. Since each of $\sigma(a_4)$ and $\sigma(a_5)$ can take five distinct values, we have 25 $Aut_H(G) = K$-orbits in $\mathcal{T}(G,H)$ each of size $\frac{|K|}{|K_1|} = 5$.

Next, assume that $|Stab_K(S)| = 2$. Since a Sylow 2-subgroup of K is cyclic, any two subgroups of K of order 2 are conjugate. Thus we may assume that $Stab_K(S) = \langle b^2 = (14)(23)\rangle = L_1$. Since $b^2ab^{-2} = a^4$, we obtain the following relations:

$$\left.\begin{array}{l}\sigma(a_0) = \sigma(a_1) = \sigma(a_2) = \sigma(a_3) = 1,\\ \sigma(a_8) = (\sigma(a_7))^4,\quad \sigma(a_9) = (\sigma(a_6))^4,\\ \sigma(a_{10}) = (\sigma(a_5))^4,\quad \sigma(a_{11}) = (\sigma(a_4))^4.\end{array}\right\} \tag{5}$$

Conversely, let $\sigma_1 : S_0 \to H$ be a map satisfying (5). Then $Stab_K(S_{\sigma_1}) \supseteq L_1$. From the relations (5), we observe that σ_1 satisfying (5) can be completely determined by assigning values of $\sigma_1(a_4)$, $\sigma_1(a_5)$, $\sigma_1(a_6)$ and $\sigma_1(a_7)$. Since each of $\sigma_1(a_i)$'s $(4 \leqslant i \leqslant 7)$ can take five distinct values, there are 625 choices of σ_1 satisfying (5). Further, from the relations (4) and (5), we observe that if a map from S_0 to H satisfies the relations (4), then it also satisfies (5). Further, since there are 25 choices of maps $\sigma : S_0 \to H$ satisfying (4), there are 600 choices of maps from $S_0 \to H$ which satisfies (5) but not (4). Let $\sigma_1 : S_0 \to H$ be a map which satisfies the relations (5) but not (4). Then $S_{\sigma_1} = \{\sigma_1(a_i)a_i \mid 0 \leqslant i \leqslant 11\} \in \mathcal{T}(G,H)$ and $Stab_K(S_{\sigma_1}) = L_1$. Assume that $T \in \mathcal{T}(G,H)$ lies in the K-orbit of S_{σ_1}. Then there exists $g \in K$ such that $gS_{\sigma_1}g^{-1} = T$. This implies that $Stab_K(T) = gL_1g^{-1}$. Since $N_K(L_1) = K_1$, if $g \notin K_1$, then $Stab_K(T) \neq L_1$. Next, if $g \in K_1 \setminus L_1$, then $gS_{\sigma_1}g^{-1} = T(\neq S_{\sigma_1})$. Since $[K_1 : L_1] = 2$, there exists a unique $T \in \mathcal{T}(G,H)$, different from S_{σ_1} which lies in the K-orbit of S_{σ_1} and $Stab_K(T) = L_1$. Thus by the discussion made above, there are 300 K-orbits in $\mathcal{T}(G,H)$ each of size $\frac{|K|}{|L_1|} = 10$.

Lastly, assume that $|Stab_K(S)| = 1$. As argued in the above paragraphs there are 125 NRTs in $\mathcal{T}(G,H)$ whose stabilizer are of order 4 and there are 3000 NRTs in $\mathcal{T}(G,H)$ whose stabilizer are of order 2, there are $5^{11} - 5^5 = 5^5(5^6 - 1)$ NRTs whose stabilizer are trivial. Hence, we have $5^4 \cdot (1+5+5^2+5^3+5^4+5^5)$, K-orbits in $\mathcal{T}(G,H)$ each of size 20. Thus $|\mathcal{I}(G,H)| = 5^2+3\cdot4\cdot5^2+5^4\cdot(1+5+5^2+5^3+5^4+5^5) = 5^2 \cdot (13 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6 + 5^7)$. □

Corollary 2.12. *There are at least,* $5^2 \cdot (13 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6 + 5^7)$ *non-isomorphic right loops of order 12.*

Proof. Let $G = Alt(5)$, the alternating group of degree 5 and H be a subgroup of G of order 5. If $S \in \mathcal{T}(G,H)$, then S is a right loop of order 12 (see [12, Proposition 4.3.3, p.102]). By Proposition 2.11, $|\mathcal{I}(G,H)|$ is precisely the number of $\mathrm{Aut}_H(G)$-orbits in $\mathcal{T}(G,H)$. Thus if $S_1, S_2 \in \mathcal{T}(G,H)$ belongs to different $\mathrm{Aut}_H(G)$-orbits, then $S_1 \ncong S_2$. This completes the proof. □

Lemma 2.13. *Let L be a subgroup of $Sym(5)$ of order 8. Then L is isomorphic to D_8, the dihedral group of order 8.*

Proof. Since $|Sym(5)| = 2^3 \cdot 3 \cdot 5$, if L is a subgroup of $Sym(5)$ of order 8, then it is a Sylow 2-subgroup of $Sym(5)$. Let $N = \langle (13), (1234) \rangle$. Then N is a subgroup of $Sym(5)$ of order 8 isomorphic to D_8. Since any two Sylow 2-subgroups of $Sym(5)$ are conjugate, the lemma follows. □

Proposition 2.14. *Let $G = Alt(5)$, the alternating group of degree 5 and H be a subgroup of G of order 2. Then $|\mathcal{I}(G,H)| = 2^{26} + 10$.*

Proof. Let H be a subgroup of G of order 2. Since any two elements of G of order 2 are conjugate, by Remark 2.10, we may assume that $H = \{I, x = (12)(34)\}$, where I is the identity element of G. Let $K = Aut_H(G)$. By Remark 2.5, we identify K with the group $N_{Sym(5)}(H) = C_{Sym(5)}(H)$, the centralizer of H in $Sym(5)$. Since there are 15 conjugates of $(12)(34)$ in $Sym(5)$, $|C_{Sym(5)}(H)| = 8$. By Lemma 2.13, $C_{Sym(5)}(H) \cong D_8$. Since $H = \{I, x = (12)(34)\}$, we have

$$K = \{I, (1324), (12)(34), (1423), (14)(23), (34), (13)(24), (12)\}.$$

Consider the subgroups $V_4 = \{I, (12)(34), (13)(24), (14)(23)\}$ (isomorphic to the Klein's four group) and $L = \{g \in G : g(5) = 5\}$ of G. Let $T_1 = \{b_0 = I,\ b_1 = (13)(24)\}$, $T_2 = \{c_0 = I, c_1 = (134), c_2 = (143)\}$ and $T_3 = \{d_0 = I, d_1 = (12345), d_2 = (13524), d_3 = (14253), d_4 = (15432)\}$. Then $T_1 \in \mathcal{T}(V_4, H)$, $T_2 \in \mathcal{T}(L, V_4)$ and $T_3 \in \mathcal{T}(G, L)$. Thus $S_0 = T_1T_2T_3 = \{b_ic_jd_k\ :\ 0 \leqslant i \leqslant 1,\ 0 \leqslant j \leqslant 2, 0 \leqslant k \leqslant 4\} \in \mathcal{T}(G, H)$.

Since G is a simple group and H is of order 2, $\langle S \rangle = G$, for every $S \in \mathcal{T}(G, H)$. Thus by Proposition 2.1, $\mathcal{I}(G, H)$ is precisely the orbits of K in $\mathcal{T}(G, H)$, where the action of K is by conjugation.

Let $S \in \mathcal{T}(G, H)$. Then there exists a unique map $\sigma\ :\ S_0 \to H$ such that $\sigma(b_0c_0d_0 = I) = I$ and $S = S_\sigma = \{\sigma(b_ic_jd_k)b_ic_jd_k\ :\ 0 \leqslant i \leqslant 1,\ 0 \leqslant j \leqslant 2, 0 \leqslant k \leqslant 4\}$. Let $g \in \{(1324), (1423), (12), (34)\} \subseteq K$. Then $g \notin Stab_K(S)$, for if $g \in Stab_K(S)$, then $g\sigma(b_1c_0d_0)b_1c_0d_0g^{-1} = \sigma(b_1c_0d_0)xb_1c_0d_0$, a contradiction as $x = (12)(34) \in H$ and $\sigma(b_1c_0d_0)b_1c_0d_0 \in S$. Let $g = (13)(24) \in K$. Then $g \notin Stab_K(S)$, for if $g \in Stab_K(S)$, then $g\sigma(b_0c_1d_0)b_0c_1d_0g^{-1} = \sigma(b_0c_1d_0)xb_0c_1d_0 \in S$ and so we have a contradiction as $x = (12)(34) \neq I$. Next, let $g = (14)(23) \in K$. Then $g \notin Stab_K(S)$, for if $g \in Stab_K(S)$, then $g\sigma(b_0c_2d_0)b_0c_2d_0g^{-1} = \sigma(b_0c_2d_0)xb_0c_2d_0 \in S$, $\sigma(b_0c_2d_0)x = \sigma(b_0c_2d_0)$, again a contradiction. The above arguments imply that stabilizer in K of an NRT of H in G is either H or $\{I\}$. Thus a K-orbit in $\mathcal{T}(G, H)$ is either of size 4 or of size 8.

Now, assume that $Stab_K(S) = H$. Then σ satisfies the following relations:

$$\left.\begin{array}{l}
\sigma(b_1c_0d_0) = I \ or\ x,\ \sigma(b_0c_1d_4)x = \sigma(b_1c_0d_3),\ \sigma(b_0c_2d_1)x = \sigma(b_0c_1d_2)\\
\sigma(b_1c_1d_1)x = \sigma(b_1c_0d_2), \sigma(b_1c_2d_2)x = \sigma(b_1c_0d_1), \sigma(b_0c_2d_3) = \sigma(b_0c_0d_4)\\
\sigma(b_0c_2d_0) = \sigma(b_1c_2d_0),\ \sigma(b_0c_1d_3) = \sigma(b_1c_2d_4),\ \sigma(b_0c_2d_2)x = \sigma(b_0c_0d_1)\\
\sigma(b_1c_1d_2)x = \sigma(b_1c_2d_1),\ \sigma(b_1c_0d_4) = \sigma(b_1c_2d_3),\ \sigma(b_0c_1d_1)x = \sigma(b_oc_0d_2)\\
\sigma(b_0c_1d_0)x = \sigma(b_1c_1d_0),\ \sigma(b_0c_2d_4) = \sigma(b_1c_1d_3),\ \sigma(b_1c_1d_4)x = \sigma(b_0c_0d_3)
\end{array}\right\} \quad (6)$$

Conversely, if a map $\sigma_1 : S_0 \to H$ with $\sigma_1(I) = I$ satisfies (6), then $Stab_K(S_{\sigma_1}) = H$. From the relations (6), we find that there are 20 K-orbits in $\mathcal{T}(G, H)$ each of size 4. Hence we have $\frac{2^{29}-80}{8} = 2^{26} - 10$, K-orbits in $\mathcal{T}(G, H)$ each of size 8. Therefore $|\mathcal{I}(G, H)| = 2^{26} - 10 + 20 = 2^{26} + 10$. □

Corollary 2.15. *There are at least,* $2^{26} + 10$ *non-isomorphic right loops of order 30.*

Proof. Let $G = Alt(5)$, the alternating group of degree 5 and H be a subgroup of G of order 2. If $S \in \mathcal{T}(G, H)$, then S is a right loop of order 30 (see [12, Proposition 4.3.3, p.102]). By Proposition 2.14, $|\mathcal{I}(G, H)|$ is precisely the number of $\mathrm{Aut}_H(G)$-orbits in $\mathcal{T}(G, H)$. Thus if $S_1, S_2 \in \mathcal{T}(G, H)$ belongs to different $\mathrm{Aut}_H(G)$-orbits, then $S_1 \ncong S_2$. This completes the proof. □

3. Graphs and isomorphism classes of transversals

In this section, we have introduced two graphs associated to the isomorphism classes of transversals of a subgroup of a finite group and studied some properties of these graphs.

Definition 3.1. Let G be a finite group and X be the set of all nontrivial proper subgroups of G. We define a graph $\Gamma_{tic}(G)$ on G whose vertex set is X and two distinct vertices H and K are adjacent in $\Gamma_{tic}(G)$ if and only if $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$. We will call this graph the *transiso-class* graph.

It is easy to observe that $\Gamma_{tic}(G)$ is complete if and only if $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$ for every $H,\ K \in X$.

Definition 3.2. Let G be a finite group. Let d be the order of a subgroup of G and X_d be the set of all subgroups of G of order d. We define a graph $\Gamma_{d,tic}(G)$ on G with vertex set X_d and two distinct vertices are adjacent in $\Gamma_{d,tic}(G)$ if and only if $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$. We call the graph $\Gamma_{d,tic}(G)$ as *d-transiso-class* graph.

We observe that $\Gamma_{d,tic}(G)$ is complete if and only if $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$ for any $H,\ K \in X_d$.

Remark 3.3. In the definitions 3.1 and 3.2, we observe that $\Gamma_{tic}(G)$ and $\Gamma_{d,tic}(G)$ both are connected if and only if they are complete.

Definition 3.4. ([11], p.143)A group G is said to be a *Dedekind* group if all the subgroups of G are normal in G.

Example 3.5. Let G be a finite Dedekind group. Since each subgroup of G is normal in G, $|\mathcal{I}(G,H)| = 1$ (see [10, Main Theorem, p.643]), for every subgroup H of G. Thus both $\Gamma_{tic}(G)$ and $\Gamma_{d,tic}(G)$ are complete, where d is the order of subgroup of G.

Proposition 3.6. *Let $G = Sym(3)$. Then $\Gamma_{d,tic}(G)$ is complete, d is the order of a subgroup of G.*

Proof. Let $X_d = \{H \leqslant G : \ |H| = d\}$. Obviously, $d \in \{1,2,3,6\}$. If $d = 1$ or $d = 3$ or $d = 6$, then $H \in X_d$ is normal in G and so $|\mathcal{I}(G,H)| = 1$. Thus $\Gamma_{d,tic}(G)$ is complete. Next, assume that $d = 2$. Since all 2-cycles in G are conjugate, any two members of X_2 are conjugate. Hence by Remark 2.10, $|\mathcal{I}(G,H)| = |\mathcal{I}(G,K)|$ for every $H,\ K \in X_2$. Thus $\Gamma_{2,tic}(G)$ is complete. □

Remark 3.7. It is easy to observe that if H is a subgroup of $G = Sym(3)$ of order 2, then $|\mathcal{I}(G,H)| = 3$. However, if $H = Alt(3)$, the alternating group of degree 3, then $|\mathcal{I}(G,H)| = 1$ (see [10]). Consequently, $\Gamma_{tic}(Sym(3))$ is not complete.

Proposition 3.8. *Let $G = Alt(4)$. Then $\Gamma_{d,tic}(G)$ is complete for every d, where d is the order of a subgroup of G.*

Proof. Let $G = Alt(4)$. Let X_d denote the set of all subgroups of G of order d. Then any two members of X_d are conjugate. By Remark 2.10, $|\mathcal{I}(G, H)| = |\mathcal{I}(G, K)|$ for every $H,\ K \in X_d$. Thus $\Gamma_{d,tic}(G)$ is complete for every d. □

Proposition 3.9. *Let $G = Alt(4)$. Then $\Gamma_{tic}(G)$ is not complete.*

Proof. Let $G = Alt(4)$. If H is a subgroup of G of order 2, then $|\mathcal{I}(G, H)| = 5$ (see [6]). Also, It is easy to observe that if K is a subgroup of order 3 of $Alt(4)$, then $|\mathcal{I}(G, K)| = 7$. Thus H and K are not adjacent in $\Gamma_{tic}(G)$. Hence $\Gamma_{tic}(G)$ is not complete. □

Lemma 3.10. *Let $G = Alt(5)$. Let X_d be the set of all subgroups of G of order d. Then any two members of X_d are conjugate.*

Proof. Let X_d be the set of all subgroups of G of order d. Since G is simple, if $H \in X_d$, then $[G : H] \geqslant 5$ (see [13, p. 308]). Hence $d \in \{1, 2, 3, 4, 5, 6, 10, 12, 60\}$. If $d = 1$ or $d = 60$, then the proof is over. Assume that $d = 2$. Let $H \in X_2$. Then H is of the form $\{I, \sigma\}$, where $\sigma \in Alt(5)$ is product of two distinct transpositions. Since all permutations of the form σ are conjugate in $Alt(5)$, any two members of X_2 are conjugate. Further, if $d \in \{3, 4, 5\}$, then any member of X_d is a Sylow d-subgroup of G. Hence any two members of X_d are conjugate.

Next, assume that $d = 6$. Since G has no permutation of order 6, a subgroup of order 6 in G is isomorphic to $Sym(3)$. If K is a subgroup of G of order 6, then $N_G(K) = K$. Hence there are 10 conjugates of K in G. Since there are exactly 10 subgroups of G of order 6, all members of X_6 form a complete conjugacy class. Now, assume that $d = 10$. Again, since G has no permutation of order 10, a subgroup of G of order 10 is isomorphic to D_{10}. If $L \in X_{10}$, then it is easy to observe that $N_G(L) = L$. Thus there are 6 conjugates of L in G. Since there are exactly 6 subgroups of G of order 10, any two subgroups of G of order 10 are conjugate. Lastly, assume that $d = 12$. By Proposition 2.9 any two subgroups of G of order 12 are conjugate. □

Proposition 3.11. *Let $G = Alt(5)$. Then $\Gamma_{d,tic}(G)$ is complete, for every d, where d is the order of a subgroup of G.*

Proof. Let $G = Alt(5)$. Let X_d denote the set of all subgroups of G of order d. Then by Lemma 3.10, any two members of X_d are conjugate. By Remark 2.10, $|\mathcal{I}(G, H)| = |\mathcal{I}(G, K)|$, for any $H,\ K \in X_d$. Hence $\Gamma_{d,tic}(G)$ is complete for every d. □

Remark 3.12. In the above proposition, we observe that $\Gamma_{d,tic}(Alt(5))$ is complete for every d, where d is the order of a subgroup of $Alt(5)$. However, $Alt(5)$ is not a Dedekind group.

Proposition 3.13. *Let $G = Alt(5)$. Then $\Gamma_{tic}(G)$ is not complete.*

Proof. Let $G = Alt(5)$. Let X be the set of all nontrivial proper subgroups of G. Let H be a subgroup of G of order 2. Then by Proposition 2.14, $|\mathcal{I}(G,H)| = 2^{26} + 10$.

Let K be a subgroup of G of order 5. Then by Proposition 2.11, $|\mathcal{I}(G,K)| \neq |\mathcal{I}(G,H)|$. Thus both H and K are in X, however they are not adjacent in $\Gamma_{tic}(G)$. Hence $\Gamma_{tic}(G)$ is not complete. □

Proposition 3.14. *Let G be a finite p-group, p is a prime. Then $\Gamma_{d,tic}(G)$ is complete if and only if each member of X_d is normal in G, where X_d is the set of all subgroups of G of order d.*

Proof. Let G be a finite p-group. Then for each divisor d of $|G|$, G contains a normal subgroup H of order d (see [9, Proposition 9.1.23]). Thus $\Gamma_{d,tic}(G)$ is complete if $|\mathcal{I}(G,K)| = 1$ for every $K \in X_d$. Consequently, each $K \in X_d$ is normal in G (see [10]). Conversely, assume that each member of X_d is normal in G. Then $|\mathcal{I}(G,H)| = 1$, for any $H \in X_d$. Hence $\Gamma_{d,tic}(G)$ is complete. □

Corollary 3.15. *Let G be a nonabelian group of order order p^3, p is a prime. Then $\Gamma_{p,tic}(G)$ is complete if and only if $G \cong Q_8$.*

Proof. Assume that $\Gamma_{p,tic}(G)$ is complete. By the above proposition each subgroup of G of order p is normal in G. Since a subgroup of G of order p^2 is maximal in G, it is normal in G. Thus if $\Gamma_{p,tic}(G)$ is complete, then all subgroups of G are normal in G. Hence G is a Dedekind group. Thus by [11, p.143], $G \cong Q_8$. Conversely, if $G = Q_8$, then $\Gamma_{2,tic}(G)$ is complete follows from the Example 3.5. □

Proposition 3.16. *Let $G = D_{2n}$. If n is even, then $\Gamma_{2,tic}(G)$ is not complete.*

Proof. Let X_2 be the set of all subgroups of G of order 2. Since the center $Z(G)$ of G is of order 2, $|\mathcal{I}(G,Z(G))| = 1$. Again if $H \in X_2$ and H is non-normal, then $|\mathcal{I}(G,H)| \neq 1$ (see [10, Main Theorem, p.643]). Thus $Z(G)$ and H are not adjacent in $\Gamma_{2,tic}(G)$. Consequently, $\Gamma_{2,tic}(G)$ is not complete. □

Let $G = D_8 = \langle a, b : a^2 = b^4 = 1,\ aba = b^{-1} \rangle$. Let $X_2 = \{H_1 = \langle a \rangle, H_2 = \langle ba \rangle, H_3 = \langle b^2 a \rangle, H_4 = \langle b^3 a \rangle, H_5 = \langle b^2 \rangle\}$ be the set of all subgroups of G of order 2 and let $X_4 = \{K_1 = \langle b \rangle, K_2 = \langle b^2, a \rangle, K_3 = \langle b^2, ba \rangle\}$ be the set of all subgroups of G of order 4. Then the connectivity of subgroups in $\Gamma_{2,tic}(D_8)$ and $\Gamma_{4,tic}(D_8)$ can be shown in following pictorial form:

(a) Γ_1 (b) Γ_2

Figure 1: $\Gamma_{2,tic}(D_8) = \Gamma_1 \cup \Gamma_2$

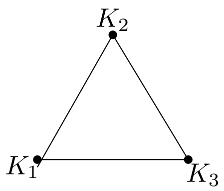

Figure 2: $\Gamma_{4,tic}(D_8)$

Proposition 3.17. *Let G be a finite group containing a nontrivial proper normal subgroup. Assume that $\Gamma_{tic}(G)$ is complete. Then G is a Dedekind group.*

Proof. Let X be the set of all nontrivial proper subgroups of G. Then there exists $H \in X$ such that $H \trianglelefteq G$ and hence $|\mathcal{I}(G, H)| = 1$ (see [10, Main Theorem, p.643]). Assume that $\Gamma_{tic}(G)$ is complete. Then $|\mathcal{I}(G, K)| = 1$, for every $K \in X$. Thus each subgroup of G is normal in G (see [10]). Hence G is a Dedekind group. □

In the Proposition 3.17, we saw that if $\Gamma_{tic}(G)$ is complete and G has a nontrivial proper normal subgroup, then G is Dedekind. Then, we may ask the following questions:

Question 1. *Does there exists a finite non-abelian simple group G such that $\Gamma_{tic}(G)$ complete ?*

Question 2. *Let G be a finite group. Let X_d be the set of all subgroups of G of order d. Assume that $\Gamma_{d,tic}$ is complete. Then what can we say about the members of X_d ?*

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