Quasigroups and Related Systems **29** (2021), 157 − 170

Annihilator graph of a commutative semigroup whose zero-divisor graph is a refinement of a star graph

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Abstract. Suppose that G is a refinement of a star graph with center c and G^* is the subgraph of G induced on the vertices $V(G) \setminus \{x \in V(G) \mid x = c \text{ or } x \text{ is an end vertex adjacent to } c\}$. Let S be a commutative semigroup with zero and $\Gamma(S)$ be the zero-divisor graph of S. In this paper, we determine the structure of the annihilator graph of S by using the zero-divisor graph $\Gamma(S)$, which is a refinement of a star graph with center c, and $\Gamma(S)^*$ has at least two components or $\Gamma(S)^*$ is isomorphic to a cycle graph or a path.

1. Introduction

Throughout the paper S is a commutative semigroup with zero whose operation is written multiplicatively. The set of all zero-divisors of S is denoted by $Z(S)$ and $Z(S)^* = Z(S) \setminus \{0\}$.

There are many papers which interlink graph theory and ring theory. Several classes of graphs associated with algebraic structures have been actively investigated (see for example, [2, 3, 4, 5, 6, 7, 8, 11, 12, 18, 19]).

For any commutative semigroup S with zero element 0, there is a simple undirected graph, which is called the zero-divisor graph and is denoted by $\Gamma(S)$ (cf. [17]). The vertex set of $\Gamma(S)$ is $Z(S)^*$ and x is adjacent to y in $\Gamma(S)$ if and only if $xy = 0$, for each two distinct elements x and y in $Z(S)^*$. It was proved that $\Gamma(S)$ is connected and the diameter of $\Gamma(S)$ is less than or equal to three. Also if $\Gamma(S)$ contains a cycle, then its girth is less than or equal to four. For more details on zero-divisor graphs see [9], [13], [15], [16], [17], [21].

In [10], A. Badawi introduced the concept of the annihilator graph for a commutative ring R, denoted by $AG(R)$, with vertices $Z(R)^*$ and $x \sim y$ is an edge in $AG(R)$ if and only if $\mathrm{ann_R}(xy) \neq \mathrm{ann_R}(x) \cup \mathrm{ann_R}(y)$, where $\mathrm{ann_R}(x) = \{r \in R \mid xr = 0\}$.

In [1], the present authors introduced the annihilator graph for a commutative semigroup S, which is denoted by $AG(S)$. The graph $AG(S)$ is an undirected

2010 Mathematics Subject Classification: 05C99, 20M14
Keywords: Zero-divisor graph, annihilator graph, isolated vertex, star graph.

graph with vertex set $Z(S)^*$ and two distinct vertices x and y are adjacent if and only if $\mathrm{ann_S}(xy) \neq \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y)$, where $\mathrm{ann_S}(x) = \{s \in S \mid xs = 0\}$. Some basic properties of $AG(S)$ are investigated in [1]. For example, it was proved that if $Z(S) \neq S$, then $\Gamma(S)$ is a subgraph of $AG(S)$, and so $AG(S)$ is connected. Also if $Z(S) = S$ and there exists $x \in S^* = S \setminus \{0\}$ such that $\mathrm{ann_S}(x) \supseteq Z(S) \setminus \{x\}$, then x is an isolated vertex in $AG(S)$.

Recall that a graph G with $n+1$ vertices is called a star graph, and is denoted by $K_{1,n}$, if there exists a vertex $x \in V(G)$ such that $\mathrm{d}(x) = n$, and for each vertex $y \in V(G) \setminus \{x\}$, we have $\mathrm{d}(y) = 1$. The vertex x is called the center of $K_{1,n}$. Suppose that G and H are two graphs. H is called a *refinement* of G if $V(G) = V(H)$ and each edge in G is an edge in H. The subgraph induced on vertices $V(G) \setminus \{x \in V(G) \mid x = c$ or x is an end vertex adjacent to $c\}$ is denoted by G^*.

In this paper, we study the annihilator graph associated to a commutative semigroup with zero by using the zero-divisor graph $\Gamma(S)$, where $\Gamma(S)$ is a refinement of a star graph with center c, and $\Gamma(S)^*$ has at least two components or $\Gamma(S)^*$ is isomorphic to a cycle graph or a path.

2. Preliminaries

Now we recall some definitions and notations of graphs. We use the standard terminology of graphs is contained in [14]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use the notation $x \sim y$ to denote that x is adjacent to y in G and edge between x and y will denote by $\{xy\}$. Also the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we use $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) :\ a$ and b are distinct vertices of $G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G, if such a cycle exists; otherwise, we use $\mathrm{gr}(G) := \infty$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote a complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. We use nK_1 to denote the totally disconnected graph with n vertices. For a vertex x of a graph G, the *neighborhood* of x, denoted by $\mathrm{N}(x)$, is the set of vertices which are adjacent to x, moreover the *degree* of x, denoted by $\mathrm{d}(x)$, is the cardinality of $\mathrm{N}(x)$. Also, a vertex u is an end vertex, if there is only one edge incident to u, and it is an *isolated* vertex if $\mathrm{d}(u) = 0$. Let G and H be two graphs. We use the notation $H \leqslant G$ (resp, $H \cong G$) to denote that H is a subgraph of G (resp, H is isomorphic to G). Also we use $G \setminus \{\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, ..., \{x_ny_n\}\}$ to denote a graph G, such that the edges $\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, ..., \{x_ny_n\}$ are deleted.

As usual P_n and C_n will denote the path of length n and the cycle of length n, respectively. Suppose that G is a graph with m components such that each component of G is isomorphic to K_n. Then we will denote G by mK_n. Let H and G be two graphs such that $V(G) \cap V(H) = \emptyset$ and $E(G) \cap E(H) = \emptyset$. Then the union of the graphs H and G, which is denoted by $H \cup G$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(H) \cup E(G)$.

Throughout the paper, we assume that $|Z(S)^*| \geqslant 3$. The case that $|Z(S)^*| \leqslant 2$ is easy. Indeed, if $|Z(S)^*| = 1$, then $AG(S) \cong \Gamma(S) \cong K_1$. Let $|Z(S)^*| = 2$. Then $\Gamma(S) \cong K_2$. Now if $Z(S) = S$, then clearly $AG(S) \cong 2K_1$, and if $Z(S) \neq S$, then $AG(S) \cong \Gamma(S) \cong K_2$. Moreover, in [1, Section 4], the case that $|Z(S)^*| = 3$ and in [20] the case that $|Z(S)^*| = 4$, have been discussed.

3. Properties of $AG(S)$

In this section, we determine the structure of the annihilator graph of a commutative semigroup S whose $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^*$ satisfies one of the properties: (1) $\Gamma(S)^*$ has at least two components, (2) $\Gamma(S)^*$ is a cycle graph, (3) $\Gamma(S)^*$ is a path. Also since $\Gamma(S)$ is a refinement of a star graph with center c, if $c^2 = 0$, then $\mathrm{ann_S}(c) = Z(S)$. Moreover, in this section, we show that if $Z(S) = S$, then 5 is sharp for the girth of $AG(S)$, while if $Z(S) \neq S$, then $\mathrm{gr}(AG(S)) \leqslant 4$.

Proposition 3.1. [22, Corollary 2.4] *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c, and $\Gamma(S)^*$ has at least two components. Then $S^2 = \{0, c\}$, where $S^2 = \{xy | x, y \in S\}$.*

By Proposition 3.1, it is clear that if $\Gamma(S)$ is a refinement of a star graph and $\Gamma(S)^*$ has at least two components, then if there exists a vertex z which is not adjacent to some vertices x and y in $\Gamma(S)$, then x and y are adjacent in $AG(S)$. Also, note that if $\Gamma(S)$ is a refinement of a star graph with center c and $S^2 = \{0, c\}$, then $\mathrm{ann_S}(xy) = Z(S)$, for all $x, y \in Z(S)$. Now, the proof of the next theorem follows from [1, Theorems 3.1 and 3.8].

Theorem 3.2. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c. Also assume that $\Gamma(S)^*$ has at least three components and $|V(\Gamma(S))| = n+1$. Then the following statements hold.*

1. *If x and y are two distinct non adjacent vertices in $\Gamma(S)$, then $x \sim y$ in $AG(S)$.*
2. *If $Z(S) \neq S$, then $AG(S) \cong K_{n+1}$.*
3. *$Z(S) = S$, then $AG(S) \cong K_n \cup K_1$, where c is an isolated vertex in $AG(S)$.*

A graph G is called a *friendship graph* (or a *fan graph*) if G is a refinement of a star graph with center c such that $G \setminus \{c\} \cong nK_2$ and it is denoted by F_n. Clearly $|V(F_n)| = 2n + 1$.

Corollary 3.3. *Suppose that $\Gamma(S) \cong F_n$ with center c and $n \geqslant 3$. Then the following statements hold.*

1. *If $Z(S) \neq S$, then $AG(S) \cong K_{2n+1}$.*
2. *If $Z(S) = S$, then $AG(S) \cong K_{2n} \cup K_1$, where c is an isolated vertex in $AG(S)$.*

Proof. Since $\Gamma(S) \cong F_n$ with center c and $n \geqslant 3$, we have $\Gamma(S)^* \cong nK_2$, and so $\Gamma(S)^*$ has at least three components. Therefore, by Theorem 3.2, the results hold. □

Lemma 3.4. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c such that $\Gamma(S)^*$ has exactly two components A and B. Then the following statements hold.*

1. *If $x, y \in A$, then $x \sim y$ in $AG(S)$. Similarly, if $x, y \in B$, then $x \sim y$ in $AG(S)$.*
2. *Suppose that x, $y \in Z(S)^* \setminus \{c\}$. Then $x \nsim y$ in $AG(S)$ if and only if there exists no end vertex adjacent to c in $\Gamma(S)$ and $x \in A$, $\mathrm{ann_S}(x) = A \cup \{0, c\}$ and $y \in B$, $\mathrm{ann_S}(y) = B \cup \{0, c\}$.*

Proof. (1). It follows by Proposition 3.1.

(2). First suppose that $x, y \in Z(S)^* \setminus \{c\}$ and $x \nsim y$ in $AG(S)$. Then, by (i), $x \in A$, $y \in B$, and so $xy \neq 0$ and, by Proposition 3.1, we have $xy = c$ which follows that $c^2 = (xy)c = x(yc) = 0$, and hence $\mathrm{ann_S}(c) = Z(S)$. Since $x \nsim y$ in $AG(S)$, we see that $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = \mathrm{ann_S}(xy) = \mathrm{ann_S}(c) = Z(S)$. If there exists u such that u is an end vertex adjacent to c in $\Gamma(S)$, then $u \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = Z(S)$, which is impossible. Thus there exists no end vertex adjacent to c in $\Gamma(S)$. Now if $x^2 \neq 0$ or $y^2 \neq 0$, then $x \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = Z(S)$, or $y \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = Z(S)$, which is impossible. Therefore $x^2 = y^2 = 0$. Finally, if there exists $a \in A$ such that $x \nsim a$ in $\Gamma(S)$, then $a \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = \mathrm{ann_S}(xy) = \mathrm{ann_S}(c) = Z(S)$, which is impossible. Hence for each $a \in A$, we have $x \sim a$ in $\Gamma(S)$, and so $\mathrm{ann_S}(x) = A \cup \{0, c\}$. Similarly, $\mathrm{ann_S}(y) = B \cup \{0, c\}$.

Conversely, since $x \in A$ and $y \in B$, which implies that $xy \neq 0$ and, by Proposition 3.1, we have $xy = c$. So $\mathrm{ann_S}(xy) = \mathrm{ann_S}(c) = Z(S)$. Since there exists no end vertex adjacent to c in $\Gamma(S)$ and $\mathrm{ann_S}(x) = A \cup \{0, c\}$ and $\mathrm{ann_S}(y) = B \cup \{0, c\}$, we have $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = A \cup B \cup \{0, c\} = Z(S) = \mathrm{ann_S}(xy)$. Therefore $x \nsim y$ in $AG(S)$. □

The next theorem follows from Lemma 3.4.

Theorem 3.5. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c such that there exists no end vertex adjacent to c in $\Gamma(S)$ and $|V(\Gamma(S)^*| = n$. Also assume that $\Gamma(S)^*$ has exactly two components A and B. Then the following statements hold.*

1. *If $Z(S) \neq S$, then $AG(S) \cong K_{n+1} \setminus \{\{xy\} |\ x \in A,\ y \in B$ and $\mathrm{ann_S}(x) = A \cup \{0, c\}$ and $\mathrm{ann_S}(y) = B \cup \{0, c\}\}$.*

2. *If* $Z(S) = S$, *then* $AG(S) \cong K_1 \cup K_n \setminus \{\{xy\}|\ x \in A,\ y \in B$ *and* $\mathrm{ann_S}(x) = A \cup \{0, c\}$ *and* $\mathrm{ann_S}(y) = B \cup \{0, c\}\}$, *where* c *is an isolated vertex in* $AG(S)$.

The next two corollaries immediately follows from Theorem 3.5 and [1, Theorems 3.1 and 3.8].

Corollary 3.6. *Suppose that* $\Gamma(S) \cong F_2$ *with center* c. *Also assume that* $Z(S) \neq S$ *and* $V(\Gamma(S)^*) = \{x, y, z, w\}$ *with* $x \sim y$ *and* $w \sim z$. *Then the following statements hold.*

1. $AG(S) \cong F_2$ *if and only if* $x^2 = y^2 = z^2 = w^2 = 0$.
2. $AG(S) \cong K_5 \setminus \{\{wy\}, \{wx\}\}$ *if and only if* $z^2 = c$ *and* $y^2 = w^2 = x^2 = 0$.
3. $AG(S) \cong K_5 \setminus \{\{yz\}\}$ *if and only if* $x^2 = w^2 = c$ *and* $y^2 = z^2 = 0$.
4. $AG(S) \cong K_5$ *if and only if* $x^2 = y^2 = c$ *or* $w^2 = z^2 = c$.

Corollary 3.7. *Suppose that* $\Gamma(S) \cong F_2$ *with center* c. *Also assume that* $Z(S) = S$ *and* $V(\Gamma(S)^*) = \{x, y, z, w\}$ *with* $x \sim y$ *and* $w \sim z$. *Then the following statements hold.*

1. $AG(S) \cong K_1 \cup 2K_2$, *where* c *is an isolated vertex in* $AG(S)$, *if and only if* $x^2 = y^2 = z^2 = w^2 = 0$.
2. $AG(S) \cong K_1 \cup K_4 \setminus \{\{wy\}, \{wx\}\}$, *where* c *is an isolated vertex in* $AG(S)$, *if and only if* $z^2 = c$ *and* $y^2 = w^2 = x^2 = 0$.
3. $AG(S) \cong K_1 \cup K_4 \setminus \{\{yz\}\}$, *where* c *is an isolated vertex in* $AG(S)$, *if and only if* $x^2 = w^2 = c$ *and* $y^2 = z^2 = 0$.
4. $AG(S) \cong K_1 \cup K_4$, *where* c *is an isolated vertex in* $AG(S)$, *if and only if* $x^2 = y^2 = c$ *or* $w^2 = z^2 = c$.

Theorem 3.8. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $T = \{u \mid u$ *is an end vertex adjacent to* c *in* $\Gamma(S)\}$ *and* $\mid T \mid = m \geqslant 1$. *Also assume that* $\Gamma(S)^*$ *has exactly two components* A *and* B *and* $|V(\Gamma(S)^*)| = n$. *Then the following statements hold.*

1. *If* $x \in A$ *and* $y \in B$, *then* $x \sim y$ *in* $AG(S)$.
2. *If* $x \in A$, $y \in B$ *and* $u \in T$, *then* $u \sim x$ *and* $u \sim y$ *in* $AG(S)$.
3. *If* u, $v \in T$, *then* $u \sim v$ *in* $AG(S)$.

The next corollary immediately follows from Theorem 3.8 and [1, Theorems 3.1 and 3.8].

Corollary 3.9. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $T = \{u \mid u$ *is an end vertex adjacent to* c *in* $\Gamma(S)\}$ *and* $\mid T \mid = m \geqslant 1$. *Also assume that* $\Gamma(S)^*$ *has exactly two components and* $|V(\Gamma(S)^*)| = n$. *Then the following statements hold.*

1. *If* $Z(S) \neq S$, *then* $AG(S) \cong K_{m+n+1}$.
2. *If* $Z(S) = S$, *then* $AG(S) \cong K_{m+n} \cup K_1$, *where* c *is an isolated vertex in* $AG(S)$.

Proposition 3.10. [22, Theorem 2.5] *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *such that* $\Gamma(S)^*$ *is isomorphic to* C_n*, where* $n \geqslant 5$*. Then* $S^2 = \{0, c\}$*.*

Lemma 3.11. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *such that there exists no end vertex adjacent to* c *in* $\Gamma(S)$*. Also assume that* $\Gamma(S)^* \cong C_n$*, where* $n \geqslant 5$ *and* x*,* $y \in Z(S)^* \setminus \{c\}$*. Then the following statements hold.*

1. *If* $x \sim y$ *in* $\Gamma(S)$*, then* $x \sim y$ *in* $AG(S)$*.*
2. *If* $x \nsim y$ *in* $\Gamma(S)$ *and* $x^2 \neq 0$ *or* $y^2 \neq 0$*, then* $x \sim y$ *in* $AG(S)$*.*
3. *If* $x \nsim y$ *in* $\Gamma(S)$ *and* $n \geqslant 7$*, then* $x \sim y$ *in* $AG(S)$*.*
4. $x \nsim y$ *in* $AG(S)$ *if and only if* $x^2 = y^2 = 0$*,* $xy = c$ *and* $n = 5$*, or* $x^2 = y^2 = 0$*,* $\mathrm{d}(x, y) = 3$ *in* $\Gamma(S)$ *and* $n = 6$*.*

Proof. The proof of (1) and (2) is clear.

(3). Since $\Gamma(S) \cong C_n$ and $n \geqslant 7$, we have $|V(\Gamma(S)^*)| \geqslant 7$, and so $|Z(S)| \geqslant 9$, since $Z(S) = C_n \cup \{0, c\}$. On the other hand, for each two distinct vertices x and y in $\Gamma(S)^*$, we see that $|\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y)| \leqslant 8$. Since $x \nsim y$ in $\Gamma(S)$, by Proposition 3.10, we have $xy = c$, and so $\mathrm{ann_S}(xy) = Z(S)$. Hence $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) \neq \mathrm{ann_S}(xy)$, and therefore $x \sim y$ in $AG(S)$.

(4). First suppose that $x \nsim y$ in $AG(S)$. Then, by (i), (ii), (iii) and Proposition 3.10, we have $x^2 = y^2 = 0$, $xy = c$ and $n = 5$, or $n = 6$. If $n = 6$ and $\mathrm{d}(x, y) = 2$ in $\Gamma(S)$, then there exists a vertex z, such that $z \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(y)$, and so $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) \neq Z(S) = \mathrm{ann_S}(c) = \mathrm{ann_S}(xy)$. Thus $x \sim y$ in $AG(S)$, which is impossible. Also if $\mathrm{d}(x, y) = 1$ in $\Gamma(S)$, then $x \sim y$ in $\Gamma(S)$ and, by (i), $x \sim y$ in $AG(S)$, which is again impossible. Therefore $\mathrm{d}(x, y) = 3$ in $\Gamma(S)$.

Conversely, first suppose that $n = 5$, $x^2 = y^2 = 0$ and $xy = c$. Then, since $x \nsim y$ in $\Gamma(S)$ and x, $y \in C_5$, we have $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = Z(S) = \mathrm{ann_S}(c) = \mathrm{ann_S}(xy)$. Thus $x \nsim y$ in $AG(S)$.
Now suppose that $x^2 = y^2 = 0$, $\mathrm{d}(x, y) = 3$ in $\Gamma(S)$ and $n = 6$. Then $Z(S) = C_6 \cup \{0, c\}$, and so $|Z(S)| = 8$. Also since $\mathrm{d}(x, y) = 3$, we see that $\mathrm{ann_S}(x) \cap \mathrm{ann_S}(y) = \{0, c\}$ and $|\mathrm{ann_S}(x)| = |\mathrm{ann_S}(y)| = 5$, and so $|\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y)| = 8 = |Z(S)| = |\mathrm{ann_S}(c)| = |\mathrm{ann_S}(xy)|$. Thus $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = \mathrm{ann_S}(xy)$. Therefore $x \nsim y$ in $AG(S)$. □

The following three theorems immediately follows from Lemma 3.11, [1, Theorems 3.1 and 3.8].

Theorem 3.12. *Assume that all the hypothesis of Lemma 3.11 hold and* $n \geqslant 7$*. Then we have the following statements.*

1. *If* $Z(S) \neq S$*, then* $AG(S) \cong K_{n+1}$*.*
2. *If* $Z(S) = S$*, then* $AG(S) \cong K_n \cup K_1$*, where* c *is an isolated vertex in* $AG(S)$*.*

Theorem 3.13. *Suppose that all the hypothesis of Lemma 3.11 hold and* $n = 6$*. Then we have the following statements.*

1. *If* $Z(S) \neq S$*, then* $AG(S) \cong K_7 \setminus \{\{xy\} | x^2 = y^2 = 0, \mathrm{d}(x, y) = 3$ *in* $\Gamma(S)\}$.
2. *If* $Z(S) = S$*, then* $AG(S) \cong K_1 \cup K_6 \setminus \{\{xy\} | x^2 = y^2 = 0, \mathrm{d}(x, y) = 3$ *in* $\Gamma(S)\}$*, where* c *is an isolated vertex in* $AG(S)$.

Theorem 3.14. *Suppose that all the hypothesis of Lemma 3.11 hold and* $n = 5$*. Then we have the following statements.*

1. *If* $Z(S) \neq S$*, then* $AG(S) \cong K_6 \setminus \{\{xy\} | x^2 = y^2 = 0, xy = c\}$.
2. *If* $Z(S) = S$*, then* $AG(S) \cong K_1 \cup K_5 \setminus \{\{xy\} | x^2 = y^2 = 0, xy = c\}$*, where* c *is an isolated vertex in* $AG(S)$.

If $Z(S) \neq S$, then, by [1, Theorem 3.1], $\Gamma(S) \leqslant AG(S)$, and since $\mathrm{gr}(\Gamma(S)) \leqslant 4$, we have $\mathrm{gr}(AG(S)) \leqslant 4$. But if $Z(S) = S$, then the following example shows that 5 is sharp for the girth of $AG(S)$.

Example 3.15. Suppose that $S = \{0, c, a_1, a_2, a_3, a_4, a_5\}$, with $a_1a_2 = a_2a_3 = a_3a_4 = a_4a_5 = a_5a_1 = 0$, $cS = 0$ and $a_i^2 = c^2 = 0$, for each $1 \leqslant i \leqslant 5$. Otherwise $a_ia_j = c$. Then $Z(S) = S$ and, by [22, Theorem 2.5], S is a semigroup and $\Gamma(S)$ is a refinement of a star graph with center c such that there exists no end vertex adjacent to c in $\Gamma(S)$ and $\Gamma(S)^* \cong C_5$.

Now, by Theorem 3.14 (ii), $AG(S) \cong K_1 \cup C_5$ which means that $gr(AG(S)) = 5$.

Theorem 3.16. *Suppose that all the hypothesis of Lemma 3.11 hold and* $n = 3$*. Then we have the following statements.* 1. *If* $Z(S) \neq S$*, then* $AG(S) \cong K_4$.

2. *If* $Z(S) = S$*, then* $AG(S) \cong 4K_1$.

Proof. Since there exists no end vertex adjacent to c in $\Gamma(S)$ and $\Gamma(S)^* \cong C_3 \cong K_3$, we have $\Gamma(S) \cong K_4$. Now, by [1, Theorems 3.1 and 3.9], the results hold. □

For the case $n = 4$, we have the following lemma.

Lemma 3.17. *Suppose that all the hypothesis of Lemma 3.11 hold and* $n = 4$*. Also assume that* $V(\Gamma(S)^*) = \{x, y, z, w\}$ *with* $x \sim y \sim z \sim w \sim x$*. Then we have the following statements.*

1. $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = \mathrm{ann_S}(y) \cup \mathrm{ann_S}(z) = \mathrm{ann_S}(z) \cup \mathrm{ann_S}(w) = \mathrm{ann_S}(w) \cup \mathrm{ann_S}(x) = Z(S)$.
2. $xz \in \{x, z, c\}$ *and* $wy \in \{w, y, c\}$.
3. $x \nsim z$ *in* $AG(S)$ *if and only if* $xz = x$*, or* $xz = z$*, or* $xz = c$ *and* $x^2 = z^2 = 0$*. Also* $w \nsim y$ *in* $AG(S)$ *if and only if* $wy = w$*, or* $wy = y$*, or* $wy = c$ *and* $w^2 = y^2 = 0$.
4. $x \sim z$ *in* $AG(S)$ *if and only if* $xz = c$ *and* $x^2 \neq 0$ *or* $z^2 \neq 0$*. Also* $w \sim y$ *in* $AG(S)$ *if and only if* $wy = c$ *and* $w^2 \neq 0$ *or* $y^2 \neq 0$.

Proof. (1). Since $V(\Gamma(S)^*) = \{x, y, z, w\}$ with $x \sim y \sim z \sim w \sim x$, we have $Z(S) = \{0, c, x, y, z, w\}$, and $\mathrm{ann_S}(x) \supseteq \{0, c, y, w\}$ and $\mathrm{ann_S}(y) \supseteq \{0, c, x, z\}$.

Thus $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(y) = Z(S)$. Similarly, $\mathrm{ann_S}(y) \cup \mathrm{ann_S}(z) = \mathrm{ann_S}(z) \cup \mathrm{ann_S}(w) = \mathrm{ann_S}(w) \cup \mathrm{ann_S}(x) = Z(S)$.

(2). Since $x \nsim z$ and $w \nsim y$ in $\Gamma(S)$, we have $xz \neq 0$ and $wy \neq 0$. If $xz = y$, then $wy = w(xz) = (wx)z = 0$, which is impossible. So $xz \neq y$. Similarly $xz \neq w$. Thus $xz \in \{x, z, c\}$. By a similar argument, $wy \in \{w, y, c\}$.

(3). Suppose that $x \nsim z$ in $AG(S)$, $xz \neq x$ and $xz \neq z$. Then, by (ii), $xz = c$. If $x^2 \neq 0$, then $x \notin \mathrm{ann_S}(x) \cup \mathrm{ann_S}(z)$, and so $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(z) \neq Z(S) = \mathrm{ann_S}(c) = \mathrm{ann_S}(xz)$. This implies that $x \sim z$ in $AG(S)$, which is impossible. Therefore $x^2 = 0$, and similarly $z^2 = 0$.

Conversely, if $xz = x$ or $xz = z$, then $x \nsim z$ in $AG(S)$. Now suppose that $xz = c$ and $x^2 = z^2 = 0$. Then $\mathrm{ann_S}(x) = \{0, c, x, y, w\}$ and $\mathrm{ann_S}(z) = \{0, c, y, z, w\}$, and so $\mathrm{ann_S}(x) \cup \mathrm{ann_S}(z) = \{0, c, x, y, z, w\} = Z(S) = \mathrm{ann_S}(c) = \mathrm{ann_S}(xz)$. Therefore $x \nsim z$ in $AG(S)$. In the same manner we can see that $w \nsim y$ in $AG(S)$ if and only if $wy = w$, or $wy = y$, or $wy = c$ and $w^2 = y^2 = 0$.

(4) By (3), it is clear. □

The following two corollaries follow from Lemma 3.17 and [1, Theorems 3.1 and 3.8].

Corollary 3.18. *Suppose that all the hypothesis of Lemma* 3.17 *hold and* $Z(S) \neq S$. *Then one of the following statements hold.*

1. $AG(S) \cong K_5$ *if and only if the conditions:*
   (1) $xz = wy = c$,
   (2) $x^2 \neq 0$ *or* $z^2 \neq 0$,
   (3) $w^2 \neq 0$ *or* $y^2 \neq 0$ *hold.*
2. $AG(S) \cong K_5 \setminus \{\{xz\}\}$ *if and only if the conditions:*
   (1) $wy = c$, *and* $w^2 \neq 0$ *or* $y^2 \neq 0$,
   (2) $xz = x$, *or* $xz = z$, *or* $xz = c$ *and* $x^2 = z^2 = 0$ *hold.*
3. $AG(S) \cong K_5 \setminus \{\{xz\}, \{wy\}\}$ *if and only if the conditions:*
   (1) $wy = w$, *or* $wy = y$, *or* $wy = c$ *and* $w^2 = y^2 = 0$,
   (2) $xz = x$, *or* $xz = z$, *or* $xz = c$ *and* $x^2 = z^2 = 0$ *hold.*

Corollary 3.19. *Suppose that all the hypothesis of Lemma* 3.17 *hold and* $Z(S) = S$. *Then one of the following statements holds.*

1. $AG(S) \cong 2K_2 \cup K_1$, *where* c *is an isolated vertex and* $x \sim z$ *and* $y \sim w$, *if and only if the conditions:*
   (1) $xz = wy = c$,
   (2) $x^2 \neq 0$ *or* $z^2 \neq 0$,
   (3) $w^2 \neq 0$ *or* $y^2 \neq 0$ *hold.*
2. $AG(S) \cong K_2 \cup 3K_1$, *where* c, x, z *are isolated vertices and* $w \sim y$ *if and only if the conditions:*
   (1) $wy = c$,
   (2) $w^2 \neq 0$ *or* $y^2 \neq 0$,
   (3) $xz = x$, *or* $xz = z$, *or* $xz = c$ *and* $x^2 = z^2 = 0$ *hold.*

3. $AG(S) \cong 5K_1$ *if and only if the conditions:*
   (1) $wy = w$, *or* $wy = y$, *or* $wy = c$ *and* $w^2 = y^2 = 0$,
   (2) $xz = x$, *or* $xz = z$, *or* $xz = c$ *and* $x^2 = z^2 = 0$ *hold.*

The next theorem follows from [1, Theorems 3.1 and 3.8].

Theorem 3.20. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $\Gamma(S)^* \cong C_n$, *where* $n \geqslant 5$. *Also assume that*

$$T = \{u \mid u \text{ is an end vertex adjacent to } c \text{ in } \Gamma(S)\}$$

and $\mid T \mid = m \geqslant 1$. *Then the following statements hold.*

1. *If* $x, y \in V(\Gamma(S)^*)$, *then* $x \sim y$ *in* $AG(S)$.
2. *If* $x \in V(\Gamma(S)^*)$ *and* $u \in T$, *then* $x \sim u$ *in* $AG(S)$.
3. *If* $u, v \in T$, *then* $u \sim v$ *in* $AG(S)$.
4. *If* $Z(S) \neq S$, *then* $AG(S) \cong K_{n+m+1}$.
5. *If* $Z(S) = S$, *then* $AG(S) \cong K_{n+m} \cup K_1$, *where* c *is an isolated vertex in* $AG(S)$.

Proposition 3.21. [22, Theorem 2.6] *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $\Gamma(S)^* \cong P_n$, *where* $n \geqslant 5$. *Then* $S^2 = \{0, c\}$ *and* $c^2 = 0$.

Theorem 3.22. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $\Gamma(S)^* \cong P_n$, *where* $n \geqslant 6$. *Also assume that*

$$T = \{u \mid u \text{ is an end vertex adjacent to } c \text{ in } \Gamma(S)\}$$

and $\mid T \mid = m \geqslant 0$. *Then we have the following statements.*

1. *If* $x, y \in V(\Gamma(S)^*)$, *then* $x \sim y$ *in* $AG(S)$.
2. *If* $x \in V(\Gamma(S)^*)$ *and* $u \in T$, *then* $x \sim u$ *in* $AG(S)$.
3. *If* $u, v \in T$, *then* $u \sim v$ *in* $AG(S)$.
4. *If* $Z(S) \neq S$, *then* $AG(S) \cong K_{n+m+2}$.
5. *If* $Z(S) = S$, *then* $AG(S) \cong K_{n+m+1} \cup K_1$, *where* c *is an isolated vertex in* $AG(S)$.

Proof. The proof follows from Proposition 3.21 and [1, Theorems 3.1 and 3.8]. □

Lemma 3.23. *Suppose that* $\Gamma(S)$ *is a refinement of a star graph with center* c *and* $\Gamma(S)^* \cong P_5$, *with* $a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5 \sim a_6$. *Also assume that there exists no end vertex adjacent to* c *in* $\Gamma(S)$. *Then* $a_2 \sim a_5$ *in* $AG(S)$ *if and only if* $a_2^2 \neq 0$ *or* $a_5^2 \neq 0$. *Otherwise,* $a_i \sim a_j$ *in* $AG(S)$, *for each* $1 \leqslant i < j \leqslant 6$.

Proof. By proposition 3.15, for each $1 \leqslant i < j \leqslant 6$, we have $a_i a_j = 0$ or $a_i a_j = c$ and $c^2 = 0$, which follows that $\mathrm{ann_S}(a_i a_j) = Z(S)$. Now if $a_2^2 \neq 0$ or $a_5^2 \neq 0$, then $ann_S(a_2) \cup \mathrm{ann_S}(a_5) \neq Z(S) = \mathrm{ann_S}(a_2 a_5)$, which implies that $a_2 \sim a_5$ in $AG(S)$.

Conversely, suppose on the contrary that $a_2 \sim a_5$ in $AG(S)$ and $a_2^2 = a_5^2 = 0$. Then $ann_S(a_2) \cup \mathrm{ann_S}(a_5) = Z(S) = \mathrm{ann_S}(a_2 a_5)$, which is a contradiction. Thus $a_2^2 \neq 0$ or $a_5^2 \neq 0$.

Finally, since $\Gamma(S)^* \cong P_5$, it implies that, for each $1 \leqslant i < j \leqslant 6$, other than the case $i = 2$ and $j = 5$, we have $\mathrm{ann_S}(a_i) \cup \mathrm{ann_S}(a_j) \neq Z(S) = \mathrm{ann_S}(a_i a_j)$, which implies that $a_i \sim a_j$ in $AG(S)$. □

Theorem 3.24. *Suppose that all the hypothesis of Lemma* 3.23 *hold. Then we have the following statements.*

1. *If $Z(S) \neq S$, then $AG(S) \cong K_7$ if and only if $a_2^2 \neq 0$ or $a_5^2 \neq 0$. Otherwise $AG(S) \cong K_7 \setminus \{a_2 a_5\}$.*
2. *If $Z(S) = S$, then $AG(S) \cong K_1 \cup K_6$ if and only if $a_2^2 \neq 0$ or $a_5^2 \neq 0$.*
   Otherwise $AG(S) \cong K_1 \cup K_6 \setminus \{a_2 a_5\}$, where c is an isolated vertex in $AG(S)$.

Proof. By Lemma 3.23 and [1, Theorems 3.1 and 3.8], it is clear. □

Lemma 3.25. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_5$, with $a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5 \sim a_6$. Also assume that $T = \{u \mid u$ is an end vertex adjacent to c in $\Gamma(S)\}$ and $\mid T \mid = m \geqslant 1$. Then we have the following statements.*

1. *If $Z(S) \neq S$, then $AG(S) \cong K_{7+m}$.*
2. *If $Z(S) = S$, then $AG(S) \cong K_{6+m} \cup K_1$, where c is an isolated vertex in $AG(S)$.*

For the case $n \leqslant 4$, Proposition 3.21 doesn't hold. For the case $n = 4$, we have the following two lemmas.

Lemma 3.26. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_4$, with $a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5$. Then the following statements hold.*

1. *$\Gamma(S)^* \leqslant AG(S)$.*
2. *$a_1 a_3 \in \{a_3, c\}$, $a_1 a_4 = c$, $a_1 a_5 \in \{a_3, c\}$, $a_2 a_4 = c$, $a_2 a_5 = c$ and $a_3 a_5 \in \{a_3, c\}$.*

Proof. (1). Since $a_5 \notin \mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_2) \cup \mathrm{ann_S}(a_3)$ and $a_1 \notin \mathrm{ann_S}(a_3) \cup \mathrm{ann_S}(a_4) \cup \mathrm{ann_S}(a_5)$, which follows that $\Gamma(S)^* \cong P_4 \leqslant AG(S)$.
(2). Since $a_1 \nsim a_3$ in $\Gamma(S)$, we have $a_1 a_3 \neq 0$. If $a_1 a_3 = a_1$, then $a_1 a_4 = (a_1 a_3) a_4 = a_1 (a_3 a_4) = 0$, and if $a_1 a_3 = a_2$, then $a_2 a_4 = 0$, which are impossible. Also if $a_1 a_3 = a_4$, then $a_2 a_4 = 0$, and if $a_1 a_3 = a_5$, then $a_2 a_5 = 0$, which are again impossible. Thus $a_1 a_3 \in \{a_3, c\}$. The similar arguments applies to the other cases. □

If $a_1 a_3 = a_3$, then $a_1 \nsim a_3$ in $AG(S)$, and if $a_1 a_3 = c$, then $a_1 \sim a_3$ in $AG(S)$, since $a_5 \notin \mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_3)$. Also if $a_1^2 = 0$ and $a_4^2 = 0$, then $\mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_4) = \{a_1, a_2, a_3, a_4, a_5, c, 0\} = \mathrm{ann_S}(c) = \mathrm{ann_S}(a_1 a_4)$. Thus $a_1 \sim a_4$ in $AG(S)$ if and only if $a_1^2 \neq 0$ or $a_4^2 \neq 0$. Since $a_3 \notin \mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_5)$ and $a_3 \in \mathrm{ann_S}(c) = \mathrm{ann_S}(a_1 a_5)$, if $a_1 a_5 = c$, then $a_1 \sim a_5$ in $AG(S)$. If $a_1 a_5 = a_3$, then $a_1^2a_5 = a_1a_3 \neq 0$ and $a_5^2a_1 = a_5a_3 \neq 0$, and so $a_1^2 \neq 0$ and $a_5^2 \neq 0$. Now if $a_3^2 \neq 0$, then $\mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_5) = \{a_2, a_4, c, 0\} = \mathrm{ann_S}(a_3) = \mathrm{ann_S}(a_1a_5)$. Hence if $a_1a_5 = a_3$, then $a_1 \sim a_5$ in $AG(S)$ if and only if $a_3^2 = 0$. Similarly, $a_2 \sim a_4$ in $AG(S)$ if and only if $a_2^2 \neq 0$ or $a_4^2 \neq 0$, and $a_2 \sim a_5$ in $AG(S)$ if and only if $a_2^2 \neq 0$ or $a_5^2 \neq 0$. Clearly, if $a_3a_5 = a_3$, then $a_3 \nsim a_5$ in $AG(S)$, and since $a_1 \notin \mathrm{ann_S}(a_3) \cup \mathrm{ann_S}(a_5)$, if $a_3a_5 = c$, then $a_3 \sim a_5$ in $AG(S)$.

For example, suppose that $S = \{0, c, a_1, a_2, a_3, a_4, a_5\}$, with $a_1a_2 = a_2a_3 = a_3a_4 = a_4a_5 = 0$, $a_1a_3 = a_1a_5 = a_3a_5 = a_3$, $a_1a_4 = a_2a_4 = a_2a_5 = c$, $a_1^2 = a_3^2 = a_5^2 = a_3$ and $a_2^2 = c$, $a_4^2 = 0$. Then, by [22, Exampe 2.7], S is a commutative semigroup and $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_4$, with $a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5$. Also there exists no end vertex adjacent to c in $\Gamma(S)$. See Figure 1.

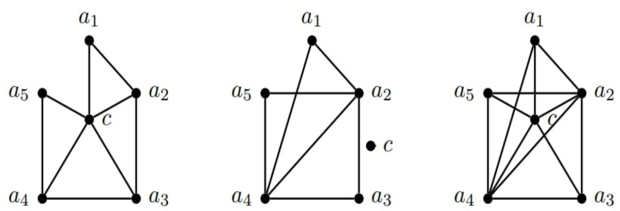

Figure 1. $\Gamma(S)$ $\quad$ $AG(S)$, $Z(S) = S$ $\quad$ $AG(S)$, $Z(S) \neq S$

Lemma 3.27. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_4$, with $a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5$. Also assume that*

$$T = \{u \mid u \text{ is an end vertex adjacent to } c \text{ in } \Gamma(S)\}$$

and $\mid T \mid= m \geqslant 1$. Then the following statements hold.

1. *For each u, $v \in T$, if $uv \notin T$, or $uv = t \in T \setminus \{u, v\}$ and $t^2 = 0$, then $u \sim v$ in $AG(S)$. Otherwise $u \nsim v$ in $AG(S)$.*
2. *For each $a_i \in V(\Gamma(S)^*)$ and $u \in T$, we have $a_iu \notin T$ and $a_i \sim u$ in $AG(S)$ if and only if $a_iu \neq a_i$, for $1 \leqslant i \leqslant 5$.*

Proof. (1). If $uv \notin T$, then $uv = c$ or $uv = a_i$, $(1 \leqslant i \leqslant 5)$. If $uv = c$, then $c^2 = 0$ and clearly $u \sim v$ in $AG(S)$. Assume that $uv = a_i$, $(1 \leqslant i \leqslant 5)$. Then there exists a_j, $(1 \leqslant j \leqslant 5$ and $j \neq i)$ such that $a_ia_j = 0$, $ua_j \neq 0$ and $va_j \neq 0$. Thus $a_j \in \mathrm{ann_S}(a_i) = \mathrm{ann_S}(uv)$ and $a_j \notin \mathrm{ann_S}(u) \cup \mathrm{ann_S}(v)$, and hence $u \sim v$ in $AG(S)$.

Now suppose that $uv = t \in T \setminus \{u, v\}$ and $t^2 = 0$. Then $u^2v = ut \neq 0$, and so $u^2 \neq 0$ also $v^2 \neq 0$. Thus $\mathrm{ann_S}(u) \cup \mathrm{ann_S}(v) = \{0, c\} \neq \{0, c, t\} = \mathrm{ann_S}(t)$, which implies that $u \sim v$ in $AG(S)$. Otherwise if $uv = u$, or $uv = v$, or $uv = t$ and $t^2 \neq 0$, then clearly $u \nsim v$ in $AG(S)$.

(2). If $a_iu = t \in T$, then there exists $a_j \in \mathrm{ann_S}(a_i)$, $j \neq i$, such that $a_jt = a_j(a_iu) = (a_ja_i)u = 0$, which is impossible. Thus $a_iu \notin T$, and so $a_iu = c$ or $a_iu = a_j$ and $1 \leqslant j \leqslant 5$. If $a_iu = c$, then clearly $a_i \sim u$ in $AG(S)$, since there exists a_j, $(1 \leqslant j \leqslant 5$ and $j \neq i)$, such that $a_ia_j \neq 0$, $ua_j \neq 0$ and $ca_j = 0$.

Now if $a_1u = a_4$, then $a_2a_4 = a_2(a_1u) = (a_2a_1)u = 0$, and if $a_1u = a_5$, then $a_2a_5 = 0$, which are impossible. Thus $a_1u \in \{c, a_1, a_2, a_3\}$. Similarly we have $a_5u \in \{c, a_3, a_4, a_5\}$, $a_2u \in \{c, a_2\}$, $a_3u \in \{c, a_3\}$, and $a_4u \in \{c, a_4\}$.

Now by the above discussion the statement (2) holds. □

In this case, by Lemma 3.26, $\Gamma(S)^* \leqslant AG(S)$ and we have $a_1 \sim a_4 \sim a_2 \sim a_5$ in $AG(S)$ and $a_1 \sim a_3$ in $AG(S)$ if and only if $a_1a_3 = c$ and $a_3 \sim a_5$ in $AG(S)$ if and only if $a_3a_5 = c$. Also $a_1 \sim a_5$ in $AG(S)$ if and only if $a_1a_5 = c$, or $a_1a_5 = a_3$ and $a_3^2 = 0$.

For the case $n = 3$, we have the following two lemmas.

Lemma 3.28. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_3$, with $a_1 \sim a_2 \sim a_3 \sim a_4$. Also assume that there exists no end vertex adjacent to c in $\Gamma(S)$. Then the following statements hold.*

1. *$a_1 \sim a_2$ and $a_3 \sim a_4$ in $AG(S)$, but if $Z(S) = S$, then $a_2 \nsim a_3$ in $AG(S)$.*
2. *$a_1a_3 \in \{a_3, c\}$, $a_1a_4 \in \{a_2, a_3, c\}$, $a_2a_4 \in \{a_2, c\}$. Also if $a_1a_4 = a_2$, then $a_2^2 = 0$, and $a_4^2 \neq 0$, and if $a_1a_4 = a_3$, then $a_3^2 = 0$ and $a_1^2 \neq 0$.*

Proof. (1). Since $a_4 \notin \mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_2)$ and $a_1 \notin \mathrm{ann_S}(a_3) \cup \mathrm{ann_S}(a_4)$, we have $a_1 \sim a_2$ and $a_3 \sim a_4$ in $AG(S)$. Also we see that $\mathrm{ann_S}(a_2) \cup \mathrm{ann_S}(a_3) = Z(S)$ and $\mathrm{ann_S}(a_2a_3) = S$, and so if $Z(S) = S$, then $a_2 \nsim a_3$ in $AG(S)$.

(2). Since $a_1 \nsim a_3$ in $\Gamma(S)$, we have $a_1a_3 \neq 0$. If $a_1a_3 = a_1$, then $a_1a_4 = (a_1a_3)a_4 = a_1(a_3a_4) = 0$, and if $a_1a_3 = a_2$, then $a_2a_4 = 0$, which are impossible. Also if $a_1a_3 = a_4$, then $a_2a_4 = 0$, which is again impossible. Thus $a_1a_3 \in \{a_3, c\}$. Since $a_1 \nsim a_4$ in $\Gamma(S)$, we have $a_1a_4 \neq 0$. If $a_1a_4 = a_1$, then $a_1a_3 = (a_1a_4)a_3 = a_1(a_4a_3) = 0$, and If $a_1a_4 = a_4$, then $a_2a_4 = 0$, which are again impossible. Thus $a_1a_4 \in \{a_2, a_3, c\}$. Similarly, $a_2a_4 \in \{a_2, c\}$. Also if $a_1a_4 = a_2$, then $a_2^2 = a_2(a_1a_4) = (a_2a_1)a_4 = 0$, and since $a_1a_4^2 = a_2a_4 \neq 0$, we have $a_4^2 \neq 0$. Similarly, if $a_1a_4 = a_3$, then $a_3^2 = 0$ and $a_1^2 \neq 0$. □

If $a_1a_3 = a_3$, then $a_1 \nsim a_3$ in $AG(S)$, and if $a_1a_3 = c$, then $a_1 \sim a_3$ in $AG(S)$ if and only if $a_1^2 \neq 0$ or $a_3^2 \neq 0$. If $a_1a_4 = c$, then $a_1 \sim a_4$ in $AG(S)$ if and only if $a_1^2 \neq 0$ or $a_4^2 \neq 0$. Assume that $a_1a_4 = a_2$. Then $a_2^2 = 0$ and $a_4^2 \neq 0$. If $a_1^2 = 0$, then $\mathrm{ann_S}(a_1) \cup \mathrm{ann_S}(a_4) = \{0, c, a_1, a_2, a_3\} = \mathrm{ann_S}(a_2)$, and so $a_1 \nsim a_4$ in $AG(S)$. Thus if $a_1a_4 = a_2$, then $a_1 \sim a_4$ in $AG(S)$ if and only if $a_1^2 \neq 0$. Similarly, if $a_1a_4 = a_3$, then $a_1 \sim a_4$ in $AG(S)$ if and only if $a_4^2 \neq 0$. Moreover $a_2 \sim a_4$ in $AG(S)$ if and only if $a_2a_4 = c$ and $a_2^2 \neq 0$ or $a_4^2 \neq 0$. Clearly, if $a_2a_4 = a_2$, then $a_2 \nsim a_4$ in $AG(S)$.

Lemma 3.29. *Suppose that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_3$, with $a_1 \sim a_2 \sim a_3 \sim a_4$. Also assume that*

$T = \{u \mid u$ *is an end vertex adjacent to c in* $\Gamma(S)\}$

and $\mid T \mid= m \geqslant 1$. Then the following statements hold.

1. *$\Gamma(S)^* \leqslant AG(S)$.*
2. *$a_1a_3 \in \{a_3, c\}$, $a_1a_4 \in \{a_2, a_3, c\}$, $a_2a_4 \in \{a_2, c\}$. Also if $a_1a_4 = a_2$, then*

$a_2^2 = 0$, and also if $a_1a_4 = a_2$, then $a_2^2 = 0$ and $a_4^2 \neq 0$, and if $a_1a_4 = a_3$, then $a_3^2 = 0$ and $a_1^2 \neq 0$.

3. *For each u, $v \in T$, if $uv \notin T$, or $uv = t \in T \setminus \{u, v\}$ and $t^2 = 0$, then $u \sim v$ in $AG(S)$. Otherwise $u \nsim v$ in $AG(S)$.*
4. *For each $a_i \in V(\Gamma(S)^*)$ and $u \in T$, we have $a_iu \notin T$ and $a_i \sim u$ in $AG(S)$ if and only if $a_iu \neq a_i$, for $1 \leqslant i \leqslant 5$.*

Proof. Since $a_2a_3 = 0$, $ua_2 \neq 0$ and $ua_3 \neq 0$, we have $u \notin \mathrm{ann_s}(a_2) \cup \mathrm{ann_s}(a_3)$ and $u \in \mathrm{ann_s}(a_2a_3)$. Thus $a_2 \sim a_3$ in $AG(S)$. Now, by using argument similar to that we used in the proof of Lemmas 3.27 and 3.28, the results hold. □

In this case, $a_1 \sim a_3$ in $AG(S)$ if and only if $a_1a_3 = c$, and if $a_1a_4 = c$, then $a_1 \sim a_4$ in $AG(S)$. Also if $a_1a_4 = a_2$, then $a_1 \sim a_4$ in $AG(S)$ if and only if $a_1^2 \neq 0$. Similarly, if $a_1a_4 = a_3$, then $a_1 \sim a_4$ in $AG(S)$ if and only if $a_4^2 \neq 0$. Moreover $a_2 \sim a_4$ in $AG(S)$ if and only if $a_2a_4 = c$ and $a_2^2 \neq 0$ or $a_4^2 \neq 0$

Assume that $\Gamma(S)$ is a refinement of a star graph with center c and $\Gamma(S)^* \cong P_2$, with $a_1 \sim a_2 \sim a_3$ such that there exists no end vertex adjacent to c in $\Gamma(S)$. Then $\Gamma(S) \cong K_4 \setminus \{a_1a_2\}$ and we can see [20, Lemmas 3.11, 3.15, 4.12, 4.16]. Also for the case $n = 1$, we can see [20, Lemmas 3.17, 3.12, 3.21, 4.9, 4.17] and [1, Section 4].

Acknowledgments. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

References

[1] **M. Afkhami, K. Khashyarmanesh and M. Sakhdari,** *On the annihilator graphs of semigroups,* J. Algebra Appl., **14** (2015), 1550015 – 1550029.

[2] **M. Afkhami and K. Khashyarmanesh,** *The cozero-divisor graph of a commutative ring,* Southeast Asian Bull. Math., **35** (2011), 753 – 762.

[3] **S. Akbari, H.R. Maimani and S. Yassemi,** *When a zero-divisor graph is planar or a complete r-partite graph,* J. Algebra, **270** (2003), 169 – 180.

[4] **D.F. Anderson and A. Badawi,** *On the zero-divisor graph of a ring,* Comm. Algebra, **36** (2008), 3073 – 3092.

[5] **D.F. Anderson and A. Badawi,** *The total graph of a commutative ring,* J. Algebra, **320** (2008), 2706 – 2719.

[6] **D.F. Anderson and A. Badawi,** *The total graph of a commutative ring without the zero element,* J. Algebra Appl., **11** (2012), 12500740 – 12500758.

[7] **D.F. Anderson and A. Badawi,** *The generelized total graph of a commutative ring,* J. Algebra Appl., **12** (2013), 1250212 – 1250230.

[8] **D.F. Anderson, A. Frazier, A. Lauve and P.S Livingston,** *The zero-divisor graph of a commutative ring. II,* Ideal Theoretic Methods in Comm. Algebra (Columbia, MO, 1999) 61 – 72.

[9] **D.F. Anderson and P.S. Livingston,** *The zero-divisor graph of a commutative ring,* J. Algebra, **217** (1999), 434 – 447.

[10] **A. Badawi,** *On the annihilator graph of a commutative ring,* Commun. Algebra, **42** (2014), 1 – 14.

[11] **A. Badawi,** *On dot-product graph of a commutative ring,* Commun. Algebra, **43** (2015), 43 – 50.

[12] **Z. Barati, K. Khashyarmanesh, F. Mohammadi and Kh. Nafar,** *On the associated graphs to a commutative ring,* J. Algebra Appl., **11** (2012), 1250037 – 1250054.

[13] **I. Beck,** *Coloring of commutative rings,* J. Algebra, **116** (1998), 208 – 226.

[14] **J.A. Bondy and U.S.R. Murty,** *Graph Theory with Applications,* American Elsevier, New York, (1976).

[15] **F.R. DeMeyer and L. DeMeyer,** *Zero-divisor graphs of semigroups,* J. Algebra, **283** (2005), 190 – 198.

[16] **L. DeMeyer, M. Dsa, I. Epstein, A. Geiser and K. Smith,** *Semigroups and the zero-divisor graph,* Bull. Inst. Comb. Appl., **57** (2009), 60 – 70.

[17] **L. Demeyer, T. Mckenzie and K. Schneider,** *The zero-divisor graph of a commutative semigroup,* Semigroup Forum, **65** (2002), 206 – 214.

[18] **A.V. Kelarev and C.E. Praeger,** *On transitive Cayley graphs of groups and semigroups,* European J. Combin., **24** (2003), 59 – 72.

[19] **K. Khashyarmanesh and M.R. Khorsandi,** *A generalization of the unit and unitary Cayley graphs of a commutative ring,* Acta Math. Hungar., **137** (2012), 242 – 253.

[20] **M. Sakhdari, K. Khashyarmanesh and M. Afkhami,** *Annihilator graphs with four vertices,* Semigroup Forum. **94** (2017), 139 – 166.

[21] **T. S. Wu and D.C. Lu,** *Subsemigroups determined by the zero-divisor graph,* Discrete Math., **308** (2008), 5122 – 5135.

[22] **T.S. Wu, Q. Liu and L. Chen,** *Zero-divisor semigroups and refinement of a star graph,* Discrete Math., **309** (2009), 2510 – 2518.

Received March 2, 2021

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