Quasigroups and Related Systems **29** (2021), 133 − 144

Characterizations of regularities on ordered semirings by idempotency of ordered ideals

Kongpop Siribute, Pakorn Palakawong na Ayutthaya and Jatuporn Eanborisoot

Abstract. We characterize regular, intra-regular, left weakly regular, right weakly regular and fully idempotent ordered semirings using idempotency of several kinds of ordered ideals including left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals, ordered bi-ideals and ordered interior ideals. Moreover, we characterize (m, n)-regular ordered semirings in terms of their ordered (m, n)-ideals.

1. Introduction

The notion of a regular semiring was defined as a similar way of a regular ring defined by von Neumann [6], i.e., for each element a of a semiring S, $a = axa$ for some $x \in S$ (equivalently, $a \in aSa$ for all $a \in S$). Later, in sense of Ahsan, Mordeson and Shabir [2], a semiring S is called intra-regular if for each $a \in S$, $a = \sum_{i \in I} x_i a^2 y_i$ for some $x_i, y_i \in S$ and finite index set I. This notion is equivalent to $a \in \Sigma S a^2 S$ for all $a \in S$ where $\Sigma S a^2 S$ is the set of all finite sums of elements in Sa^2S. In 1993, Ahsan [1] called a semiring S to be fully idempotent if every ideal of S is idempotent. We are able to study the fully idempotency of a semiring as a kind of regularities due to the fact that a semiring S is fully idempotent if and only if $a \in \Sigma SaSaS$ for all $a \in S$. Later, Shabir and Anjum [12] studied the concept of a right k-weakly regular hemiring in terms of its fuzzy ideals. They defined a hemiring to be right k-weakly regular if $a \in \overline{\Sigma aSaS}$ for all $a \in S$.

An ordered semiring is a notable generalization of a semiring, in other words, a semiring S is an ordered semiring together with the relation $\{(x, x) \mid x \in S\}$. In sense of Gan and Jiang [4], an ordered semiring is a semiring S together with a partial order on S satisfying the compatibility property. In [4], the notion of an ordered ideal of an ordered semiring was defined. In 2012, Mandal [5] introduced the notion of a regular ordered semiring by for each $a \in S$, $a \leqslant axa$ for some $x \in S$, i.e., $a \in (aSa]$ for all $a \in S$.

In this work, as generalizations of intra-regular semirings [2] and fully idempotent semirings [1], we give the notions of intra-regular ordered semirings and

2010 Mathematics Subject Classification: 16Y60, 06F25.
Keywords: ordered semiring, regular ordered semiring, ordered ideal, fully idempotent ordered semiring, (m, n)-regular ordered semiring, ordered (m, n)-ideal.

fully idempotent ordered semirings. In addition, as a similar way of [12], we study the notion of left weakly regular and right weakly regular ordered semirings in the form $a \in (\Sigma SaSa]$ and $a \in (\Sigma aSaS]$ for all $a \in S$, respectively. Then, we use idempotency of left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals, ordered bi-ideals and ordered interior ideals to characterize mentioned kinds of regularities on ordered semirings. Moreover, we define an ordered (m,n)-ideals of an ordered semiring in a similar way of an (m,n)-ideals of an ordered semigroup defined by Sanborisoot and Changphas [11] and also study it on an (m,n)-regular ordered semiring as an analogous way on an (m,n)-regular ordered semigroup [3]. In conclusion, we have that the idempotency of each kind of ordered ideals of an ordered semiring is able to lead the ordered semiring to be different kinds of regularities.

2. Preliminaries

An *ordered semiring* [4] is a system $(S,+,\cdot,\leqslant)$ such that $(S,+,\cdot)$ is a semiring and $(S,\leqslant)$ is a poset satisfying the compatibility property, i.e., if $a \leqslant b$, then $a+c \leqslant b+c$, $c+a \leqslant c+b$, $ac \leqslant bc$ and $ca \leqslant cb$ for all $a,b,c \in S$. An element 0 of an ordered semiring $(S,+,\cdot,\leqslant)$ is called an *absorbing zero* if $x+0 = x = 0+x$ and $x0 = 0 = 0x$ for all $x \in S$.

Throughout this work, we simply write S instead of an ordered semiring $(S,+,\cdot,\leqslant)$ and always assume that S is additively commutative (i.e., $a+b = b+a$ for all $a,b \in S$) together with an absorbing zero 0.

For $\emptyset \neq A, B \subseteq S$, we denote that $A+B = \{a+b \mid a \in A, b \in B\}$, $AB = \{ab \mid a \in A, b \in B\}$ and $(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in S\}$. The set of all finite sums of elements in $\emptyset \neq A \subseteq S$ is denoted by $\Sigma A = \left\{\sum_{i\in I} a_i \mid a_i \in A \text{ and } I \text{ is a finitie set}\right\}$. If $I = \emptyset$, then we set $\sum_{i\in I} a_i = 0$ for all $a_i \in S$.

For basic properties of the finite sums Σ and the operator $(\;\;]$, we refer to [8–10] However, we give the following useful remark which will be used in the main results.

Remark 2.1. Let A and B be nonempty subsets of an ordered semiring S. Then $(\Sigma(A](B]] \subseteq (\Sigma AB]$.

Definition 2.2. Let A be a nonempty subset of an ordered semiring S such that $A+A \subseteq A$ and $A = (A]$. Then A is called:

(i) a *left* (*right*) *ordered ideal* [4] of S if $SA \subseteq A$ $(AS \subseteq A)$;
(ii) an *ordered ideal* [4] of S if A is both a left and a right ordered ideal of S;
(iii) an *ordered quasi-ideal* [7] of S if $(\Sigma AS] \cap (\Sigma SA] \subseteq A$;
(iv) an *ordered bi-ideal* of S if $A^2 \subseteq A$ and $ASA \subseteq A$;
(v) an *ordered interior ideal* of S if $A^2 \subseteq A$ and $SAS \subseteq A$.

Let A be a nonempty subset of an ordered semiring S. We denote the notation $L(A)$, $R(A)$, $J(A)$, $Q(A)$ and $I(A)$ to be the smallest left ordered ideals, right ordered ideals, ordered ideals, ordered quasi-ideals and ordered interior ideals of S containing A, respectively. We recall constructions of $L(A)$, $R(A)$ $J(A)$ and $Q(A)$ which occur in [7] as follows.

Lemma 2.3. *Let A be a nonempty subset of an ordered semiring S. The following statements hold:*

(*i*) $L(A) = (\Sigma A + \Sigma SA]$;

(*ii*) $R(A) = (\Sigma A + \Sigma AS]$;

(*iii*) $J(A) = (\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]$;

(*iv*) $Q(A) = (\Sigma A + ((\Sigma AS] \cap (\Sigma SA])]$.

Lemma 2.4. *Let A be a nonempty subset of an ordered semiring S. Then $I(A) = (\Sigma A + \Sigma A^2 + \Sigma SAS]$.*

Proof. Let $\emptyset \neq A \subseteq S$. and $I = (\Sigma A + \Sigma A^2 + \Sigma SAS]$. Clearly, $I + I \subseteq I$, $I = (I]$ and $A \subseteq I$. We have

$$\begin{aligned} I^2 &= (\Sigma A + \Sigma A^2 + \Sigma SAS](\Sigma A + \Sigma A^2 + \Sigma SAS] \\ &\subseteq ((\Sigma A + \Sigma A^2 + \Sigma SAS)(\Sigma A + \Sigma A^2 + \Sigma SAS)] \\ &\subseteq (\Sigma A^2 + \Sigma SAS] \subseteq I \quad \text{and} \\ SIS &= S(\Sigma A + \Sigma A^2 + \Sigma SAS]S \\ &\subseteq (S(\Sigma A + \Sigma A^2 + \Sigma SAS)S] \\ &\subseteq (\Sigma SAS + \Sigma SA^2S + \Sigma SSASS] \\ &\subseteq (\Sigma SAS + \Sigma SAS + \Sigma SAS] = (\Sigma SAS] \subseteq I. \end{aligned}$$

So, I is an ordered interior-ideal of S containing A. If J is an ordered interior-ideal of S containing A, then $I = (\Sigma A + \Sigma A^2 + \Sigma SAS] \subseteq (\Sigma J + \Sigma J^2 + \Sigma SJS] \subseteq (\Sigma J + \Sigma J + \Sigma J] = (\Sigma J] = J$. □

In a particular case of $A = \{a\}$ for some $a \in S$, we write $L(a)$, $R(a)$, $J(a)$, $Q(a)$ and $I(a)$ instead of $L(\{a\})$, $R(\{a\})$, $J(\{a\})$, $Q(\{a\})$ and $I(\{a\})$, respectively. The following corollary is directly obtained by Lemma 2.3 and 2.4.

Corollary 2.5. *Let a be an element of an ordered semiring S. The following statements hold:*

(*i*) $L(a) = (\Sigma a + Sa]$;

(*ii*) $R(a) = (\Sigma a + aS]$;

(*iii*) $J(a) = (\Sigma a + Sa + aS + \Sigma SaS]$;

(*iv*) $Q(a) = (\Sigma a + ((aS] \cap (Sa])]$;

(*v*) $I(A) = (\Sigma a + \Sigma a^2 + \Sigma SaS]$.

To define the notion of an ordered $(m.n)$-ideal of an ordered semiring S, for any $\emptyset \neq A, B \subseteq S$, we set $A^mBA^0 = A^mB$, $A^0BA^n = BA^n$ and $A^0BA^0 = B$ for all non-negative integers m, n.

Definition 2.6. Let m and n be non-negative integers. A subsemiring A of an ordered semiring S such that $A = (A]$ is called an *ordered* (m, n)*-ideal* of S if $A^mSA^n \subseteq A$.

Clearly, $\emptyset \neq A \subseteq S$ is an ordered $(0, 0)$-ideal of S if and only if $A = S$. It is easy to see that a left ordered ideal, a right ordered ideal and an ordered bi-ideal of an ordered semiring is an ordered $(0, 1)$-ideal, an ordered $(1, 0)$-ideal and an ordered $(1, 1)$-ideal, respectively.

For a nonempty subset A of an ordered semiring S, we denote the notation $[A]_{(m,n)}$ to be the smallest ordered (m, n)-ideal of S containing A.

Theorem 2.7. *Let A be a nonempty subset of an ordered semiring S. Then*

$$[A]_{(m,n)} = (\Sigma A + \Sigma A^2 + \ldots + \Sigma A^{m+n} + \Sigma A^mSA^n]$$

for all non-negative integers m and n.

Proof. Let $\emptyset \neq A \subseteq S$ and $X = (\Sigma A + \Sigma A^2 + \ldots + \Sigma A^{m+n} + \Sigma A^mSA^n]$. It is clear that $A \subseteq X \neq \emptyset$, $X = (X]$ and $X + X \subseteq X$. We obtain that

$$\begin{aligned} X^2 &= (\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n]\cdot(\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n] \\ &\subseteq ((\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n)\cdot(\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n)] \\ &\subseteq (\Sigma A^2+\Sigma A^3+\ldots+\Sigma A^{m+n}+\Sigma A^{m+n+1}+\ldots+\Sigma A^{2m+2n}+\Sigma A^mSA^n] \\ &\subseteq (\Sigma A^2+\Sigma A^3+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n+\ldots+\Sigma A^mSA^n+\Sigma A^mSA^n] \\ &= (\Sigma A^2+\Sigma A^3+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n] \subseteq X \quad \text{and} \end{aligned}$$

$$\begin{aligned} X^mSX^n &= ((\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n])^mS \\ &\quad \cdot((\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n])^n \\ &\subseteq ((\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n)^m]S \\ &\quad \cdot((\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n)^n] \\ &\subseteq (\Sigma A^m+\Sigma A^mS]S(\Sigma A^n+\Sigma SA^n] \\ &\subseteq (\Sigma A^mS+\Sigma A^mS](\Sigma A^n+\Sigma SA^n] \\ &= (\Sigma A^mS](\Sigma A^n+\Sigma SA^n] \subseteq ((\Sigma A^mS)(\Sigma A^n+\Sigma SA^n)] \\ &\subseteq (\Sigma A^mSA^n+\Sigma A^mSSA^n] \subseteq (\Sigma A^mSA^n+\Sigma A^mSA^n] \\ &= (\Sigma A^mSA^n+\Sigma A^mSA^n] = (\Sigma A^mSA^n] \subseteq X. \end{aligned}$$

Now, X is an ordered (m, n)-ideal of S containing A. Let Y be an ordered (m, n)-ideal of S containing A. Then $X = (\Sigma A+\Sigma A^2+\ldots+\Sigma A^{m+n}+\Sigma A^mSA^n] \subseteq (\Sigma Y + \Sigma Y^2 + \ldots + \Sigma Y^{m+n} + \Sigma Y^mSY^n] \subseteq (\Sigma Y + \Sigma Y + \ldots + \Sigma Y + \Sigma Y] = (\Sigma Y] = Y$. $\square$

In a particular case of $A = \{a\}$ for some $a \in S$, we write $[a]_{(m,n)}$ instead of $[\{a\}]_{(m,n)}$. The following corollary is directly obtained by Theorem 2.7.

Corollary 2.8. *Let a be an element of an ordered semiring S. Then*

$$[a]_{(m,n)} = (\Sigma a + \Sigma a^2 + \ldots + \Sigma a^{m+n} + a^m S a^n]$$

for all non-negative integers m and n.

We immediately get that $[a]_{(1,1)}$ is the smallest ordered bi-ideal of S containing a for any elements a of an ordered semiring S; accordingly, we use the notation $B(a)$ instead of $[a]_{(1,1)}$.

Corollary 2.9. *Let a be an element of an ordered semiring S. Then $B(a) = (\Sigma a + \Sigma a^2 + aSa]$.*

3. Main Results

Definition 3.1. We call an ordered semiring S to be:

(*i*) *regular* [5] if $a \in (aSa]$ for all $a \in S$;

(*ii*) *intra-regular* if $a \in (\Sigma S a^2 S]$ for all $a \in S$;

(*iii*) *left weakly regular* if $a \in (\Sigma SaSa]$ for all $a \in S$;

(*iv*) *right weakly regular* if $a \in (\Sigma aSaS]$ for all $a \in S$.

Lemma 3.2. *An ordered semiring S is both regular and intra-regular if and only if $a \in (aSa^2Sa]$ for all $a \in S$.*

Proof. Assume that S is both regular and intra-regular. Let $a \in S$. It follows that $a \in (aSa] \subseteq (aS(aSa]] \subseteq (aSaSa] \subseteq (aS(\Sigma Sa^2S]Sa] \subseteq (a(\Sigma Sa^2S]a] \subseteq ((\Sigma aSa^2Sa]] = (aSa^2Sa]$. Conversely, if $a \in (aSa^2Sa]$ for all $a \in S$, then $a \in (aSa^2Sa] \subseteq (aSa]$ and $a \in (aSa^2Sa] \subseteq (Sa^2S] \subseteq (\Sigma Sa^2S]$. Hence, S is regular and intra-regular. □

Definition 3.3. A nonempty subset T of an ordered semiring S is called *idempotent* if $T = (\Sigma T^2]$.

Definition 3.4. An ordered semiring S is called *fully idempotent* if every ordered ideal of S is idempotent.

Example 3.5. (i) $(\mathbb{N}, +, \cdot, =)$ is an ordered semiring where $\mathbb{N}$ is the set of all natural numbers, $+$ is the usual addition, $\cdot$ is the usual multiplication and $=$ is the equal relation. We have that the ordered ideal $2\mathbb{N}$ is idempotent in $(\mathbb{N}, +, \cdot, =)$.

(ii) $(\mathbb{N}, +, \cdot, \leqslant)$ is an ordered semiring where $\leq$ is the natural ordered relation. Since $(\mathbb{N}, +, \cdot, \leqslant)$ has no proper ordered ideal and $\mathbb{N} = (\Sigma \mathbb{N}^2]$, it is fully idempotent.

We characterize fully idempotent ordered semirings by idempotency of ordered ideals.

Theorem 3.6. *The following statements are equivalent:*

(i) *S is fully idempotent;*

(ii) *$J_1 \cap J_2 = (\Sigma J_1 J_2]$ for all ordered ideals J_1 and J_2 of S;*

(iii) *$J(a) \cap J(b) = (\Sigma J(a)J(b)]$ for all $a, b \in S$;*

(iv) *$J(a) = (\Sigma(J(a))^2]$ for all $a \in S$;*

(v) *$a \in (\Sigma SaSaS]$ for all $a \in S$.*

Proof. $(i) \Rightarrow (ii)$. Let J_1 and J_2 be ordered ideals of S. Then $(\Sigma J_1 J_2] \subseteq (\Sigma J_1] = J_1$ and $(\Sigma J_1 J_2] \subseteq (\Sigma J_2] = J_2$. It follows that $(\Sigma J_1 J_2] \subseteq J_1 \cap J_2$. It is easy to show that $J_1 \cap J_2$ is an ordered ideal of S. By (i), we get $J_1 \cap J_2 = (\Sigma(J_1 \cap J_2)^2] = (\Sigma(J_1 \cap J_2)(J_1 \cap J_2)] \subseteq (\Sigma J_1 J_2]$.

$(ii) \Rightarrow (iii)$. and $(iii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (v)$. Let $a \in S$. By (iv), we obtain that

$$\begin{aligned} a \in J(a) &= (\Sigma(J(a))^2] = (\Sigma(\Sigma a + Sa + aS + \Sigma SaS](\Sigma a + Sa + aS + \Sigma SaS]] \\ &\subseteq (\Sigma(\Sigma a + Sa + aS + \Sigma SaS)(\Sigma a + Sa + aS + \Sigma SaS)] \\ &\subseteq (\Sigma(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS)] \\ &\subseteq (\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]. \end{aligned} \tag{1}$$

Using equation (1), we get that

$$a \in (aS + \Sigma SaS + \Sigma SaSaS] \tag{2}$$

$$a \in (Sa + \Sigma SaS + \Sigma SaSaS]. \tag{3}$$

Using equations (2) and (3), we get that

$$\begin{aligned} a^2 = aa &\in (aS + \Sigma SaS + \Sigma SaSaS](Sa + \Sigma SaS + \Sigma SaSaS] \\ &\subseteq ((aS + \Sigma SaS + \Sigma SaSaS)(Sa + \Sigma SaS + \Sigma SaSaS)] \\ &\subseteq (aSa + \Sigma aSaS + \Sigma SaSa + \Sigma SaSaS]. \end{aligned} \tag{4}$$

Using equations (2) and (3) again, we get that

$$\begin{aligned} aSa &\subseteq (Sa + \Sigma SaS + \Sigma SaSaS]S(aS + \Sigma SaS + \Sigma SaSaS] \\ &\subseteq ((Sa + \Sigma SaS + \Sigma SaSaS)S](aS + \Sigma SaS + \Sigma SaSaS] \\ &\subseteq (\Sigma SaS + \Sigma SaSaS](aS + \Sigma SaS + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaS + \Sigma SaSaS)(aS + \Sigma SaS + \Sigma SaSaS)] \\ &\subseteq (\Sigma SaSaS]. \end{aligned} \tag{5}$$

Using equation (2), we get that

$$SaSa \subseteq SaS(aS + \Sigma SaS + \Sigma SaSaS] \subseteq (\Sigma SaSaS]. \tag{6}$$

Using equation (3), we get that

$$aSaS \subseteq (Sa + \Sigma SaS + \Sigma SaSaS]SaS \subseteq (\Sigma SaSaS]. \tag{7}$$

Using equations (4), (5), (6) and (7) we get that

$$\begin{aligned} a^2 &\in (aSa + \Sigma aSaS + \Sigma SaSa + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + \Sigma(\Sigma SaSaS] + \Sigma(\Sigma SaSaS] + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS]] \\ &\subseteq ((\Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS]] \\ &= (\Sigma SaSaS]. \end{aligned} \tag{8}$$

Using equation (8), we get that

$$a^2S \subseteq (\Sigma SaSaS]S \subseteq (\Sigma SaSaS] \tag{9}$$

$$Sa^2 \subseteq S(\Sigma SaSaS] \subseteq (\Sigma SaSaS] \tag{10}$$

$$Sa^2S \subseteq S(\Sigma SaSaS]S \subseteq (\Sigma SaSaS]. \tag{11}$$

Using equations (1) and (5)–(11), we obtain that

$$\begin{aligned} a &\in (\Sigma a + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS] \\ &\subseteq (\Sigma(\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + \Sigma(\Sigma SaSaS] \\ &\quad + (\Sigma SaSaS] + \Sigma(\Sigma SaSaS] + \Sigma(\Sigma SaSaS] + \Sigma SaSaS] \\ &\subseteq ((\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] \\ &\quad + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS] + (\Sigma SaSaS]] \\ &\subseteq ((\Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS \\ &\quad + \Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS + \Sigma SaSaS]] \\ &= (\Sigma SaSaS]. \end{aligned}$$

$(v) \Rightarrow (i)$. Let J be an ordered ideal of S. Clearly, $(\Sigma J^2] \subseteq (\Sigma J] = J$. If $x \in J$, then by (v), we get $x \in (\Sigma SxSxS] \subseteq (\Sigma SJSJS] \subseteq (\Sigma JJ] = (\Sigma J^2]$ and so $J \subseteq (\Sigma J^2]$. Hence, $J = (\Sigma J^2]$ and thus S is fully idempotent. □

In general, every ordered ideal of an ordered semiring is an ordered interior ideal but not conversely [7]. However, they are same in fully idempotent ordered semirings.

Proposition 3.7. *Ordered ideals and ordered interior ideals coincide in fully idempotent ordered semirings.*

Proof. Let I be an ordered interior ideal of an ordered semiring S. Assume that S is fully idempotent. If $x \in IS$, then by Theorem 3.6, $x \in (\Sigma SxSxS] \subseteq (\Sigma SxS] \subseteq (\Sigma S(IS)S] \subseteq (\Sigma SIS] \subseteq (\Sigma I] = I$. Similarly, we have that $SI \subseteq I$. Hence, I is an ordered ideal of S. □

Using Theorem 3.6 and Proposition 3.7, we directly obtain the following corollary as characterizations of fully idempotent ordered semirings by idempotency of ordered interior ideals.

Corollary 3.8. *The following statements are equivalent:*
(i) *S is fully idempotent;*
(ii) *$I_1 \cap I_2 = (\Sigma I_1 I_2]$ for all ordered interior ideals I_1 and I_2 of S;*
(iii) *$I(a) \cap I(b) = (\Sigma I(a)I(b)]$ for all $a, b \in S$;*
(iv) *$I(a) = (\Sigma(I(a))^2]$ for all $a \in S$.*

Now, we use idempotency of ordered quasi-ideals to characterize an ordered semiring which is both regular and intra-regular.

Theorem 3.9. *The following statements are equivalent:*
(i) *S is both regular and intra-regular;*
(ii) *every ordered quasi-ideal of S is idempotent, i.e., $Q = (\Sigma Q^2]$ for all ordered quasi-ideals Q of S;*
(iii) *$Q(a) = (\Sigma(Q(a))^2]$ for all $a \in S$.*

Proof. (i) $\Rightarrow$ (ii). Assume that S is both regular and intra-regular. Let Q be an ordered quasi-ideal of S. Obviously, $(\Sigma Q^2] \subseteq (\Sigma Q] = Q$ (every ordered quasi-ideal is always a subsemiring [7]). If $x \in Q$, then using Remark 3.2, we get $x \in (xSx^2Sx] = (xSxxSx] \subseteq (QSQQSQ] \subseteq (QQ] = (Q^2] \subseteq (\Sigma Q^2]$ (every ordered quasi-ideal is an ordered bi-ideal [7] and so $QSQ \subseteq Q$ for every ordered quasi-ideal Q of an ordered semiring S). Hence, $Q = (\Sigma Q^2]$ and so Q is idempotent.

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (i). Assume that (iii) holds and let $a \in S$. Since every left (right) ordered ideal is an ordered quasi-ideal [7], we obtain that

$$a \in Q(a) = (\Sigma Q(a)Q(a)] \subseteq (\Sigma R(a)L(a)] \tag{12}$$

$$a \in Q(a) = (\Sigma Q(a)Q(a)] \subseteq (\Sigma L(a)R(a)]. \tag{13}$$

We consider equation (12). Using Corollary 2.5, it turns out that

$$\begin{aligned} a \in (\Sigma R(a)L(a)] &= (\Sigma(\Sigma a + aS](\Sigma a + Sa]] \\ &\subseteq (\Sigma(\Sigma a + aS)(\Sigma a + Sa)] \subseteq (\Sigma(\Sigma a^2 + aSa)] \\ &\subseteq (\Sigma a^2 + aSa]. \end{aligned} \tag{14}$$

Using equation (14), we get that

$$a^2 = aa \in a(\Sigma a^2 + aSa] \subseteq (\Sigma a^3 + aSa] \subseteq (\Sigma aSa + aSa] = (aSa + aSa] = (aSa].$$

Using equation (14) again, we obtain that

$$\begin{aligned} a \in (\Sigma a^2 + aSa] &\subseteq (\Sigma(aSa] + aSa] \subseteq ((\Sigma aSa] + aSa] \\ &= ((aSa] + aSa] \subseteq ((aSa + aSa]] = (aSa]. \end{aligned}$$

Now, S is regular. We consider equation (13). Using Corollary 2.5. We get

$$\begin{aligned} a \in (\Sigma L(a)R(a)] &= (\Sigma(\Sigma a + Sa](\Sigma a + aS]] \\ &\subseteq (\Sigma(\Sigma a + Sa)(\Sigma a + aS)] \subseteq (\Sigma(\Sigma a^2 + \Sigma a^2 S + \Sigma Sa^2 + \Sigma Sa^2 S)] \\ &\subseteq (\Sigma a^2 + \Sigma a^2 S + \Sigma Sa^2 + \Sigma Sa^2 S] = (\Sigma a^2 + a^2 S + Sa^2 + \Sigma Sa^2 S]. \end{aligned} \tag{15}$$

Using equation (15), we get that

$$\begin{aligned} a^2 = aa &\in (\Sigma a^2 + a^2S + Sa^2 + \Sigma Sa^2S](\Sigma a^2 + a^2S + Sa^2 + \Sigma Sa^2S] \\ &\subseteq ((\Sigma a^2 + a^2S + Sa^2 + \Sigma Sa^2S)(\Sigma a^2 + a^2S + Sa^2 + \Sigma Sa^2S)] \\ &\subseteq (\Sigma a^4 + \Sigma Sa^2S] \subseteq (\Sigma Sa^2S + \Sigma Sa^2S] = (\Sigma Sa^2S]. \qquad (16) \end{aligned}$$

Using equation (16), we get that

$$a^2S \subseteq (\Sigma Sa^2S]S \subseteq (\Sigma Sa^2S] \text{ and } Sa^2 \subseteq S(\Sigma Sa^2S] \subseteq (\Sigma Sa^2S]. \qquad (17)$$

Using equations (15), (16) and (17), we have that

$$\begin{aligned} a &\in (\Sigma a^2 + a^2S + Sa^2 + \Sigma Sa^2S] \\ &\subseteq (\Sigma(\Sigma Sa^2S] + (\Sigma Sa^2S] + (\Sigma Sa^2S] + \Sigma Sa^2S] \\ &\subseteq ((\Sigma Sa^2S] + (\Sigma Sa^2S] + (\Sigma Sa^2S] + (\Sigma Sa^2S]] \\ &\subseteq (\Sigma Sa^2S + \Sigma Sa^2S + \Sigma Sa^2S + \Sigma Sa^2S] = (\Sigma Sa^2S]. \end{aligned}$$

Now, S is intra-regular. Therefore, S is both regular and intra-regular. □

In general, every ordered quasi-ideal of an ordered semiring is an ordered bi-ideal but not conversely [7]. However, they coincide in regular ordered semirings. Using this fact and Theorem 3.9, we obtain the following corollary as characterizations of an ordered semiring which is both regular and intra-regular by idempotency of ordered bi-ideals.

Corollary 3.10. *The following statements are equivalent:*

(i) *S is both regular and intra-regular;*

(ii) *every ordered bi-ideal of S is idempotent, i.e., $B = (\Sigma B^2]$ for all ordered bi-ideals B of S;*

(iii) *$B(a) = (\Sigma(B(a))^2]$ for all $a \in S$.*

Now, we use idempotency of left ordered ideals to characterize a left weakly regular ordered semiring.

Theorem 3.11. *The following statements are equivalent:*

(i) *S is left weakly regular;*

(ii) *every left ordered ideal of S is idempotent, i.e., $L = (\Sigma L^2]$ for all left ordered ideals L of S;*

(iii) *$L(a) = (\Sigma(L(a))^2]$ for all $a \in S$.*

Proof. (i) ⇒ (ii). Let L be a left ordered ideal of S. Clearly, $(\Sigma L^2] \subseteq (\Sigma L] = L$. If $x \in L$, then by (i), we get that $x \in (\Sigma SaSa] \subseteq (\Sigma SLSL] \subseteq (\Sigma LL] = (\Sigma L^2]$. Hence, $L = (\Sigma L^2]$.

(ii) ⇒ (iii). It is obvious.

$(iii) \Rightarrow (i)$. Let $a \in S$. Using Corollary 2.5, we obtain that

$$\begin{aligned} a \in L(a) &= (\Sigma L(a)L(a)] = (\Sigma(\Sigma a + Sa](\Sigma a + Sa]] \\ &\subseteq (\Sigma(\Sigma a + Sa)(\Sigma a + Sa)] \subseteq (\Sigma(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa)] \\ &\subseteq (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \end{aligned} \tag{18}$$

Using equation (18), we get that

$$a \in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \subseteq (Sa + \Sigma SaSa]. \tag{19}$$

Using equation (19), we get that

$$\begin{aligned} a^2 = aa &\in (Sa + \Sigma SaSa](Sa + \Sigma SaSa] \\ &\subseteq ((Sa + \Sigma SaSa)(Sa + \Sigma SaSa)] \subseteq (\Sigma SaSa]. \end{aligned} \tag{20}$$

Using equation (19) again, we get that

$$\begin{aligned} aSa &\subseteq (Sa + \Sigma SaSa]S(Sa + \Sigma SaSa] \subseteq ((Sa + \Sigma SaSa)S](Sa + \Sigma SaSa] \\ &\subseteq (\Sigma SaS](Sa + \Sigma SaSa] \subseteq ((\Sigma SaS)(Sa + \Sigma SaSa)] \subseteq (\Sigma SaSa]. \end{aligned} \tag{21}$$

Using equation (20), we get that

$$Sa^2 \subseteq S(\Sigma SaSa] \subseteq (S(\Sigma SaSa)] \subseteq (\Sigma SaSa]. \tag{22}$$

Using equations (18) and (20)–(22), it turns out that

$$\begin{aligned} a &\in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \\ &\subseteq (\Sigma(\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + \Sigma SaSa] \\ &\subseteq ((\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa]] \\ &\subseteq ((\Sigma SaSa + \Sigma SaSa + \Sigma SaSa + \Sigma SaSa]] = (\Sigma SaSa]. \end{aligned}$$

Therefore, S is left weakly regular. □

As a duality of Theorem 3.11, we obtain characterizations of right weakly regular ordered semirings in terms of idempotency of right ordered ideals analogously.

Theorem 3.12. *The following statements are equivalent:*

(i) *S is right weakly regular;*

(ii) *every right ordered ideal of S is idempotent, i.e., $R = (\Sigma R^2]$ for all right ordered ideals R of S;*

(iii) *$R(a) = (\Sigma(R(a))^2]$ for all $a \in S$.*

To define the notion of an (m, n)-regular ordered semiring, for any elements a and for any nonempty subsets B of an ordered semiring S, we set $a^m Ba^0 = a^m B$, $a^0 Ba^n = Ba^n$ and $a^0 Ba^0 = B$ for all non-negative integers m and n.

Definition 3.13. Let m and n be non-negative integers. An ordered semiring S is called *(m, n)-regular* if $a \in (a^m Sa^n]$ for all $a \in S$.

Theorem 3.14. *Let S be an ordered semiring and m, n be positive integers. Then the following statements hold:*

(*i*) *S is $(m,0)$-regular if and only if $R = (R^mS]$ for each ordered $(m,0)$-ideal R of S;*

(*ii*) *S is $(0,n)$-regular if and only if $L = (SL^n]$ for each ordered $(0,n)$-ideal L of S.*

Proof. (i). Assume that S is $(m,0)$-regular. Let R be an ordered $(m,0)$-ideal of S. If $a \in R$, then $a \in (a^mS] \subseteq (R^mS]$ implies $R \subseteq (R^mS]$. Clearly, $(R^mS] \subseteq R$. Hence, $R = (R^mS]$.

Conversely. let $a \in S$. By assumption and Corollary 2.8, we obtain that

$$\begin{aligned} a \in [a]_{(m,0)} &= (([a]_{(m,0)})^mS] = (((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS])^mS] \\ &\subseteq (((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^m]S] \\ &\subseteq (((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^mS]] \\ &= ((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^mS] \\ &\subseteq ((\Sigma a^m + \Sigma a^mS)S] \subseteq (\Sigma a^mS + \Sigma a^mS] = (\Sigma a^mS] = (a^mS]. \end{aligned}$$

Therefore, S is $(m,0)$-regular.

(ii). It can be proved in a similar way of (i). □

It is not interesting to characterize a $(0,0)$-regular ordered semiring S because $a \in (a^0Sa^0] = (S] = S$ for all $a \in S$. Consequently, we obtain the following theorem as a characterization of an (m,n)-regular ordered semiring where m and n are not being zero at the same time.

Theorem 3.15. *Let m and n be non-negative integers where m and n are not being zero at the same time. An ordered semigroup S is (m,n)-regular if and only if $R \cap L = (R^mL^n]$ for every ordered $(m,0)$-ideal R and every ordered $(0,n)$-ideal L of S.*

Proof. The case of $m \neq 0$, $n = 0$ and $m = 0$, $n \neq 0$ is directly follows from Theorem 3.14(i) and (ii), respectively. Hence, we assume that $m \neq 0$ and $n \neq 0$.

Assume that S is (m,n)-regular. Let R and L be an ordered $(m,0)$-ideal and an ordered $(0,n)$-ideal of S, respectively. If $x \in R \cap L$, then by assumption, $x \in (a^mSa^n] \subseteq (R^mSL^n] \subseteq (R^mL^n]$ implies $R \cap L \subseteq (R^mL^m]$. Clearly, $(R^mL^n] \subseteq R \cap L$. Hence, $R \cap L = (R^mL^n]$.

Conversely, let $a \in S$. Then by assumption and Corollary 2.8, we get

$$\begin{aligned} a \in [a]_{(m,0)} \cap [a]_{(0.n)} &= ([a]^m_{(m,0)}[a]^n_{(0.n)}] \\ &= ((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS]^m(\Sigma a + \Sigma a^2 + \ldots + \Sigma a^n + Sa^n]^n] \\ &\subseteq ((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^m](\Sigma a + \Sigma a^2 + \ldots + \Sigma a^n + Sa^n)^n]] \\ &\subseteq (((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^m)(\Sigma a + \Sigma a^2 + \ldots + \Sigma a^n + Sa^n)^n)]] \\ &= ((\Sigma a + \Sigma a^2 + \ldots + \Sigma a^m + a^mS)^m)(\Sigma a + \Sigma a^2 + \ldots + \Sigma a^n + Sa^n)^n)] \end{aligned}$$

$\subseteq ((\Sigma a^m + a^m S)(\Sigma a^n + Sa^n)] \subseteq (\Sigma a^{m+n} + a^m Sa^n]$
$\subseteq (\Sigma a^{m+n-1}(\Sigma a^{m+n} + a^m Sa^n] + a^m Sa^n] \subseteq (\Sigma(a^m Sa^n + a^m Sa^n] + a^m Sa^n]$
$\subseteq ((a^m Sa^n] + a^m Sa^n] \subseteq ((a^m Sa^n + a^m Sa^n]] = (a^m Sa^n]$.

Therefore, S is (m, n)-regular. □

References

[1] **J. Ahsan**, *Fully idempotent semirings*, Proc. Japan Acad. **69** (1993), no. 6, 185−188.

[2] **J. Ahsan**, **J.N. Mordeson and M. Shabir**, *Fuzzy semirings with applications to automata theory*, Studies in Fuzziness and Soft Computing, Springer, 2012.

[3] **L. Bussaban and T. Changphas**, *On (m, n)-ideals and (m, n)-regular ordered semigroups*, Songklanakarin J. Sci. Technol. **38** (2016), no. 2, 199 − 206.

[4] **A.P. Gan and Y.L. Jiang**, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition **31** (2011), no. 6, 989 − 996.

[5] **D. Mandal**, *Fuzzy ideals and fuzzy interior ideals in ordered semirings*, Fuzzy Inf. Eng. **6** (2014), no. 1, 101 − 114.

[6] **J. von Neumann**, *On regular rings*, Proc. Natl. Acad. Sci. USA **22** (1936), 707 − 113.

[7] **P. Palakawong na Ayuthaya and B. Pibaljommee**, *Characterizations of regular ordered semirings by ordered quasi-ideals*, Int. J. Math. Math. Sci. **2016** (2016), Article ID. 4272451.

[8] **P. Palakawong na Ayuthaya and B. Pibaljommee**, *Characterizations of ordered k-regular semirings by ordered quasi k-ideals*, Quasigroups Related Systems **25** (2017), 109 − 120.

[9] **P. Palakawong na Ayuthaya and B. Pibaljommee**, *Characterizations of ordered intra k-regular semirings by ordered k-ideals*, Commun. Korean Math. Soc. **33** (2018), no. 1, 1 − 12.

[10] **P. Palakawong na Ayuthaya and B. Pibaljommee**, *Characterizations of completely ordered k-regular semirings*, Songklanakarin J. Sci. Technol. **41** (2019), no. 3, 501 − 505.

[11] **J. Sanborisoot and T. Changphas**, *On characterizations of (m, n)-regular ordered semigroups*, Far East J. Math. Sci. **65** (2012), no. 1, 75 − 86.

[12] **M. Shabir and R. Anjum**, *Right k-weakly regular hemirings*, Quasigroups Related Systems **20** (2012), no. 1, 97 − 112.

Received August 03, 2020

K. Siribute
Department of Curriculum and Instruction (Mathematics), Faculty of Education, Sakon Nakhon Rajabhat University, Sakon Nakhon, Thailand, 47000, E-ail: ozilthaipu@gmail.com

P. Palakawong na Ayutthaya
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand, E-mail:pakorn1702@gmail.com

J. Eanborisoot
Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham, Thailand, 44150, E-mail: jatuporn.san@msu.ac.th