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Characterizations of ordered k-regularities on ordered semirings

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Abstract. We investigate the connections among some types of ordered k-regularities of ordered semirings and give some of their characterizations using their ordered k-ideals, prime ordered k-ideals, semiprime ordered k-ideals and pure ordered k-ideals.

1. Introduction

Regularities are important and interesting properties to research on algebraic structures, especially, semigroups and semirings. Some notable types of regularities defined by Kehayopulu [7, 8] and Kehayopulu and Tsingelis [9] on semigroups and ordered semigroups are the bases of many works about regularities on semirings and ordered semirings. A semiring, a well-known generalization of a ring, is an algebraic system $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by a distributive law. Originally, the regular property of a semiring $(S, +, \cdot)$ is defined on $(S, \cdot)$ as a similar way of a regular ring defined by von Neumann [11]. He called a semiring $(S, +, \cdot)$ to be regular if the semigroup $(S, \cdot)$ is regular, i.e., for each $a \in S$, $a = axa$ for some $x \in S$. However, in the sense of Bourne [3], a semiring $(S, +, \cdot)$ is regular if for each $a \in S$, $a + axa = aya$ for some $x, y \in S$. Later, Adhikari, Sen and Weinert [1] renamed Bourne regular semirings to be k-regular semirings. It is easy to obtain that a k-regular semiring is a generalization of a regular semiring. In 1958, Henriksen [5] defined a more restricted class of ideals in a semiring, which he called k-ideals, a considerably useful kind of ideals to characterize k-regular semirings. Afterwards, Bhuniya and Jana [2, 6] defined the notions of quasi-k-ideals and k-bi-ideals of semirings and use them to characterize k-regular and intra k-regular semirings.

A notable generalization of semirings is an ordered semiring. In the sense of Gan and Jiang [4], an ordered semiring $(S, +, \cdot, \leqslant)$ is a semiring $(S, +, \cdot)$ together with a partially ordered relation $\leqslant$ on S satisfying the compatibility property. In 2014, Mandal [10] defined an ordered semiring $(S, +, \cdot, \leqslant)$ to be regular and k-regular if for each $a \in S$, $a \leqslant axa$ and $a + axa \leqslant aya$ for some $x, y \in S$, respectively. In 2016, we gave some characterizations of regular, left regular, right regular, and intra-regular ordered semirings using many kinds of their ordered ideals in [12].

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Later, Patchakheio and Pibaljommee [16] defined an ordered semiring $(S, +, \cdot, \leqslant)$ to be ordered k-regular if $a \in \overline{(aSa]}$ for all $a \in S$. This notion is a generalization of k-regular ordered semirings defined by Mandal. Moreover, in [16] they gave the notions of left ordered k-regular, right ordered k-regular, left weakly ordered k-regular and right weakly ordered k-regular semirings and characterize them using their ordered k-ideals. In 2017, Senarat and Pibaljommee [18] used prime and irreducible ordered k-bi-ideals to characterize left and right weakly ordered k-regular semirings.

In our previous works [13–15, 17], we characterized ordered k-regular, left ordered k-regular, right ordered k-regular, ordered intra k-regular, completely ordered k-regular, left weakly ordered k-regular, right weakly ordered k-regular and fully ordered k-idempotent semirings in terms of many kinds of their ordered k-ideals. In this work, we recollect all types of mentioned kinds of ordered k-regularities, investigate connections among them and left generalized ordered k-regular, right generalized ordered k-regular and generalized ordered k-regular semirings and give some more their characterizations. Furthermore, we use the concepts of prime ordered k-ideals, semiprime ordered k-ideals and pure ordered k-ideals of ordered semirings to characterize some kinds of ordered k-regularities.

2. Preliminaries

An *ordered semiring* [4] is a system $(S, +, \cdot, \leqslant)$ consisting of the semiring $(S, +, \cdot)$ and the partially ordered set $(S, \leqslant)$ connected by the compatibility property. If $(S, +)$ is commutative, $(S, +, \cdot, \leqslant)$ is called *additively commutative* [1]. Throughout this work, we simple write S instead of an ordered semiring $(S, +, \cdot, \leqslant)$ and always assume that it is additively commutative.

For any $\emptyset \neq A, B \subseteq S$, we denote $A + B = \{a + b \in S \mid a \in A, b \in B\}$, $AB = \{ab \in S \mid a \in A, b \in B\}$, $(A] = \{x \in S \mid x \leqslant a$ for some $a \in A\}$ and

$$\Sigma A = \left\{ \sum_{i \in I} a_i \mid a_i \in A \text{ and } I \text{ is a finite nonempty set} \right\}.$$

The k-closure [16] of $\emptyset \neq A \subseteq S$ is denoted by $\overline{A} = \{x \in S \mid x + a \leqslant b$ for some $a, b \in A\}$. By the elementary properties of the finite sums Σ, the operator $(\ ]$ and the k-closure of a nonempty subset of an ordered semiring, we refer to [13–16]. Nevertheless, we give the following lemma to be useful accessories for reaching the main results.

Lemma 2.1. *Let A and B be nonempty subsets of an ordered semiring S. The following statements hold:*

(i) $\Sigma\overline{(A]} \subseteq \overline{(\Sigma A]}$;

(ii) $\overline{(A]} = \overline{(\overline{(A]}]}$;

(*iii*) $A\overline{(B]} \subseteq \overline{(A]}\,\overline{(B]} \subseteq \overline{(\Sigma AB]}$ *and* $\overline{(A]}B \subseteq \overline{(A]}\,\overline{(B]} \subseteq \overline{(\Sigma AB]}$*;*

(*iv*) $A + \overline{(B]} \subseteq \overline{(A]} + \overline{(B]} \subseteq \overline{(A+B]}$*;*

(*v*) $\overline{(A\overline{(B]}]} \subseteq \overline{(\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$ *and* $\overline{(\overline{(A]}B]} \subseteq \overline{(\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$*;*

(*vi*) $\overline{(A + \overline{(B]}]} \subseteq \overline{(\overline{(A]} + \overline{(B]}]} \subseteq \overline{(A+B]}$.

A nonempty subset A of an ordered semiring S such that $A + A \subseteq A$ is called a *left* (resp. *right*) *ordered k-ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$) and $A = \overline{A}$. If A is both a left and a right ordered k-ideal of S, then A is called an *ordered k-ideal* [16] of S. A nonempty subset Q of S is called an *ordered quasi-k-ideal* [13] of S if $\overline{(\Sigma QS]} \cap \overline{(\Sigma SQ]} \subseteq Q$ and $Q = \overline{Q}$. A nonempty subset B of S such that $B + B \subseteq B$, $B^2 \subseteq B$ and $B = \overline{B}$ is said to be an *ordered k-bi-ideal* [18] (resp. *ordered k-interior ideal*) [14] of S if $BSB \subseteq B$ (resp. $SBS \subseteq B$).

For $a \in S$, by the notations $L(a)$, $R(a)$, $J(a)$, $Q(a)$, $B(a)$ and $I(a)$, we mean the intersection of all left ordered k-ideals, right ordered k-ideals, ordered k-ideals, ordered quasi-k-ideals, ordered k-bi-ideals and ordered k-interior ideals of S containing a, respectively. Now, we recollect their constructions which occur in [13–16] as follows.

Lemma 2.2. *For $\emptyset \neq A \subseteq S$, the following statements hold:*

(*i*) $L(a) = \overline{(\Sigma a + Sa]}$*;*

(*ii*) $R(a) = \overline{(\Sigma a + aS]}$*;*

(*iii*) $J(a) = \overline{(\Sigma a + Sa + aS + \Sigma SaS]}$*;*

(*iv*) $Q(a) = \overline{(\Sigma a + (\overline{(aS]} \cap \overline{(Sa]})]}$*;*

(*v*) $B(a) = \overline{(\Sigma a + \Sigma a^2 + aSa]}$*;*

(*vi*) $I(a) = \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$.

We define the relations $\mathcal{L}$ and $\mathcal{R}$ on an ordered semiring S by

$$\mathcal{L} := \{(x,y) \in S \times S \mid L(x) = L(y)\} \quad \text{and} \quad \mathcal{R} := \{(x,y) \in S \times S \mid R(x) = R(y)\}.$$

3. Ordered k-regularities of Ordered Semirings

We recall the notions of some types of ordered k-regularities of ordered semirings as the following definition.

Definition 3.1. An ordered semiring S is called:

(*i*) *ordered k-regular* if $a \in \overline{(aSa]}$ for all $a \in S$ (cf. [16]);

(*ii*) *left ordered k-regular* if $a \in \overline{(Sa^2]}$ for all $a \in S$ (cf. [16]);

(*iii*) *right ordered k-regular* if $a \in \overline{(a^2S]}$ for all $a \in S$ (cf. [16]);

(*iv*) *completely ordered k-regular* if S is ordered k-regular, left ordered k-regular and right ordered k-regular (cf. [15]);

(*v*) *ordered intra k-regular* if $a \in \overline{(\Sigma Sa^2S]}$ for all $a \in S$ (cf. [14]);

(*vi*) *left weakly ordered k-regular* if $a \in \overline{(\Sigma SaSa]}$ for all $a \in S$ (cf. [16]);

(*vii*) *right weakly ordered k-regular* f $a \in \overline{(\Sigma aSaS]}$ for all $a \in S$ (cf. [16]);

(*viii*) *fully ordered k-idempotent* if $I = \overline{(\Sigma I^2]}$ for each ordered k-ideal I of S (cf. [15]).

According to Definition 3.1(*viii*), we note that an ordered semiring S is fully ordered k-idempotent if and only if $a \in \overline{(\Sigma SaSaS]}$ for all $a \in S$ [15].

Here, we give two lemmas which will be significantly used later.

Lemma 3.2. *An ordered semiring S is ordered intra k-regular if $a \in \overline{(\Sigma a^2 + \Sigma Sa^2S]}$ for all $a \in S$.*

Proof. Let $a \in S$. Assume that

$$a \in \overline{(\Sigma a^2 + \Sigma Sa^2S]}. \tag{1}$$

Using (1), we get

$$\begin{aligned} a^2 = aa &\in \overline{(\Sigma a^2 + \Sigma Sa^2S]}\,\overline{(\Sigma a^2 + \Sigma Sa^2S]} \subseteq \overline{(\Sigma(\Sigma a^2 + \Sigma Sa^2S)(\Sigma a^2 + \Sigma Sa^2S)]} \\ &\subseteq \overline{(\Sigma(\Sigma a^4 + \Sigma Sa^2S]]} \subseteq \overline{(\Sigma(\Sigma Sa^2S]]} = \overline{(\Sigma Sa^2S]}. \end{aligned} \tag{2}$$

Using (1) and (2), we obtain

$$\begin{aligned} a &\in \overline{(\Sigma a^2 + \Sigma Sa^2S]} \subseteq \overline{(\Sigma\overline{(\Sigma Sa^2S]} + \Sigma Sa^2S]} \subseteq \overline{(\overline{(\Sigma Sa^2S]} + \overline{(\Sigma Sa^2S]}]} \\ &\subseteq \overline{(\overline{(\Sigma Sa^2S + \Sigma Sa^2S]}]} = \overline{(\Sigma Sa^2S]}. \end{aligned}$$

Therefore, S is ordered intra k-regular. □

Lemma 3.3. *If an ordered semiring S is ordered intra k-regular, then $J(a) = \overline{(\Sigma SaS]}$ for all $a \in S$.*

Proof. Let $a \in S$. Assume that S is ordered intra k-regular. Then

$$\begin{aligned} J(a) &= \overline{(\Sigma a + Sa + aS + \Sigma SaS]} \\ &\subseteq \overline{(\Sigma\overline{(\Sigma Sa^2S]} + S\overline{(\Sigma Sa^2S]} + \overline{(\Sigma Sa^2S]}S + \Sigma S\overline{(\Sigma Sa^2S]}S]} \\ &\subseteq \overline{(\overline{(\Sigma Sa^2S]} + \overline{(\Sigma Sa^2S]} + \overline{(\Sigma Sa^2S]} + \overline{(\Sigma Sa^2S]}]} \\ &\subseteq \overline{(\overline{(\Sigma Sa^2S + \Sigma Sa^2S + \Sigma Sa^2S + \Sigma Sa^2S]}]} \\ &= \overline{(\overline{(\Sigma Sa^2S]}]} = \overline{(\Sigma Sa^2S]} \subseteq \overline{(\Sigma SaS]}. \end{aligned}$$

On the other hand, we show that $\overline{(\Sigma SaS]} \subseteq J(a)$. Let $s \in \Sigma SaS$ and $t \in \Sigma a + Sa + aS$. Then $s + (t + s) \leqslant t + s + s$ such that $t + s, t + s + s \in \Sigma a + Sa + aS + \Sigma SaS$ and so $s \in \overline{\Sigma a + Sa + aS + \Sigma SaS} \subseteq \overline{(\Sigma a + Sa + aS + \Sigma SaS]} = J(a)$. This means that $\Sigma SaS \subseteq J(a)$. It follows that $\overline{(\Sigma SaS]} \subseteq \overline{(J(a)]} = J(a)$. □

Theorem 3.4. [16] *An ordered semiring S is ordered k-regular if and only if $R \cap L = \overline{(RL]}$ for every right ordered k-ideal R and left ordered k-ideal L of S.*

Corollary 3.5. [13] *An ordered semiring S is ordered k-regular if and only if $a \in \overline{(R(a)L(a)]}$ for all $a \in S$.*

Now, we give more characterizations of an ordered k-regular semiring in terms of many kinds of their ordered k-ideals.

Theorem 3.6. *The following conditions are equivalent:*

(i) *S is ordered k-regular;*

(ii) *$B \cap L \subseteq \overline{(BL]}$ for every ordered k-bi-ideal B and left ordered k-ideal L of S;*

(iii) *$R \cap B \subseteq \overline{(RB]}$ for every right ordered k-ideal R and ordered k-bi-ideal B of S;*

(iv) *$R \cap B \cap L \subseteq \overline{(RBL]}$ for every right ordered k-ideal R, ordered k-bi-ideal B and left ordered k-ideal L of S;*

(v) *$B \cap I = \overline{(BIB]}$ for every ordered k-bi-ideal B and ordered k-interior ideal I of S;*

(vi) *$B \cap J = \overline{(BJB]}$ for every ordered k-bi-ideal B and ordered k-ideal J of S;*

(vii) *$B \cap I \cap L \subseteq \overline{(BIL]}$ for every ordered k-bi-ideal B, ordered k-interior ideal I and left ordered k-ideal L of S;*

$(viii)$ *$Q \cap I \cap L \subseteq \overline{(QIL]}$ for every ordered quasi-k-ideal Q, ordered k-interior ideal I and left ordered k-ideal L of S;*

(ix) *$R \cap I \cap L \subseteq \overline{(RIL]}$ for every right ordered k-ideal R, ordered k-interior ideal I and left ordered k-ideal L of S;*

(x) *$B \cap J \cap L \subseteq \overline{(BJL]}$ for every ordered k-bi-ideal B, ordered k-ideal J and left ordered k-ideal L of S;*

(xi) *$Q \cap J \cap L \subseteq \overline{(QJL]}$ for every ordered quasi-k-ideal Q, ordered k-ideal J and left ordered k-ideal L of S;*

(xii) *$R \cap J \cap L \subseteq \overline{(RJL]}$ for every right ordered k-ideal R, ordered k-ideal J and left ordered k-ideal L of S;*

$(xiii)$ *$R \cap I \cap B \subseteq \overline{(RIB]}$ for every right ordered k-ideal R, ordered k-interior ideal I and ordered k-bi-ideal B of S;*

(xiv) *$R \cap I \cap Q \subseteq \overline{(RIQ]}$ for every right ordered k-ideal R, ordered k-interior ideal I and ordered quasi-k-ideal Q of S;*

(xv) *$R \cap J \cap B \subseteq \overline{(RJB]}$ for every right ordered k-ideal R, ordered k-ideal J and ordered k-bi-ideal B of S;*

(xvi) *$R \cap J \cap Q \subseteq \overline{(RJQ]}$ for every right ordered k-ideal R, ordered k-ideal J and ordered quasi-k-ideal Q of S.*

Proof. $(i) \Rightarrow (ii)$. Let B and L be an ordered k-bi-ideal and a left ordered k-ideal of S, respectively. If $x \in B \cap L$ then by (i), $x \in \overline{(xSx]} \subseteq \overline{(BSL]} \subseteq \overline{(BL]}$.

$(ii) \Rightarrow (i)$. Let $a \in S$. By (ii), $a \in B(a) \cap L(a) \subseteq \overline{(B(a)L(a)]}$. Since every right ordered k-ideal is an ordered k-bi-ideal [13], we get $a \in \overline{(B(a)L(a)]} \subseteq \overline{(R(a)L(a)]}$. By Corollary 3.5, S is ordered k-regular.

$(i) \Rightarrow (iii)$. and $(iii) \Rightarrow (i)$ can be proved in a similar way of $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$, respectively.

$(i) \Rightarrow (iv)$. Let R, B and L be a right ordered k-ideal, an ordered k-bi-ideal and a left ordered k-ideal of S, respectively. If $x \in R \cap B \cap L$ then by (i), $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(RSBSL]} \subseteq \overline{(RBL]}$.

$(iv) \Rightarrow (i)$. Let $a \in S$. By (iv), $a \in R(a) \cap B(a) \cap L(a) \subseteq \overline{(R(a)B(a)L(a)]} \subseteq \overline{(R(a)L(a)]}$. Using Corollary 3.5, S is ordered k-regular.

$(i) \Rightarrow (v)$. Let B and I be an ordered k-bi-ideal and an ordered k-interior ideal of S, respectively. If $x \in B \cap I$ then by (i), $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(BSISB]} \subseteq \overline{(BIB]}$. Clearly, $\overline{(BIB]} \subseteq B \cap I$. Hence, $B \cap I = \overline{(BIB]}$.

$(v) \Rightarrow (vi)$. It follows from the fact that every ordered k-ideal is an ordered k-interior ideal [14].

$(vi) \Rightarrow (i)$. Let $a \in S$. By (vi), $a \in B(a) \cap J(a) = \overline{(B(a)J(a)B(a)]}$. Since every one-sided ordered k-ideal is an ordered k-bi-ideal [13], $a \in \overline{(B(a)J(a)B(a)]} \subseteq \overline{(R(a)J(a)L(a)]} \subseteq \overline{(R(a)L(a)]}$. Using Corollary 3.5, S is ordered k-regular.

$(i) \Rightarrow (vii)$. Let B, I and L be an ordered k-bi-ideal, an ordered k-interior ideal and a left ordered k-ideal of S, respectively. If $x \in B \cap I \cap L$ then by (i), $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(BSISL]} \subseteq \overline{(BIL]}$.

$(vii) \Rightarrow (viii)$. It follows from the fact that every ordered quasi-k-ideal is an ordered k-bi-ideal [12].

$(viii) \Rightarrow (ix)$. It follows from the fact that every right ordered k-ideal is an ordered quasi-k-ideal [12].

$(ix) \Rightarrow (i)$. Let $a \in S$. By (ix), $a \in R(a) \cap I(a) \cap L(a) \subseteq \overline{(R(a)I(a)L(a)]} \subseteq \overline{(R(a)L(a)]}$. Using Corollary 3.5, S is ordered k-regular.

$(i) \Rightarrow (x) \Rightarrow (xi) \Rightarrow (xii) \Rightarrow (i)$ can be proved in a similar way of $(i) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (ix) \Rightarrow (i)$.

$(i) \Rightarrow (xiii)$. Let R, I and B be a right ordered ideal, an ordered k-interior ideal and an ordered k-bi-ideal of S, respectively. If $x \in R \cap I \cap B$ then by (i), $x \in \overline{(xSx]} \subseteq \overline{(\overline{(xSx]}Sx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(RSISB]} \subseteq \overline{(RIB]}$.

$(xiii) \Rightarrow (xiv)$. It follows from the fact that every ordered quasi-k-ideal is an ordered k-bi-ideal [13].

$(xiv) \Rightarrow (i)$. Let $a \in S$. By (xiv), $a \in R(a) \cap I(a) \cap Q(a) \subseteq \overline{(R(a)I(a)Q(a)]} \subseteq \overline{(R(a)Q(a)]}$. Using the fact that every left ordered k-ideal is an ordered quasi-k-ideal [13], $a \in \overline{(R(a)Q(a)]} \subseteq \overline{(R(a)L(a)]}$. By Corollary 3.5, S is ordered k-regular.

$(i) \Rightarrow (xv) \Rightarrow (xvi) \Rightarrow (i)$ can be proved in a similar way of $(i) \Rightarrow (xiii) \Rightarrow (xiv) \Rightarrow (i)$. □

Definition 3.7. Let a be an element of an ordered semiring S. Then a is called: *left generalized ordered k-regular* (resp. *right generalized ordered k-regular*, *generalized ordered k-regular*) if $a \in \overline{(Sa]}$ (resp. $a \in \overline{(aS]}$, $a \in \overline{(\Sigma SaS]}$).

If a is left generalized ordered k-regular (resp. right generalized ordered k-regular, generalized ordered k-regular) for all $a \in S$, then S is called *left generalized ordered k-regular* (resp. *right generalized ordered k-regular, generalized ordered k-regular*).

Remark 3.8. Let a and b be elements of an ordered semiring S. If a is left (resp. right) generalized ordered k-regular and $a\mathcal{L}b$ ($a\mathcal{R}b$), then b is also left (resp. right) generalized ordered k-regular.

Proof. Let $a, b \in S$. If a is left generalized ordered k-regular and $a\mathcal{L}b$, then

$$\begin{aligned} b \in L(a) = \overline{(\Sigma a + Sa]} &\subseteq \overline{(\Sigma\overline{(Sa]} + Sa]} \subseteq \overline{(Sa]} \subseteq \overline{(SL(b)]} \\ &\subseteq \overline{(S\overline{(\Sigma b + Sb]}]} \subseteq \overline{(\Sigma Sb + Sb]} = \overline{(Sb + Sb]} = \overline{(Sb]}. \end{aligned}$$

Hence, b is also left generalized ordered k-regular. □

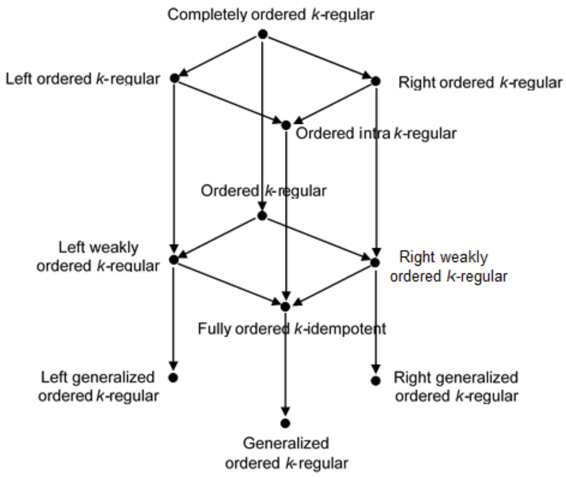
Remark 3.9. Let a and b be elements of an ordered semiring S such that a is generalized ordered k-regular. If $a\mathcal{L}b$ or $a\mathcal{R}b$, then b is also generalized ordered k-regular.

Proof. Let $a, b \in S$. Assume that a is generalized ordered k-regular and $a\mathcal{L}b$. Then

$$\begin{aligned} b \in L(a) = \overline{(\Sigma a + Sa]} &\subseteq \overline{(\Sigma\overline{(\Sigma SaS]} + S\overline{(\Sigma SaS]}]} \subseteq \overline{(\overline{(\Sigma SaS]} + \overline{(\Sigma SaS]}]} \subseteq \overline{(\Sigma SaS]} \\ &\subseteq \overline{(\Sigma SL(b)S]} \subseteq \overline{(\Sigma S\overline{(\Sigma b + Sb]}S]} \subseteq \overline{(\Sigma SbS + \Sigma SbS]} = \overline{(\Sigma SbS]}. \end{aligned}$$

Hence, b is generalized ordered k-regular. The case of $a\mathcal{R}b$ can be proved similarly. □

Connections among eleven types of ordered k-regularities can be summarized by the following diagram. Each arrow represents the implication between two regularities and its converse is not generally true.



Example 3.10. Let $S = \{a, b, c, d\}$. Define two binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

and

$\cdot$	a	b	c	d
a	a	a	c	d
b	a	a	c	d
c	a	a	c	d
d	a	a	c	d

Define a binary relation $\leqslant$ on S by $\leqslant := \{(a, a), (b, b), (c, c), (d, d), (a, d), (b, d), (c, d)\}$.

Then $(S, +, \cdot, \leqslant)$ is an ordered semiring.

Since $x \in \overline{(\Sigma S x^2 S]} = S$ for all $x \in S$, we have that S is ordered intra k-regular and hence S is fully ordered k-idempotent and generalized ordered k-regular.

Since $x \in \overline{(\Sigma x^2 S]} = S$ for all $x \in S$, we have that S is right ordered k-regular and hence S is right weakly ordered k-regular and right generalized ordered k-regular.

However, $b \notin \overline{(Sb]} = \{a\}$ and so S is not left generalized ordered k-regular. Consequently, S is not left weakly ordered k-regular and also neither left ordered k-regular nor ordered k-regular.

Example 3.11. Consider the set $S = \{a, b, c, d\}$ together with the operation $+$ and the relation $\leqslant$ of Example 3.10. Define a binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	c	c	c	c
d	d	d	d	d

Then $(S, +, \cdot, \leqslant)$ is an ordered semiring.

Since $x \in \overline{(\Sigma S x^2 S]} = S$ for all $x \in S$, we have that S is ordered intra k-regular and hence S is fully ordered k-idempotent and generalized ordered k-regular.

Since $x \in \overline{(\Sigma S x^2]} = S$ for all $x \in S$, we have that S is left ordered k-regular and hence S is left weakly ordered k-regular and left generalized ordered k-regular.

However, $b \notin \overline{(bS]} = \{a\}$ and so S is not right generalized ordered k-regular. Consequently, S is not right weakly ordered k-regular and also neither right ordered k-regular nor ordered k-regular.

Example 3.12. Let $S = \{a, b, c\}$. Define two binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

and

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a binary relation $\leqslant$ on S by $\leqslant := \{(a,a),(b,b),(c,c),(a,b),(a,c),(b,c)\}$. Then $(S,+,\cdot,\leqslant)$ is an ordered semiring. Since $a \in \overline{(Sa]} = \{a\}$, $b \in \overline{(Sb]} = \{a,b\}$ and $c \in \overline{(Sc]} = S$, we get that S is left generalized ordered k-regular. However, $b \notin \overline{(\Sigma SbS]} = \{a\}$ and so S is not generalized ordered k-regular. Consequently, S is not fully ordered k-idempotent and also not left weakly ordered k-regular.

Example 3.13. Consider the set $S = \{a,b,c\}$ together with the operation $+$ and the relation $\leqslant$ of Example 3.12. Define a binary operation $\cdot$ on S by the following table;

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	a	c

Then $(S,+,\cdot,\leqslant)$ is an ordered semiring. Since $a \in \overline{(aS]} = \{a\}$, $b \in \overline{(bS]} = \{a,b\}$ and $c \in \overline{(cS]} = S$, we get that S is right generalized ordered k-regular. However, $b \notin \overline{(\Sigma SbS]} = \{a\}$ and so S is not generalized ordered k-regular. Consequently, S is not fully ordered k-idempotent and also not left weakly ordered k-regular.

Example 3.14. Consider the set $S = \{a,b,c\}$ together with the operation $+$ and the relation $\leqslant$ of Example 3.12. Define a binary operation $\cdot$ on S by the following table;

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

Then $(S,+,\cdot,\leqslant)$ is an ordered semiring. Since $a \in \overline{(\Sigma SaS]} = \{a\}$, $b \in \overline{(\Sigma SbS]} = \{a,b\}$ and $c \in \overline{(\Sigma ScS]} = S$, we get that S is generalized ordered k-regular. However, S is not fully ordered k-idempotent because $b \notin \overline{(\Sigma SbSbS]} = \{a\}$.

4. Prime and Semiprime Ordered k-ideals

Now, we use the concepts of prime and semiprime ordered k-ideals to characterize several kinds of ordered k-regularities on ordered semirings.

Definition 4.1. A nonempty subset T of an ordered semiring S is said to be *prime* if for any $a,b \in S$, $ab \in T$ implies $a \in T$ or $b \in T$.

Definition 4.2. A nonempty subset T of an ordered semiring S is said to be *semiprime* if for any $a \in S$, $a^2 \in T$ implies $a \in T$.

It is clear that every prime subset of an ordered semiring is semiprime but not conversely.

Example 4.3. Consider the ordered semiring $(\mathbb{N}, +, \cdot, \leqslant)$ such that $\mathbb{N}$ is the set of all natural numbers, $+$ is the usual addition, $\cdot$ is the usual multiplication and $\leqslant$ is the natural order. We easily get that $2\mathbb{N}$ is a prime subset and $6\mathbb{N}$ is a semiprime subset of $(\mathbb{N}, +, \cdot, \leqslant)$. However, $6\mathbb{N}$ is not prime because $2 \cdot 3 \in 6\mathbb{N}$ but $2, 3 \notin 6\mathbb{N}$.

Theorem 4.4. *An ordered semiring S is left (right) ordered k-regular if and only if every left (right) ordered k-ideal of S is semiprime.*

Proof. Assume that S is left ordered k-regular. Let L be a left ordered k-ideal of S and $x \in S$. If $x^2 \in L$ then by assumption, $x \in \overline{(Sx^2]} \subseteq \overline{(SL]} \subseteq \overline{(L]} = L$. Hence, L is semiprime.

Conversely, assume that every left ordered k-ideal of S is semiprime. Let $a \in S$. Since a^2 belongs to $L(a^2)$ a semiprime left ordered k-ideal, we get

$$a \in L(a^2) = \overline{(\Sigma a^2 + Sa^2]} \tag{3}$$

Using (3), we obtain

$$a^2 = aa \in a\overline{(\Sigma a^2 + Sa^2]} \subseteq \overline{(\Sigma a^3 + Sa^2]} \subseteq \overline{(Sa^2]} \tag{4}$$

Using (3) and (4), we obtain

$$a \in \overline{(\Sigma a^2 + Sa^2]} \subseteq \overline{(\Sigma \overline{(Sa^2]} + Sa^2]} \subseteq \overline{(Sa^2]}.$$

Therefore, S is left ordered k-regular. □

Theorem 4.5. [15] *An ordered semiring S is completely ordered k-regular if and only if every ordered k-bi-ideal of S is semiprime.*

Theorem 4.6. [15] *An ordered semiring S is both left and right ordered k-regular if and only if every ordered quasi-k-ideal of S is semiprime.*

Theorem 4.7. *An ordered semiring S is ordered intra k-regular if and only if every ordered k-interior ideal of S is semiprime.*

Proof. Assume that S is ordered intra k-regular. Let I be an ordered k-interior ideal of S and $x \in S$. If $x^2 \in I$ then by assumption, $x \in \overline{(\Sigma Sx^2S]} \subseteq \overline{(\Sigma SIS]} \subseteq \overline{(\Sigma I]} = I$. Hence, I is semiprime.

Conversely, assume that every ordered k-interior ideal I of S is semiprime. Let $a \in S$. Since a^2 belongs to $I(a^2)$ a semiprime ordered k-interior ideal, we get $a \in I(a^2) = \overline{(\Sigma a^2 + \Sigma a^4 + \Sigma Sa^2S]} \subseteq \overline{(\Sigma a^2 + \Sigma Sa^2S]}$. By Lemma 3.2, S is ordered intra k-regular. □

We note that every ordered k-ideal of an ordered semiring is an ordered k-interior ideal [13, 14] and they coincide in ordered intra k-regular semirings [14]. As a consequence of Theorem 4.7 and using the above fact, we obtain the following corollary.

Corollary 4.8. *An ordered semiring S is ordered intra k-regular if and only if every ordered k-ideal of S is semiprime.*

Theorem 4.9. *An ordered semiring S is ordered intra k-regular and the set of all ordered k-ideals of S forms a chain if and only if every ordered k-ideal of S is prime.*

Proof. Let T be an ordered k-ideal of S and let $a, b \in S$ be such that $ab \in T$. Using Lemma 3.3, we have $J(a) = \overline{(\Sigma SaS]}$, $J(b) = \overline{(\Sigma SbS]}$ and $J(ab) = \overline{(\Sigma SabS]}$. We show that $J(a) \cap J(b) \subseteq J(ab)$. Let $z \in J(a) \cap J(b)$. Then

$$z^2 \in J(b)J(a) = \overline{(\Sigma SbS]}\,\overline{(\Sigma SaS]} \subseteq \overline{(\Sigma SbSaS]}. \tag{5}$$

If $w \in bSa$, then $w^2 \in bSabSa \subseteq SabS \subseteq \overline{(\Sigma SabS]} = J(ab)$. By assumption and Theorem 4.7, $J(ab)$ is semiprime and so $w \in J(ab)$. Thus, $bSa \subseteq J(ab)$. By (5), it turns out that $z^2 \in \overline{(\Sigma S(bSa)S]} \subseteq \overline{(\Sigma SJ(ab)S]} \subseteq \overline{(\Sigma J(ab)]} = J(ab)$. Since $J(ab)$ is semiprime, $z \in J(ab)$. Hence, $J(a) \cap J(b) \subseteq J(ab)$. Since the set of all ordered k-ideals of S is a chain, $J(a) \subseteq J(b)$ or $J(b) \subseteq J(a)$. If $J(a) \subseteq J(b)$, then $a \in J(a) = J(a) \cap J(b) \subseteq J(ab) = \overline{(\Sigma SabS]} \subseteq \overline{(\Sigma STS]} \subseteq T$. If $J(b) \subseteq J(a)$, then $b \in J(b) = J(a) \cap J(b) \subseteq J(ab) = \overline{(\Sigma SabS]} \subseteq \overline{(\Sigma STS]} \subseteq T$. Therefore, T is prime.

Conversely, assume that every ordered k-ideal of S is prime. Let A and B be ordered k-ideals of S. We want to show that $A \subseteq \overline{(\Sigma AB]}$ or $B \subseteq \overline{(\Sigma AB]}$. Suppose that $B \nsubseteq \overline{(\Sigma AB]}$. There exists $b \in B$ such that $b \notin \overline{(\Sigma AB]}$. Then for any $a \in A$, we have that $ab \in AB \subseteq \overline{(\Sigma AB]}$. Since $\overline{(\Sigma AB]}$ is prime, $a \in \overline{(\Sigma AB]}$ and so $A \subseteq \overline{(\Sigma AB]}$. Hence, $A \subseteq \overline{(\Sigma AB]} \subseteq \overline{(\Sigma B]} = B$ or $B \subseteq \overline{(\Sigma AB]} \subseteq \overline{(\Sigma A]} = A$. It follows that the set of all ordered k-ideals of S forms a chain. By assumption, every ordered k-ideal of S is also semiprime. Hence, by Theorem 4.7, S is ordered intra k-regular. □

Using the fact that every ordered k-ideal is an ordered k-interior ideal, together with Theorem 4.9, we directly obtain the following corollary.

Corollary 4.10. *An ordered semiring S is ordered intra k-regular and the set of all ordered k-ideals of S forms a chain if and only if every ordered k-interior ideal of S is prime.*

5. Pure Ordered k-ideals

In this section, we present the notions of left pure, right pure, quasi-pure, bi-pure, left weakly pure and right weakly pure ordered k-ideals of ordered semirings and use them to characterize ordered k-regular, left weakly ordered k-regular, right weakly ordered k-regular and fully ordered k-idempotent semirings.

Definition 5.1. An ordered k-ideal A of an ordered semiring S is called *left pure* (resp. *right pure*) if $x \in \overline{(Ax]}$ (resp. $x \in \overline{(xA]}$) for all $x \in A$.

Theorem 5.2. *Let A be an ordered k-ideal of an ordered semiring S. Then A is left pure (resp. right pure) if and only if $A \cap L = \overline{(AL]}$ for every left ordered k-ideal L (resp. $R \cap A = \overline{(RA]}$ for every right ordered k-ideal R) of S.*

Proof. (i) Assume that A is left pure. Let L be a left ordered k-ideal of S. If $x \in A \cap L$, then $x \in \overline{(Ax]} \subseteq \overline{(AL]}$. Therefore, $A \cap L \subseteq \overline{(AL]}$. Clearly, $\overline{(AL]} \subseteq A \cap L$. Hence, $A \cap L = \overline{(AL]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$\begin{aligned} x \in A \cap L(x) &= \overline{(AL(x)]} = \overline{(A\overline{(\Sigma x + Sx]}]} \subseteq \overline{(\overline{(\Sigma Ax + ASx]}]} \\ &\subseteq \overline{(Ax + Ax]} \subseteq \overline{(Ax]}. \end{aligned}$$

Hence, A is a left pure ordered k-ideal of S.

(ii) It can be proved similarly. □

Definition 5.3. An ordered k-ideal A of an ordered semiring S is called *quasi-pure* if $x \in \overline{(xA]} \cap \overline{(Ax]}$ for all $x \in A$.

It is clear that every quasi-pure ordered k-ideal of an ordered semiring is both left pure and right pure.

Theorem 5.4. *An ordered k-ideal A of an ordered semiring S is quasi-pure if and only if $A \cap Q = \overline{(QA]} \cap \overline{(AQ]}$ for every ordered quasi-k-ideal Q of S.*

Proof. Assume that A is quasi-pure. Let Q be an ordered quasi-k-ideal of S. If $x \in A \cap Q$, then $x \in \overline{(xA]} \cap \overline{(Ax]} \subseteq \overline{(QA]} \cap \overline{(AQ]}$. Thus, $A \cap Q \subseteq \overline{(QA]} \cap \overline{(AQ]}$. Clearly, $\overline{(QA]} \cap \overline{(AQ]} \subseteq A \cap Q$. Hence, $A \cap Q = \overline{(QA]} \cap \overline{(AQ]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$\begin{aligned} x \in A \cap Q(x) &= \overline{(Q(x)A]} \cap \overline{(AQ(x)]} \\ &= \overline{(\overline{(\Sigma x + (\overline{(xS]} \cap \overline{(Sx]})]}A]} \cap \overline{(A\overline{(\Sigma x + (\overline{(xS]} \cap \overline{(Sx]})]}]} \\ &\subseteq \overline{(\overline{(\Sigma x + \overline{(xS]}]}A]} \cap \overline{(A\overline{(\Sigma x + \overline{(Sx]}]}]} \\ &\subseteq \overline{(\overline{(\Sigma x + xS]}A]} \cap \overline{(A\overline{(\Sigma x + Sx]}]} \\ &\subseteq \overline{(\Sigma xA + xSA]} \cap \overline{(\Sigma Ax + ASx]} \\ &\subseteq \overline{(xA + xA]} \cap \overline{(Ax + Ax]} \subseteq \overline{(xA]} \cap \overline{(Ax]}. \end{aligned}$$

Hence, A is a quasi-pure ordered k-ideal of S. □

Definition 5.5. An ordered k-ideal A of an ordered semiring S is called *bi-pure* if $x \in \overline{(xAx]}$ for all $x \in A$.

It is easy to obtain that every bi-pure ordered k-ideal of an ordered semiring is quasi-pure.

Theorem 5.6. *An ordered k-ideal A of an ordered semiring S is bi-pure if and only if $A \cap B = \overline{(BAB]}$ for every ordered k-bi-ideal B of S.*

Proof. Assume that A is bi-pure. Let B be an ordered k-bi-ideal of S. If $x \in A \cap B$, then $x \in \overline{(xAx]} \subseteq \overline{(BAB]}$. Thus, $A \cap B \subseteq \overline{(BAB]}$. Clearly, $\overline{(BAB]} \subseteq A \cap B$. Hence, $A \cap B = \overline{(BAB]}$.

Conversely, let $x \in A$. Using assumption and Lemmas 2.1 and 2.2, we get

$$\begin{aligned} x \in A \cap B(x) = \overline{(B(x)AB(x)]} &= \overline{(\overline{(\Sigma x + \Sigma x^2 + xSx]}A\overline{(\Sigma x + \Sigma x^2 + xSx]}]} \\ &\subseteq \overline{(\Sigma xAx]} = \overline{(xAx]}. \end{aligned}$$

Hence, A is a bi-pure ordered k-ideal of S. □

Definition 5.7. An ordered k-ideal A of S is called *left weakly pure* (resp. *right weakly pure*) if $A \cap I = \overline{(\Sigma AI]}$ (resp. $I \cap A = \overline{(\Sigma IA]}$) for every ordered k-ideal I of S.

We note that every left (resp. right) pure ordered k-ideal of an ordered semiring is left (resp. right) weakly pure.

Now, we characterize some kinds of ordered k-regularities by pure and weakly pure ordered k-ideals of ordered semirings.

Lemma 5.8. [17] *Let S be an ordered semiring. Then the following statements hold:*

(i) *if $a \in \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}$ for any $a \in S$, then S is left weakly ordered k-regular;*
(ii) *if $a \in \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS]}$ for any $a \in S$, then S is right weakly ordered k-regular.*

Theorem 5.9. *An ordered semiring S is left (resp. right) weakly ordered k-regular if and only if every ordered k-ideal of S is left (resp. right) pure.*

Proof. Assume that S is left weakly ordered k-regular. Let A be an ordered k-ideal of S and let $x \in A$. By assumption, $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SASx]} \subseteq \overline{(\Sigma Ax]} = \overline{(Ax]}$. Hence, A is left pure.

Conversely, let $a \in S$. By assumption, we obtain that $J(a)$ is left pure. Using Lemmas 2.1 and 2.2 and Theorem 5.2, we obtain that

$$\begin{aligned} a \in J(a) \cap L(a) = \overline{(J(a)L(a)]} &= \overline{(\overline{(\Sigma a + aS + Sa + \Sigma SaS]}\,\overline{(\Sigma a + Sa]}]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 5.8(i), we get that S is left weakly ordered k-regular. □

As a consequence of Theorem 5.9 and the fact that every quasi-pure ordered k-ideal is both left pure and right pure, we directly obtain the following corollary.

Corollary 5.10. *An ordered semiring S is both left and right weakly ordered k-regular if and only if every ordered k-ideal of S is quasi-pure.*

We note that an ordered k-ideal of an ordered semiring is bi-pure if and only if it is an ordered k-regular subsemiring. Accordingly, we obtain the following remark.

Remark 5.11. An ordered semiring S is ordered k-regular if and only if every ordered k-ideal of S is bi-pure.

Proof. Assume that S is ordered k-regular. Let A be an ordered k-ideal of S and let $x \in A$. By the ordered k-regularity of S, we have that $x \in \overline{(xSx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(xSASx]} \subseteq \overline{(xSAx]} \subseteq \overline{(xAx]}$. Hence, A is bi-pure.

The converse is obvious since S itself is a bi-pure ordered k-ideal and so S is ordered k-regular. □

Corollary 5.12. [15] *Let S be an ordered semiring. If*

$$a \in \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]}$$

for all $a \in S$, then S is fully ordered k-idempotent.

Theorem 5.13. *Let S be an ordered semiring. Then*

(*i*) *if S is fully ordered k-idempotent, then every ordered k-ideal of S is both left and right weakly pure;*

(*ii*) *if every ordered k-ideal of S is left weakly pure or right weakly pure, then S is fully ordered k-idempotent.*

Proof. (*i*). Assume that S is fully ordered k-idempotent. Let A and I be any ordered k-ideals of S. By assumption, it turns out that if $x \in A \cap I$, then

$$x \in \overline{(\Sigma SxSxS]} \subseteq \overline{(\Sigma SASIS]} \subseteq \overline{(\Sigma ASI]} \subseteq \overline{(\Sigma AI]} \quad \text{and}$$
$$x \in \overline{(\Sigma SxSxS]} \subseteq \overline{(\Sigma SISAS]} \subseteq \overline{(\Sigma ISA]} \subseteq \overline{(\Sigma IA]}.$$

So, $A \cap I \subseteq \overline{(\Sigma AI]}$ and $A \cap I \subseteq \overline{(\Sigma IA]}$. Clearly, $\overline{(\Sigma AI]} \subseteq A \cap I$ and $\overline{(\Sigma IA]} \subseteq A \cap I$. Hence, $A \cap I = \overline{(\Sigma AI]} = \overline{(\Sigma IA]}$ and thus A is both left and right weakly pure.

(*ii*). Assume that every ordered k-ideal of S is left weakly pure. Let $a \in S$. Then $J(a)$ is left weakly pure. It follows that $J(a) = \overline{(\Sigma J(a)J(a)]}$. By Lemmas 2.1 and 2.2, we obtain that

$$\begin{aligned} a \in J(a) = \overline{(\Sigma J(a)J(a)]} &= \overline{(\Sigma \overline{(\Sigma a + Sa + aS + \Sigma SaS]}\,\overline{(\Sigma a + Sa + aS + \Sigma SaS]}]} \\ &= \overline{(\Sigma \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]}]} \\ &= \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS + Sa^2 + \Sigma SaSa + \Sigma Sa^2S + \Sigma SaSaS]}. \end{aligned}$$

By Corollary 5.12, we obtain that S is fully ordered k-idempotent.

It can be proved analogously if every ordered k-ideal of S is right weakly pure. □

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