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On completely regular 2-duo semigroups

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Abstract. We present characterizations of completely regular 2-duo semigroups using $(2,0)$-ideals, $(0,2)$-ideals, $(2,2)$-ideals and $(2,2)$-quasi-ideals of semigroups. We then consider 2-duo semigroups when every $(2,2)$-ideal is quasi-prime.

1. Introduction

Let S be a semigroup. An element $a \in S$ is said to be *regular* if there exits $x \in S$ such that $a = axa$, and S is said to be *regular* if every element of S is regular. Let A be a nonempty subset of S. We say that A is a *left ideal* (respectively, *right ideal*) of S if $SA \subseteq A$ (respectively, $AS \subseteq A$). A is called a *two-sided ideal* of S if it is both a left and a right ideal of S. S is called a *duo semigroup* if its left ideals and right ideals are two-sided. In [6], the author characterized regular duo semigroups by left ideals and right ideals.

Let m and n be non-negative integers. A subsemigroup A of a semigroup S is said to be an *(m,n)-ideal of S* if $A^mSA^n \subseteq A$. Here, $A^0S = SA^0 = S$. An (m,n)-ideal was firstly introduced by S. Lajos in [4]; the author considered (m,n)-ideals on regular duo semigroups in [5]. The results were extended to ordered semigroups by L. Bussaban and T. Changphas in [1].

In this paper, we define an n-duo semigroup extending the concept of duo semigroups. We then characterize completely regular 2-duo semigroups by $(2,2)$-ideals. Moreover, we consider when $(2,2)$-ideals of 2-duo semigroups are all quasi-prime.

2. Main Results

Definition 2.1. (cf. [2],[3],[8]) Let S be a semigroup and let $a \in S$. We say that a is *completely regular* if $a \in a^2Sa^2$. A semigroup S is *completely regular* if every element of S is completely regular.

Definition 2.2. Let S be any semigroup and let n be a non-negative integer. We say that S is an *n-duo semigroup* if it satisfies the following conditions:

(i) Every $(n,0)$-ideal of S is a $(0,n)$-ideal of S;

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(ii) Every $(0, n)$-ideal of S is an $(n, 0)$-ideal of S.

Let A be a nonempty subset of a semigroup S. The set $L(A)$ (respectively, $R(A)$) is a left (respectively, right) ideal of S *generated* by A. It is well known that $L(A) = A \cup SA$ and $R(A) = A \cup AS$. Moreover, the set $L(A)$ coincide the set $R(A)$ on duo semigroups. By Theorem 2.4 and Example 2.3, we show that every duo semigroup is an n-duo semigroup ($n \geqslant 2$), but the converse is not generally true.

Example 2.3. Let $S = \{a, b, c, d\}$. Consider a semigroup S with an associative operation defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	a

Then $(S, \cdot)$ is a 2-duo semigroup, but it is not a duo semigroup.

Theorem 2.4. *Let S be a semigroup. If S is a duo semigroup, then S is an n-duo semigroup where $n \geqslant 2$.*

Proof. Assume that S is a duo semigroup. Let A be an $(n, 0)$-ideal of S. Then

$$\begin{aligned} A^n S &\subseteq A^{n-1}(A \cup AS) \\ &= A^{n-1} R(A) \\ &= A^{n-1} L(A) \\ &= A^{n-1}(A \cup SA) \\ &= A^n \cup A^{n-1} SA \\ &\subseteq A \cup A^{n-1} SA. \end{aligned}$$

Continue in the same manner, we obtain that

$$A^n S \subseteq A \cup SA^n \subseteq A.$$

Thus, A is a $(0, n)$-ideal of S. Similarly, we have that every $(0, n)$-ideal of S is an $(n, 0)$-ideal of S. Therefore, S is an n-duo semigroup. □

Let S be a semigroup. For each $a \in S$, the symbol $J_{(m,n)}(a)$ stands for the (m, n)-ideal of S generated by a. S. Lajos proved in [4] that

$$J(a)_{(m,n)} = \bigcup_{i=1}^{m+n} a^i \bigcup a^m S a^n.$$

It is observed that $J_{(0,2)}(a) = J_{(2,0)}(a)$ for all $a \in S$ if S is a 2-duo semigroup.

Theorem 2.5. *Let S be a semigroup. Then S is a completely regular 2-duo semigroup if and only if the following conditions hold:*

(1) $(A^2 \cup A^2S)^2 = A$ *for all* $(0,2)$*-ideals* A *of* S*;*

(2) $(B^2 \cup SB^2)^2 = B$ *for all* $(2,0)$*-ideals* B *of* S*.*

Proof. Assume that S is a completely regular 2-duo semigroup. Let A be a $(0,2)$-ideal of S. Then $A = A^2$ because

$$A \subseteq A^2SA^2 \subseteq A^3 \subseteq A^2 \subseteq A.$$

Next, we prove the main equation of this theorem. Consider

$$\begin{aligned} A &= A^2 \\ &= (A \cup A)^2 \\ &\subseteq (A^2 \cup A^2SA^2)^2 \\ &\subseteq (A^2 \cup A^2S)^2 \\ &\subseteq A^2 \\ &= A. \end{aligned}$$

Therefore, $(A^2 \cup A^2S)^2 = A$. If B is a $(2,0)$-ideal of S, we can proceed similarly and then we obtain $(B^2 \cup SB^2)^2 = B$.

Conversely, assume that (1) and (2) hold. Let A be a $(0,2)$-ideal of S. Then

$$\begin{aligned} A^2S &= (A^2 \cup A^2S)^2(A^2 \cup A^2S)^2S \\ &\subseteq (A^2 \cup A^2S)^2S \\ &\subseteq (A^2 \cup A^2S)(A^2S) \\ &\subseteq (A^2 \cup A^2S)(A^2 \cup A^2S) \\ &= A. \end{aligned}$$

Thus, A is a $(2,0)$-ideal of S. Similarly, if B is a $(2,0)$-ideal of S, then by (2) we obtain B is a $(0,2)$-ideal of S. Therefore, S is 2-duo.

To prove that S is completely regular, let $a \in S$. Consider

$$\begin{aligned} a \in J(a)_{(2,0)} &= ((J(a)_{(2,0)})^2 \cup (J(a)_{(2,0)})^2S)^2 \\ &= \left((J(a)_{(2,0)})^2 \cup (J(a)_{(2,0)})^2S\right)\left((J(a)_{(0,2)})^2 \cup (J(a)_{(2,0)})^2S\right) \\ &\subseteq \left(a^2 \cup a^2S\right)\left((J(a)_{(0,2)})^2 \cup J(a)_{(2,0)}\right) \\ &\subseteq \left(a^2 \cup a^2S\right)\left(a^2 \cup Sa^2 \cup J(a)_{(0,2)}\right) \\ &= \left(a^2 \cup a^2S\right)(a \cup a^2 \cup Sa^2) \\ &\subseteq a^3 \cup a^4 \cup a^2Sa^2. \end{aligned}$$

Thus, a is completely regular. □

Theorem 2.6. *Let S be any semigroup. Then S is a completely regular 2-duo semigroup if and only if*

$$(B^2 \cup B^2S)^2 = B = (B^2 \cup SB^2)^2$$

for all $(2,2)$-ideal B of S.

Proof. Assume that S is a completely regular 2-duo semigroup. Let B be a $(2,2)$-ideal of S. Then

$$\begin{aligned}(B^2 \cup B^2S)^2 &\subseteq B^4 \cup B^4S \cup B^2SB^2 \cup B^2SB^2S \\ &\subseteq B \cup B^4S \\ &\subseteq B \cup BS \\ &\subseteq B \cup B^2SB^2S \\ &\subseteq B \cup B(B^2SB^2)SB^2S.\end{aligned}$$

Since SB^2 is a $(0,2)$-ideal of S and S is a 2-duo semigroup, it follows that

$$\begin{aligned}B \cup B(B^2SB^2)SB^2S &= B \cup B^3(SB^2SB^2)S \\ &\subseteq B \cup B^3SB^2 \\ &\subseteq B \cup B^2SB^2 \\ &\subseteq B.\end{aligned}$$

According to the proof of Theorem 2.5, we have that $B = B^2$. Thus,

$$B = B^4 \subseteq (B^2 \cup B^2S)^2.$$

Therefore, $B = (B^2 \cup B^S)^2$. Similarly, $B = (B^2 \cup B^2S)$. Hence,

$$(B^2 \cup B^2S)^2 = B = (B^2 \cup B^2S).$$

Conversely, let A be a $(0,2)$-ideal of S. Then A is a $(2,2)$ ideal of S. By assumption,

$$A = (A^2 \cup A^2S)^2.$$

On the same way, we obtain that

$$B = (B^2 \cup SB^2)$$

for every $(2,0)$-ideal B of S. By Theorem 2.5, S is a completely regular 2-duo semigroup. □

Definition 2.7. Let S be a semigroup and let m, n be non-negative integers. A subsemigroup Q of S is said to be an (m,n)-quasi-ideal of S if $SQ^m \cap Q^nS \subseteq Q$. Here, $Q^0S = SQ^0 = S$.

Theorem 2.8. *Let S be a semigroup. Then S is a completely regular 2-duo semigroup if and only if*

$$(Q^2 \cup Q^2S)^2 = Q = (Q^2 \cup SQ^2)^2$$

for all $(2,2)$-quasi-ideal Q of S.

Proof. Assume that S is a completely regular 2-duo semigroup. Let Q be a $(2,2)$-quasi-ideal of S. Then

$$(Q^2 \cup Q^2S)^2 = Q^2 \cup Q^4S \cup Q^2SQ^2 \cup Q^2SQ^2S \subseteq Q^2S$$

and

$$\begin{aligned}(Q^2 \cup Q^2S)^2 &= Q^4 \cup Q^4S \cup Q^2SQ^2 \cup Q^2SQ^2S\\ &\subseteq Q^2SQ^2 \cup SQ^2S\\ &\subseteq SQ^2 \cup SQ^2S\\ &\subseteq SQ^2 \cup SQ(Q^2SQ^2)S\\ &\subseteq SQ^2 \cup SQ^2SQ^2S\\ &\subseteq SQ^2 \cup SQ^2\\ &= SQ^2.\end{aligned}$$

Thus, $(Q^2 \cup Q^2S)^2 \subseteq QS^2 \cap SQ^2 \subseteq Q$. The opposite inclusion is obtained by the following equation:

$$Q \subseteq Q^2SQ^2 \subseteq (Q^2 \cup Q^2S)^2.$$

Similarly, we have

$$Q = (Q^2 \cup SQ^2)^2.$$

This implement has been proven.

Conversely, let A and B be a $(0,2)$-ideal of S and a $(2,0)$-ideal of S, respectively. Then A and B are $(2,2)$-quasi-ideals as well. By assumption, we have

$$A = (A^2 \cup A^2S)^2$$

and

$$B = (B^2 \cup SB^2)^2.$$

By Theorem 2.5, we have that S is a completely regular 2-duo semigroup. □

Example 2.9. Let $S = \{0, 1, 2, 3\}$ and defined a binary operation on S by

$\cdot$	a	b	c	d
a	a	b	a	d
b	b	a	b	d
c	a	b	c	d
d	d	d	d	d.

Then $(S, \cdot)$ is a semigroup. We have that $\{d\}, \{a, b, d\}$ and S are only $(2,2)$-ideals of S. Moreover every $(2,2)$-ideal B of S satisfies the equation

$$(B^2 \cup B^2S)^2 = B = (B^2 \cup SB^2)^2.$$

Thus, S is a completely regular 2-duo semigroup.

Lemma 2.10. *Let S be a semigroup. Then S is completely regular if and only if $A = A^2$ for every $(2,2)$-ideal A of S.*

Proof. Assume that S is completely regular. Let A be a $(2,2)$-ideal of S. Then

$$A \subseteq A^2SA^2 \subseteq A^2S(A^2SA^2)(A^2SA^2) \subseteq (A^2SA^2)(A^2SA^2) \subseteq A^2 \subseteq A.$$

Thus, $A = A^2$.

Conversely, assume that $A = A^2$ for every $(2,2)$-ideal A of S. Let $a \in S$. Then

$$\begin{aligned} a &\in J_{(2,2)}(a) \\ &= (J_{(2,2)}(a))^2 \\ &\subseteq a^2 \cup a^3 \cup a^4 \cup a^2Sa^2. \end{aligned}$$

Thus, $a \in a^2Sa^2$. This implies that S is completely regular. □

Remark 2.11. If S is completely regular, then AB is a $(2,2)$-ideal of S for all $(2,2)$-ideals A, B of S.

Definition 2.12. Let S be a semigroup. A $(2,2)$-ideal P of S is said to be *quasi-prime* if

$$AB \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P$$

for all $(2,2)$-ideals A, B of S.

Definition 2.13. Let S be a semigroup. A $(2,2)$-ideal P of S is said to be *quasi-semiprime* if

$$A^2 \subseteq P \Rightarrow A \subseteq P$$

for every $(2,2)$-ideal A of S.

Recall and apply Lemma 2.11 in [7], we have the following lemma:

Lemma 2.14. *Let S be a semigroup. Then $A = A^2$ for every $(2,2)$-ideal A of S if and only if every $(2,2)$-ideal of S is quasi-semiprime.*

Theorem 2.15. *Let S be a 2-duo semigroup. Then every $(2,2)$-ideal of S is quasi-prime if and only if S is completely regular and $(2,2)$-ideals of S form a chain by inclusion.*

Proof. Assume that every $(2,2)$-ideal of S is quasi-prime. Then they are quasi-semiprime as well. By Lemma 2.14, we have that $A = A^2$ for every $(2,2)$-ideal A of S. By Theorem 2.10, S is completely regular. Next, we show that $(2,2)$-ideals of S form a chain by inclusion. Let A, B be $(2,2)$-ideals of S. By Remark2.11, we obtain that AB is also a $(2,2)$-ideal of S. By assumption, AB is quasi-prime. Then we have two cases to consider:

Case 1: $A \subseteq AB$. Then

$$A \subseteq AB \subseteq A(B^2SB^2) \subseteq AB^2SB(B^2SB^2) \subseteq AB^2SB^2SB^2.$$

Since B^2S is a $(2,0)$-ideal of S and S is a 2-duo semigroup, it follows that B^2S is a $(0,2)$-ideal of S. Thus,

$$AB^2SB^2SB^2 \subseteq B^2SB^2 \subseteq B.$$

These imply that $A \subseteq B$.

Case 2: $B \subseteq A$. Then

$$B \subseteq AB \subseteq (A^2SA^2)B \subseteq (A^2SA^2)ASA^2B \subseteq ASA^2SA^2B.$$

Since SA^2 is a $(0,2)$-ideal of S and S is a 2-duo semigroup, it follows that SA^2 is a $(2,0)$-ideal of S. Thus,

$$ASA^2SA^2B \subseteq A^2SA^2 \subseteq A.$$

These imply that $B \subseteq A$. From Case 1 and Case 2, we conclude that $(2,2)$-ideals of S form a chain by inclusion.

To prove the opposite direction, let P be a $(2,2)$-ideal of S. Assume that A, B are $(2,2)$-ideals of S such that $AB \subseteq P$. If $A \subseteq B$, then

$$A = A^2 \subseteq AB \subseteq P.$$

Otherwise, $B \subseteq A$ implies that

$$B = B^2 \subseteq AB \subseteq P.$$

Thus, P is quasi-prime. □

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