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w-supplemented property in the lattices

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Abstract

Let L be a lattice with the greatest element 1. In this paper, we introduce and investigate the latticial counterpart of the filter-theoretical concepts of w-supplemented. The basic properties and possible structures of such filters are studied.

1. Introduction

The notion of an order plays an important role not only throughout mathematics but also in adjacent such as logic and computer science, hence, ought to be in the literature. The beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least upper and greatest lower bounds. In structure, lattices lie between semigroups and rings. In this respect, it closely resembles group theory. Thus lattices and groups provide two of the most basic tools of universal algebra, and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices. In this paper, we extend several concepts from module theory to lattice theory. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 4, 5, 6, 7]).

The notion of a supplement submodule was introduced in [10] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules U and V of a module M, V is said to be a *supplement* of U in M or U is said to *have a supplement* V in M if $U+V=M$ and $U \cap V \ll V$. The module M is called *supplemented* if every submodule of M has a supplement in M. See [4] and [12] for results and the definitions related to supplements and supplemented modules. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [14]. Supplemented modules are also discussed in [11]. Recently, several authors have studied different generalizations of supplemented modules. Rad-supplemented modules have been studied in [13] and [3]. See [13]; these modules are called generalized supplemented modules. For submodules U and V of a module M, V is said to be a rad-*supplement* of U in M or U is said to have a rad-*supplement* V in M if

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$U + V = M$ and $U \cap V \subseteq \mathrm{rad}(V)$. M is called a rad-*supplemented module* if every submodule of M has a rad-supplement in M. We shall say that a module M is w-supplemented if every semisimple submodule of M has a supplement in M [1]. Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular (see for example [1, 3, 4, 8, 9, 12, 13]. Supplemented property in the lattices have already been investigated in [8]. This paper is based on another variation of supplemented filters. In fact, in the present paper, we are interested in investigating (amply) w-supplemented filters in a distributive lattice with 1 to use other notions of w-supplemented, and associate which exist in the literature as laid forth in [1] (see Sections 2 and Section 3).

Let us briefly review some definitions and tools that will be used later [2]. By a *lattice* we mean a poset $(L, \leqslant)$ in which every couple elements x, y has a g.l.b. (called the *meet* of x and y, and written $x \wedge y$) and a l.u.b. (called the *join* of x and y, and written $x \vee y$). A lattice L is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a *least element* 0 and *greatest element* 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a *filter*, if for $a \in F$, $b \in L$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be *maximal* if E is a filter in L with $P \subsetneqq E$, then $E = L$. If F is a filter of a lattice L, then the *radical* of F, denoted by $\mathrm{rad}(F)$, is the intersection of all maximal subfilters of F.

Let L be a lattice. If A is a subset of L, then the *filter generated by* A, denoted by $T(A)$, is the intersection of all filters that is containing A. A filter F is called *finitely generated* if there is a finite subset A of F such that $F = T(A)$. A subfilter G of a filter F of L is called *small* in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$ [8]. A subfilter G of F is called *essential* in F, written $G \trianglelefteq F$, if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. Let G be a subfilter of a filter F of L. A subfilter H of F is called a *supplement* of G in F if $F = T(G \cup H)$ and H is minimal with respect to this property, or equivalently, $F = T(G \cup H)$ and $G \cap H \ll H$. H is said to be a *supplement subfilter* of F if H is a supplement of some subfilter of F. F is called a *supplemented* filter if every subfilter of F has a supplemented in F. A subfilter G of a filter F of L has *ample supplements* in F if, for every subfilter H of F with $F = T(H \cup G)$, there is a supplement H' of G with $H' \subseteq H$. If every subfilter of a filter F has ample supplements in F, then we call F *amply supplemented*. Let G, H be subfilters of a filter F of L. If $F = T(G \cup H)$ and $G \cap H \ll F$, then H is called a weak supplement of G in F. If every subfilter of F has a weak supplement in F, then F is called a weak supplemented filter. If $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{rad}(H)$, then H is called a *rad-supplement* of G in F. If every subfilter of F has a *rad-supplement* in F, then F is called a *rad-supplemented filter*.

A lattice L is called *semisimple*, if for each proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$). In this case, we say that

F is a *direct summand* of L, and we write $L = F \oplus G$. A filter F of L is called a *semisimple* filter, if every subfilter of F is a direct summand. A *simple* lattice (resp. filter), is a lattice (resp. filter) that has no filters besides the $\{1\}$ and itself. For a filter F, $\mathrm{Soc}(F) = T(\cup_{i\in\Lambda} F_i)$, where $\{F_i\}_{i\in\Lambda}$ is the set of all simple filters of L contained in F.

Proposition 1.1. [6, 7] *A non-empty subset F of a lattice L is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

Proposition 1.2. [8, Lemma 2.4, Theorem 2.6 and Theorem 2.9] *Let F be a filter of a distributive lattice L with* 1.

(1) *If $A \ll F$ and $C \subseteq A$, then $C \ll F$.*
(2) *If A, B are subfilters of F with $A \ll B$, then then A is a small subfilter in subfilters of F that contains the subfilter of B. In particular, $A \ll F$.*
(3) *If $F_1, F_2, \ldots, F_n$ are small subfilters of F, then $T(F_1 \cup F_2 \cup \cdots \cup F_n)$ is also small in F.*
(4) *If A, B, C and D are subfilters of F with $A \ll B$ and $C \ll D$, then $T(A \cup C) \ll T(B \cup D)$.*
(5) *Let G, H be subfilters of F such that H is a supplement of G in F. If $F = T(U \cup H)$ for some subfilter U of G, then H is a supplement of U in F.*
(6) $\mathrm{rad}(F) = T(\cup_{G \ll F} G)$.
(7) *Every finitely generated subfilter of* $\mathrm{rad}(F)$ *is small in* $\mathrm{rad}(F)$.
(8) $x \in \mathrm{rad}(F)$ *if and only if* $T(\{x\}) \ll \mathrm{rad}(F)$.

Lemma 1.3. [8, Proposition 2.1]

(1) *Let A be an arbitrary non-empty subset of L. Then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x$ for some $a_i \in A$ $(1 \leqslant i \leqslant n)\}$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.*
(2) *Let A, B and C be subfilters of a filter F of L. Then $T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C))$. In particular, if $F = T(T(A \cup B) \cup C)$, then $F = T(T(C \cup B) \cup A) = T(T(A \cup C) \cup B)$.*
(3) (*Modular law*) *If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.*

2. Basic Properties of w-supplemented Filters

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. In this section we collect some basic properties concerning w-supplemented filters of L. Our starting point is the following lemma.

Lemma 2.1. *Every subfilter of a semisimple filter of L is semisimple.*

Proof. Assume that K is a subfilter of a semisimple filter F of L and let G be a subfilter of K. By assumption, $F = T(G \cup H)$ and $H \cap G = \{1\}$ for some subfilter H of F. Then by modular law, $K = K \cap T(G \cup H) = T(G \cup (H \cap K))$ and $G \cap (H \cap K) = \{1\}$, as required. □

Lemma 2.2. *Let F be a filter of L such that $F = T(G \cup H)$, where H is a subfilter of F and G is a semisimple subfilter of F. Then $F = G' \oplus H$ for some subfilter G' of G.*

Proof. By assumption, $G = (G \cap H) \oplus G'$ for some subfilter G' of G. Then by Lemma 1.3, $F = T(H \cup T((G \cap H) \cup G')) = T(G' \cup T(H \cup (G \cap H))) = T(G' \cup H)$ and $G' \cap H = G \cap G' \cap H = \{1\}$. So $F = G' \oplus H$. □

Lemma 2.3. *Let U, V be subfilters of a filter F of L such that V is a direct summand of F with $U \subseteq V$. Then $U \ll F$ if and only if $U \ll V$.*

Proof. If $U \ll V$, then $U \ll F$ by Proposition 1.2 (2). Conversely, assume that $U \ll F$ and $F = T(V \cup V')$ with $V \cap V' = \{1\}$. Let $V = T(U \cup K)$ for some subfilter K of V. It follows from Lemma 1.3 that

$$F = T(V' \cup T(U \cup K)) = T(U \cup T(V' \cup K));$$

hence $F = T(V' \cup K)$ since $U \ll F$. Now it is enough to show that $V \subseteq K$. Let $x \in V$. Then $x \in T(V' \cup K)$ gives $x = x \vee (v' \wedge k) = (x \vee v') \wedge (x \vee k)$ for some $v' \in V'$ and $k \in K$. Since $x \vee v' \in V \cap V' = \{1\}$, we get $x = x \vee k \in K$, as required. □

Lemma 2.4. *Let F be a filter of L. Then the following hold:*

(1) $\mathrm{Soc}(\mathrm{rad}(F)) \ll F$.

(2) *If G is a semisimple subfilter of F such that $G \subseteq \mathrm{rad}(F)$, then $G \ll F$.*

Proof. (1). Put $G = \mathrm{Soc}(\mathrm{rad}(F))$ and suppose that $F = T(G \cup K)$ for some subfilter K of F. Set $H = G \cap K$. Then we have $G = H \oplus H'$ for some subfilter H' of $G \subseteq \mathrm{rad}(F)$, $F = T(K \cup T(H \cup H')) = T(H' \cup T(H \cup K)) = T(H' \cup K)$ and $\{1\} = H \cap H' = (G \cap K) \cap H' = H' \cap K$; hence $F = H' \oplus K$. We claim that $H' = \{1\}$. To see this, it suffices to show that every simple subfilter of H' is $\{1\}$. If S is any simple subfilter of $H' \subseteq \mathrm{rad}(F)$, then S is a direct summand of H'; hence it is a direct summand of F. By Proposition 1.2 (7), $S \ll \mathrm{rad}(F)$ which implies that $S \ll F$ by Proposition 1.2 (2). Thus S is a direct summand of F and is small in F and hence $S = \{1\}$. Thus $F = T(K) = K$. This completes the proof.

(2). By assumption, $G \subseteq \mathrm{rad}(F)$ gives $G = \mathrm{Soc}(G) \subseteq \mathrm{Soc}(\mathrm{rad}(F))$. Now the assertion follows from (1) and Proposition 1.2 (1). □

Definition 2.5. A filter F of L is called *w-supplemented*, if every semisimple subfilter of F has a supplement in F.

We next give three other characterizations of w-supplemented filters.

Theorem 2.6. *Let F be a filter of L. Then the following statements are equivalent.*

(1) *F is w-supplemented.*

(2) *Every semisimple subfilter of F has a supplement that is a direct summand.*

(3) *Every semisimple subfilter of F has a weak supplement.*

(4) *Every semisimple submodule of F has a* rad-*supplement.*

Proof. (1) $\Rightarrow$ (2) Let G be a semisimple subfilter of F. Then $F = T(G \cup H)$ and $G \cap H \ll H$ for some subfilter H of F. By Lemma 2.2, there exists a subfilter G' of G such that $F = G' \oplus H$.

(2) $\Rightarrow$ (3). Let G be a semisimple subfilter of F. Then By (2), $F = T(G \cup H)$ and $G \cap H \ll H$ for some subfilter H of F. By Proposition 1.2 (2), $G \cap H \ll H$ gives $G \cap H \ll F$; hence G has a weak supplement.

(3) $\Rightarrow$ (4). Let G be a semisimple subfilter of F. By assumption, $F = T(G \cup H)$ and $G \cap H \ll F$ for some subfilter H of F. By Lemma 2.2, there exists a subfilter G' of G such that $F = G' \oplus H$. Since $G \cap H \subseteq H$ and $G \cap H \ll F$, we get $G \cap H \ll H$ by lemma 2.3. This implies $G \cap H \subseteq \mathrm{rad}(H)$ by Proposition 2.1 (6). Thus H is rad-supplement of G in F.

(4) $\Rightarrow$ (1). Let G be a semisimple subfilter of F. By (4), $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{rad}(H)$ for some subfilter H of F. Since $G \cap H \subseteq \mathrm{rad}(H) \subseteq \mathrm{rad}(F)$, Lemma 2.4 and Lemma 2.1 gives $G \cap H \ll F$. Since G is semisimple, by Lemma 2.2, $F = G' \oplus H$ for some subfilter G' of G. So we get $G \cap H \ll H$ by Lemma 2.3. This completes the proof. □

Corollary 2.7. *Let F be a filter of L. Then F is w-supplemented if and only if for each semisimple submodule G of F, there exists a decomposition $F = F_1 \oplus F_2$ such that F_1 is a subfilter of G and $G \cap F_2 \ll F_2$.*

Proof. Apply Theorem 2.6 □

Proposition 2.8. *If F is a w-supplemented filter of L, then $F = X \oplus S$ for some semisimple subfilter X and a subfilter S of F.*

Proof. Let G be a semisimple subfilter of F. If there is no $G \neq \{1\}$, then $F = F \oplus \{1\}$ and result follows. Otherwise, by assumption, $F = T(G \cup S)$ and $G \cap S \ll S$ for some subfilter S of F. By Lemma 2.1, there exists a semisimple subfilter X of G such that $G = (G \cap S) \oplus X$; hence by Lemma 1.3, $F = T(G \cup S) = T(S \cup T((G \cap S) \cup X)) = T(X \cup T(S \cup (G \cap S))) = T(X \cup T(S)) = T(X \cup S)$ and $X \cap S = (G \cap S) \cap X = \{1\}$. So $F = X \oplus S$. □

Theorem 2.9. *Every direct summand of a w-supplemented filter F of L is w-supplemented.*

Proof. Let G be a direct summand of F. Then $F = T(G \cup H)$ and $G \cap H = \{1\}$ for some subfilter H of F. Let S be a semisimple subfilter of G. If $S = \{1\}$, then G trivially w-supplemented. So we may assume that $S \neq \{1\}$. Since S is a semisimple subfilter of F, we have $F = T(S \cup K)$ and $S \cap K \ll K$ for some subfilter K of F. Then by modular law, $G = G \cap T(S \cup K) = T(S \cup (K \cap G))$. It is enough to show that $S \cap (K \cap G) = K \cap S \ll K \cap G$. By Lemma 2.2, $G = T(S' \cup (K \cap G))$ and $S' \cap (K \cap G) = S' \cap K = \{1\}$ for some subfilter S' of S. That is, $K \cap G$ is a direct summand of G. By Proposition 1.2 (2), $S \cap K \ll K$ gives $S \cap K \ll F$. Since $S \cap K \subseteq G$ and G is a direct summand of F, we get $K \cap S \ll G$ by Lemma 2.3. As $K \cap G$ is a direct summand of G, $K \cap S \subseteq K \cap G$ and $K \cap S \ll G$, we obtain $K \cap S \ll K \cap G$ by Lemma 2.3. This completes the proof. □

Lemma 2.10. *Let H, G be subfilters of F such that $T(H \cup G)$ has a supplement of U in F and $H \cap T(U \cup G)$ has a supplement V in H. Then $T(U \cup V)$ is a supplement of G in F.*

Proof. To simplify our notation let $B = T(U \cup G) \cap H \subseteq T(U \cup G)$. By hypothesis, $T(U \cup T(H \cup G)) = F$, $U \cap T(H \cup G) \ll U$, $T(V \cup B) = H$ and $V \cap B = V \cap T(U \cup G) = A \ll V$. By Lemma 1.3, we have $F = T(U \cup T(H \cup G)) = T(H \cup T(U \cup G)) = T(T(B \cup V) \cup T(U \cup G)) =$

$$T(V \cup T(B \cup T(U \cup G))) = T(V \cup T(U \cup G)) \subseteq T(G \cup T(U \cup V)) \subseteq F;$$

hence $F = T(G \cup T(U \cup V))$. It is enough to show that $T(U \cup V) \cap G \ll T(U \cup V)$. Since $T(G \cup V) \subseteq T(H \cup G)$ and $F = T(G \cup T(U \cup V)) = T(U \cup T(G \cup V))$, Proposition 1.2 (5) gives U also is a supplement of $T(G \cup V)$ in F which implies that $C = T(G \cup V) \cap U \ll U$. Now by Proposition 1.2 (4), $T(U \cup V) \cap G \subseteq T(A \cup C)) \ll T(U \cup V)$; hence $T(U \cup V) \cap G \ll T(U \cup V)$ by Proposition 1.2 (1). □

Theorem 2.11. *Let F_1, F_2 and F be filters of L such that $F = F_1 \oplus F_2$. If F_1 and F_2 are w-supplemented, then F is w-supplemented.*

Proof. Let K be a semisimple subfilter of F. At the first we show that $F_1 \cap T(K \cup F_2)$ is a semisimple subfilter of F_1. Assume that G is a subfilter of $F_1 \cap T(K \cup F_2)$ and let $x \in G$. Then there are elements $h \in K$ and $f_2 \in F_2$ such that $x = x \vee (h \wedge f_2) = (x \vee h) \wedge (x \vee f_2)$. Then $x \vee f_2 \in G \cap F_2 \subseteq F_1 \cap F_2 = \{1\}$ which implies that $x = x \vee h \in K$; hence $G \subseteq K$. By Lemma 2.1, G is semisimple. If $G = F_1 \cap T(K \cup F_2)$, we are done. So we may assume that $G \neq F_1 \cap T(K \cup F_2)$. There exists a subfilter G' of K such that $K = G \oplus G'$. Then by Lemma 1.3,

$$F_1 \cap T(K \cup F_2) = F_1 \cap T(T(G \cup G') \cup F_2) \subseteq T(G \cup T(G' \cup F_2)) \cap F_1$$
$$= T(G \cup (F_1 \cap T(G' \cup F_2))) \text{ with } F_1 \cap T(G' \cup F_2) \neq \{1\}.$$

As

$$G \cup (F_1 \cap T(G' \cup F_2)) \subseteq F_1 \cap T(K \cup F_2),$$

we get $T(G \cup (F_1 \cap T(G' \cup F_2))) \subseteq F_1 \cap T(K \cup F_2)$; hence

$$F_1 \cap T(K \cup F_2) = T(G \cup (F_1 \cap T(G' \cup F_2)).$$

It is enough to show that $G \cap (F_1 \cap T(G' \cup F_2)) = G \cap T(G' \cup F_2) = \{1\}$. Let $z \in G$ and $z \in T(G' \cup F_2)$. Thus $z = z \vee (g' \wedge f) = (z \vee g') \wedge (z \vee f)$ for some $g' \in G'$ and $f \in F_2$. Since $z \vee g' \in G \cap G' = \{1\}$ and $z \vee f \in G \cap F_2 = \{1\}$, we get $z = 1$. Thus $A = F_1 \cap T(K \cup F_2)$ is a semisimple subfilter of F_1. Similarly, $B = F_2 \cap T(K \cup F_1)$ is a semisimple subfilter of F_2. Then A and B have supplements V_1 and V_2 in F_1 and F_2, respectively. Clearly, $F = T(F \cup \{1\}) = T(T(F_1 \cup F_2) \cup K) = T(F_2 \cup T(F_1 \cup K))$ has a supplement $\{1\}$ in F. If $G = T(F_1 \cup K)$ and $H = F_2$, then V_2 is a supplement $T(F_1 \cup K)$ in F by Lemma 2.10. Also $F_1 \cap T(K \cup V_2) \subseteq F_1 \cap T(F_2 \cup K)$ gives $F_1 \cap T(V_2 \cup K)$ is semisimple by Lemma 2.1 which implies that it has a supplement S in F_1. Again applying Lemma 2.10, $T(V_2 \cup S)$ is a supplement of K in F. Thus F is w-supplemented. □

Corollary 2.12. *Every finite direct sum of w-supplemented filters of L is w-supplemented.*

Proof. Apply Theorem 2.11. □

Proposition 2.13. *Let G be a subfilter of a filter F of L. Then the following hold:*

(1) *If A is the intersection of filters of L which are essential in F, then* $A = \mathrm{Soc}(F)$.

(2) $\mathrm{Soc}(G) = G \cap \mathrm{Soc}(F)$ *and* $\mathrm{Soc}(\mathrm{Soc}(F)) = \mathrm{Soc}(F)$.

Proof. (1). Let $\mathrm{Soc}(F) = T(\cup_{i \in I} F_i)$, where $\{F_i\}_{i \in I}$ is the set of all simple filters of L contained in F. Let $G \trianglelefteq F$. For each $i \in I$, $F_i \cap G \neq 1$ which implies that $F_i \subseteq G$; hence $\mathrm{Soc}(F) \subseteq A$. For the reverse inclusion, it is enough to show that A is semisimple. Let G be a filter of L such that $G \subseteq A$. If $G \trianglelefteq F$, then $A \subseteq G$; so $G = A$. So we may assume that G is not essential in F. Let G' be a complement of G in F; so $T(G \cup G') \trianglelefteq F$ by [8, Lemma 2.15 (3)]. It follows that $G \subseteq A \subseteq T(G \cup G')$; thus $A = A \cap T(G \cup G') = T(G \cup (A \cap G'))$ by Lemma 1.3 which implies that $A = G \oplus (A \cap G')$; hence A is semisimple. Thus $A \subseteq \mathrm{Soc}(F)$, and so we have equality.

(2). Let $\mathrm{Soc}(F) = T(\cup_{i \in I} F_i)$, where $\{F_i\}_{i \in I}$ is the set of all simple filters of L contained in F. Since the inclusion $\mathrm{Soc}(G) \subseteq G \cap \mathrm{Soc}(F)$ is clear, we will prove the reverse inclusion. Let $x \in G \cap \mathrm{Soc}(F)$. So $x = x \vee (f_1 \wedge f_2 \wedge \cdots \wedge f_t) = (x \vee f_1) \wedge \cdots \wedge (x \vee f_t)$ for some $f_j \in F_{i_j}$ $(1 \leqslant j \leqslant t)$. If for each $1 \leqslant j \leqslant t$, $F_{i_j} \subseteq G$, then we are done. Therefore, without loss of generality, we can assume that $F_{i_1}, F_{i_2}, \cdots, F_{i_m} \not\subseteq G$ (so $G \cap F_{i_1} = \{1\}, \cdots G \cap F_{i_m} = \{1\}$) and $F_{i_{m+1}}, \cdots, F_{i_t} \subseteq G$. As for each $1 \leqslant j \leqslant m$, F_{i_j}, G are filters, we get $x \vee f_{i_j} = 1$; hence $x = (x \vee f_{m+1}) \wedge \cdots \wedge (x \vee f_t) \in T(F_{m+1} \cup \cdots \cup F_t) \subseteq \mathrm{Soc}(G)$, and so we have equality. Finally, if $G = \mathrm{Soc}(F)$, then $\mathrm{Soc}(\mathrm{Soc}(F)) = \mathrm{Soc}(F)$. □

Lemma 2.14. *If F is a filter of L with* $\mathrm{Soc}(F) \ll F$*, then F is w-supplemented.*

Proof. It is clear that if $\mathrm{Soc}(F) = \{1\}$, then F is w-supplemented. Let G be a semisimple subfilter of F. Since $\mathrm{Soc}(F)$ is the largest semisimple subfilter of F, then $G \subseteq \mathrm{Soc}(F) \ll F$ which implies that $G \ll F$ by Proposition 1.2 (1). Now $F = T(F \cup G)$ and $G \cap F = G \ll F$ gives G has a supplement in F. Thus F is w-supplemented. □

Theorem 2.15. *Let F be a filter of L. Then F is w-supplemented if and only if* $\mathrm{Soc}(F)$ *has a supplement in F.*

Proof. If F is w-supplemented, then $\mathrm{Soc}(F)$ has a supplement in F since it is semisimple. Conversely, let H be a supplement of $\mathrm{Soc}(F)$ in F. Then by Proposition 2.13, $F = T(H \cup \mathrm{Soc}(F))$ and $\mathrm{Soc}(H) = H \cap \mathrm{Soc}(F) \ll H$; hence H is w-supplemented by Lemma 2.14. By Lemma 2.2, $F = H \oplus S$, where S is a semisimple subfilter of F (so it is w-supplemented). Thus F is w-supplemented by Theorem 2.11. □

A subfilter G of a filter F of L is said to be radical if $\mathrm{rad}(G) = G$.

Proposition 2.16. *Every radical filter F of L is w-supplemented.*

Proof. Since $\mathrm{Soc}(F) = \mathrm{Soc}(\mathrm{rad}(F)) \ll F$ by Lemma 2.4, we get F is w-supplemented by Lemma 2.14. □

Definition 2.17. A filter F of L is called *amply w-supplemented*, if $F = T(A \cup B)$, where A is a semisimple subfilter of F, then B contains a supplement of A.

Theorem 2.18. *Let F be a filter of L. Then F is w-supplemented if and only if F is amply w-supplemented.*

Proof. Clearly, if F is amply w-supplemented, then it is w-supplemented. Conversely, let A be a semisimple subfilter of F such that $F = T(A \cup B)$. It suffices to show that B contains a supplement of A in F. Since $A \cap B$ is semisimple, we have $F = T(H \cup (A \cap B))$ and $A \cap B \cap H \ll H$ for some subfilter H of F. By Lemma 2.2, $F = H \oplus F_1$ for some subfilter F_1 of $A \cap B$. Then by Lemma 1.3, $B = B \cap T(H \cup F_1) = T(F_1 \cup (H \cap B))$ and

$$F = T(A \cup T(F_1 \cup (B \cap H))) = T((B \cap H) \cup T(A \cup F_1)) = T((B \cap H) \cup A)$$

with $B \cap H \subseteq B$. It follows that $H = H \cap T((B \cap H) \cup A) = T((B \cap H) \cup (H \cap A))$. Since $H \cap A$ is semisimple, by Lemma 2.2, $H = (B \cap H) \oplus K$ for some subfilter K of $H \cap A$. Now $B \cap H$ is a direct summand of H and $A \cap B \cap H \ll H$ gives $A \cap B \cap H \ll H \cap B$ by Lemma 2.3, as required. □

Lemma 2.19. *Let F be a filter of L such that* $\mathrm{rad}(F) \trianglelefteq F$*. Let $K \subseteq G \subseteq F$ be subfilters of F and assume K to be a direct summand of F. Then* $\mathrm{rad}(K) = \mathrm{rad}(G)$ *if and only if $G = K$.*

Proof. Let $\mathrm{rad}(K) = \mathrm{rad}(G)$. By assumption, $F = K \oplus K'$ for some subfilter K' of F. Then by modular law, $G = G \cap T(K \cup K') = T(K \cup (K' \cap G))$ with $K \cap (K' \cap G) = \{1\}$. Then $\mathrm{rad}(G) = T(\mathrm{rad}(K) \cup \mathrm{rad}(G \cap K'))$ with $\mathrm{rad}(G) \cap \mathrm{rad}(G \cap K') = \{1\}$ by [8, Proposition 2.16] (so $\mathrm{rad}(G \cap K') = \{1\}$). Clearly, $G \cap K'$ is a supplement of K in G. If $\mathrm{rad}(G) = \mathrm{rad}(K)$, then by [8, Theorem 2.9 (3)], $\{1\} = \mathrm{rad}(G \cap K') = (G \cap K') \cap \mathrm{rad}(F)$ which implies that $G \cap K' = \{1\}$ since $\mathrm{rad}(F) \trianglelefteq F$, and so $G = K$. The other implication is clear. □

Theorem 2.20. *Let F be a filter of L such that* $\mathrm{rad}(F) \trianglelefteq F$. *Then the following statements are equivalent:*

(1) *F is w-supplemented;*

(2) *Every semisimple submodule of F is a direct summand;*

(3) $\mathrm{Soc}(F)$ *is a direct summand of F.*

Proof. (1) $\Rightarrow$ (2). Let G be a semisimple subfilter of F. By (1), there is a subfilter K of F such that $F = T(G \cup K)$ and $G \cap K \ll K$. By Lemma 2.2, $F = K \oplus G'$ for some subfilter G' of G. By [8, Proposition 2.16], we have $\mathrm{rad}(G) = \mathrm{rad}(G') = \{1\}$ which implies that $G = G'$ by Lemma 2.19. Thus $F = K \oplus G$.

(2) $\Rightarrow$ (3). Since $\mathrm{Soc}(F)$ is semisimple subfilter of F, we get it is a direct summand of F by (2).

(3) $\Rightarrow$ (1). Let G be a semisimple subfilter of F. So G is a subfilter and a direct summand of $\mathrm{Soc}(F)$; hence G is a direct summand of F by (3). So $F = G \oplus H$ and $G \cap H = \{1\} \ll H$ for some subfilter H of F. Thus F is w-supplemented. □

Corollary 2.21. *Let F be a filter of L such that* $\mathrm{rad}(F) \trianglelefteq F$. *If F is w supplemented, then every subfilter of F is w-supplemented.*

Proof. Assume that G is a subfilter of F and let K be a semisimple subfilter of G. By Theorem 2.20, there exists a subfilter H of F such that $F = K \oplus H$. By modularity, $G = G \cap T(K \cup H) = T(K \cup (G \cap H))$ with $K \cap (G \cap H) = \{1\}$, that is, $G = K \oplus (G \cap H)$. Therefore G is w-supplemented. □

Definition 2.22. We say that a filter F of L is *totally w-supplemented*, if every subfilter of F is w-supplemented. A lattice L is called a *V-lattice* if $\mathrm{rad}(F) = \{1\}$ for every filter F of L.

Proposition 2.23. *For a V-lattice L and a filter F of L, the following statements are equivalent:*

(1) *F is w-supplemented;*

(2) *F is amply w-supplemented;*

(3) *F is totally w-supplemented.*

Proof. (1) $\Leftrightarrow$ (2). The proof is followed from Theorem 2.18.

(1) $\Leftrightarrow$ (3). Let F be totally w-supplemented. Since $F \subseteq F$, F is also w-supplemented. Conversely, assume that F is w-supplemented and G be a subfilter of F. We will show that G is w-supplemented. Let K be a semisimple subfilter of G. By assumption, there is a subfilter H of F such that $F = T(H \cup K)$ and $H \cap K \ll H$. So $H \cap K \subseteq \text{rad}(H) \subseteq \text{rad}(F) = \{1\}$; hence $F = H \oplus K$. By modularity, $G = G \cap T(K \cup H) = T(K \cup (G \cap H))$ with $K \cap (G \cap H) = \{1\}$; so $G = K \oplus (G \cap H)$. Thus G is w-supplemented. □

Proposition 2.24. *Let $F = F_1 \oplus F_2$ be a filter of L such that* $\text{rad}(F) \trianglelefteq F$, *where F_1 and F_2 are totally w-supplemented filters. Then F is totally w-supplemented.*

Proof. Let G be a subfilter of F and K be a semisimple subfilter of G. Clearly, F_1 and F_2 are w-supplemented; so F is w-supplemented by Theorem 2.11. Then $F = T(K \cup H)$ and $K \cap H \ll H$ for some subfilter H of F. By Lemma 2.2, $F = K' \oplus H$ for some subfilter K' of K. By Lemma 2.19, $K = K'$ which implies that $F = K \oplus H$. So by modular law, $G = G \cap T(H \cup K) = T(K \cup (G \cap H))$ and $K \cap (G \cap H) = K \cap H \ll H$. Hence G is w-supplemented. □

Theorem 2.25. *Let $F = F_1 \oplus F_2$ be a filter of L such that F_2 is semisimple. Then F is totally w-supplemented if and only if F_1 is totally w-supplemented.*

Proof. It suffices to show that if F_1 is totally w-supplemented, then F is totally w-supplemented. Let G be a subfilter of F. Since F_2 is semisimple, there is a subfilter H of F_2 such that $F_2 = (G \cap F_2) \oplus H$ (so $G \cap H = \{1\}$ and $H \cap F_1 = \{1\}$). By Lemma 1.3, since

$$F = T(F_1 \cup T((G \cap F_2) \cup H)) = T((G \cap F_2) \cup (F_1 \cup H)),$$

we get $G = T((G \cap F_2) \cup (G \cap T(F_! \cup H)))$ with $(G \cap F_2) \cap (G \cap T(F_1 \cup H)) = \{1\}$, that is, $G = (G \cap F_2) \oplus (G \cap (F_1 \oplus H))$. If $x \in G \cap T(F_1 \cup H)$, then $x = (x \vee f_1) \wedge (x \vee h)$ for some $f_1 \in F_1$ and $h \in H$. As $x \vee h \in G \cap H = \{1\}$, we get $x \in F_1$, and so $G \cap (F_1 \oplus H)$ is a subfilter of F_1; hence it is w-supplemented. Also, $G \cap F_2$ is w-supplemented since it is semisimple. Now the assertion follows from Theorem 2.11. □

3. W-supplemented Quotient Filters

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leqslant)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $(\frac{L}{F}, \leqslant_Q)$ is a poset. The following notation below will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leqslant_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leqslant_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $(\frac{L}{F}, \leqslant_Q)$ is a lattice.

Remark 3.1. Let G be a subfilter of a filter F of L.

(1) If $a \in F$, then $a \wedge F = F$. By the definition of $\leqslant_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition $\leqslant_Q$, we can easily show that if L is distributive, then $\frac{L}{F}$ is distributive.

(4) $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.

(5) If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L.

(6) If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.

(7) If H and V are filters of L containing G, then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.

(8) If H is a filter of L containing G, then $\frac{T(F \cup H)}{G} = T(\frac{H}{G} \cup \frac{F}{G})$.

Proposition 3.2. *Every quotient of a semisimple filter of L is semisimple.*

Proof. Let K be a subfilter of a semisimple filter F. We show that $\frac{F}{K}$ is semisimple. Let $\frac{G}{K}$ be a subfilter of $\frac{F}{K}$. Since F is semisimple, $F = T(G \cup H)$ with $G \cap H = \{1\}$ for some subfilter H of F. Then we have $\frac{F}{K} = \frac{T(G \cup H)}{K} =$

$$\frac{T(G \cup T(H \cup K))}{K} = T(\frac{G}{K} \cup \frac{T(H \cup K)}{K})$$

and $\frac{G}{K} \cap \frac{T(H \cup K)}{K} = \frac{G \cap T(H \cup K)}{K}$. It is enough to show that $G \cap T(H \cup K) = K$. Clearly, $K \subseteq G \cap T(H \cup K)$. For the reverse inclusion, suppose that $x \in G \cap T(H \cup K)$. Then $x = x \vee (h \wedge k) = (x \vee h) \wedge (x \vee k)$ for some $h \in H$ and $k \in K$. As $x \vee h \in G \cap H = \{1\}$, we get $x = x \vee k \in K$, and so we have equality. Thus $\frac{F}{K} = \frac{G}{K} \oplus \frac{T(H \cup K)}{K}$. □

Proposition 3.3. *Let H and G be subfilters of a filter F of L. IF H is semisimple, then $\frac{T(H \cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$.*

Proof. Let $\frac{U}{G}$ be a subfilter of $\frac{T(H \cup G)}{G}$ (so $U \subseteq T(H \cup G)$). By assumption, $H = (H \cap U) \oplus K$ for some subfilter K of H (so $U \cap K = \{1\}$). At first we show that $T(U \cup K) = T(H \cup G)$. Since $U \subseteq T(H \cup G)$ and $K \subseteq H$, we get $T(U \cup K) \subseteq T(H \cup G)$. For the reverse inclusion, by Lemma 1.3, we have

$$T(H \cup G) = T(G \cup T(K \cup (H \cap U))) \subseteq T(G \cup T(U \cup K)) \subseteq T(K \cup T(U \cup G)) = T(U \cup K),$$

and so we have equality. Next we show that $T(U\cup K)=T(U\cup T(G\cup K))$. Since the inclusion $T(U\cup K)\subseteq T(U\cup T(G\cup K))$ is clear, we will prove the reverse containment. Let $x\in T(U\cup T(G\cup K))$. Then $x=(x\vee u)\wedge(x\vee t)$ for some $u\in U$ and $t\in T(G\cup K)$ which implies that $x=(x\vee u)\wedge(x\vee t\vee g)\wedge(x\vee t\vee k)\in T(K\cup U)$ for some $g\in G$ and $k\in K$. Thus $T(U\cup T(G\cup K))=T(U\cup K)=T(G\cup H))$. Clearly, $G\subseteq U\cap T(G\cup K)$. If $z\in U\cap T(G\cup K)$, then $z=(z\vee g)\wedge(z\vee k)$ for some $g\in G$ and $k\in K$. As $z\vee k\in U\cap K=\{1\}$, we get $z=z\vee g\in G$; hence $G=U\cap T(G\cup U)$. Now we have

$$T(\frac{U}{G}\cup\frac{T(G\cup K}{G})=\frac{T(U\cup T(G\cup K))}{G}=\frac{T(H\cup G)}{G}$$

and $\frac{U}{G}\cap\frac{T(G\cup K)}{G}=\frac{U\cap T(G\cup K)}{G}=\frac{G}{G}=\{G\}$. Thus $\frac{T(H\cup G)}{G}=\frac{U}{G}\oplus\frac{T(G\cup K)}{G}$. □

Theorem 3.4. *Let G be a subfilter a filter F of L such that $G\ll F$. If G and $\frac{F}{G}$ are w-supplemented, then F is w-supplemented.*

Proof. If H is any semisimple subfilter of F, then $\frac{T(H\cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$ by Proposition 3.3. If $\frac{F}{G}=\frac{T(H\cup G)}{G}$, then $F=T(H\cup G)$. By Lemma 2.2, $F=H'\oplus G$ for some subfilter H' of H which implies that F is w-supplemented as a finite direct sum of w-supplemented filters. So we may assume that $\frac{F}{G}\neq\frac{T(H\cup G)}{G}$. By assumption, there exists a subfilter $\frac{K}{G}$ of $\frac{F}{G}$ such that $\frac{F}{G}=T(\frac{T(H\cup G)}{G}\cup\frac{K}{G})=\frac{T(K\cup T(H\cup G))}{G}=\frac{T(K\cup H)}{G}$ and $\frac{T(H\cup G)}{G}\cap\frac{K}{G}=\frac{T(H\cup G)\cap K}{G}=\frac{T(G\cup(H\cap K)}{G}\ll\frac{K}{G}$. Since $F=T(K\cup H)$, it is enough to show that $H\cap K\ll K$. Let $K=T(X\cup(H\cap K))$ for some subfilter X of K. Then $\frac{K}{G}=T(\frac{T(G\cup(H\cap K)}{G}\cup\frac{T(X\cup G)}{G})$. Since $\frac{T(G\cup(H\cap K)}{G}\ll\frac{K}{G}$, then $\frac{K}{G}=\frac{T(X\cup G)}{G}$; hence $K=T(X\cup G)$. As $F=T(K\cup H)$, there is a subfilter U of H such that $F=K\oplus U$ by Lemma 2.2. As K is a direct summand of F and $G\subseteq K$, $G\ll F$ gives $G\ll K$ by Lemma 2.3; hence $K=X$. Thus $H\cap K\ll K$. This completes the proof. □

Theorem 3.5. *Let F be a filter of L. If every semisimple subfilter of $\frac{F}{\mathrm{rad}(F)}$ is a direct summand, then F is (amply) w-supplemented. In particular, if $\frac{F}{\mathrm{rad}(F)}$ is semisimple, then F is (amply) w-supplemented.*

Proof. Let G be a semisimple subfilter of F. Then by Proposition 3.3, $\frac{T(G\cup\mathrm{rad}(F))}{\mathrm{rad}(F)}$ is a semisimple subfilter of $\frac{F}{\mathrm{rad}(F)}$. If $\frac{T(G\cup\mathrm{rad}(F))}{\mathrm{rad}(F)}=\frac{F}{\mathrm{rad}(F)}$, then $T(G\cup\mathrm{rad}(F))=F$. Thus $F=\mathrm{rad}(F)\oplus G'$ for some subfilter G' of G by Lemma 2.2. Since $G\cap\mathrm{rad}(F)$ is semisimple and $G\cap\mathrm{rad}(F)\subseteq\mathrm{rad}(F)$, $G\cap\mathrm{rad}(F)\ll F$ by Lemma 2.4 and also by Lemma 2.3, $G\cap\mathrm{rad}(F)\ll\mathrm{rad}(F)$ since $\mathrm{rad}(F)$ is a direct summand of F. Thus F is w-supplemented. So we may assume that $\frac{T(G\cup\mathrm{rad}(F))}{\mathrm{rad}(F)}\neq\frac{F}{\mathrm{rad}(F)}$. By assumption and Lemma 1.3, there is a subfilter $\frac{H}{\mathrm{rad}(F)}$ of $\frac{F}{\mathrm{rad}(F)}$ such that

$$\frac{F}{\mathrm{rad}(F)}=T(\frac{T(G\cup\mathrm{rad}(F))}{\mathrm{rad}(F)}\cup\frac{H}{\mathrm{rad}(F)})=\frac{T(G\cup H)}{\mathrm{rad}(F)}$$

and $\frac{T(G \cup \mathrm{rad}(F))}{\mathrm{rad}(F)} \cap \frac{H}{\mathrm{rad}(F)} = \frac{T(\mathrm{rad}(F) \cup (G \cap H))}{\mathrm{rad}(F)} = \{\mathrm{rad}(F)\}$; so $F = T(G \cup H)$ and $T(\mathrm{rad}(F) \cup (G \cap H)) = \mathrm{rad}(F)$ (so $G \cap H \subseteq \mathrm{rad}(F)$). By Lemma 2.2, $F = H \oplus K$ for some subfilter K of G. Since $G \cap H$ is semisimple, by Lemma 2.4, $G \cap H \ll F$. By Lemma 2.3, since H is a direct summand of F and $G \cap H \ll F$, we get $G \cap H \ll H$. Therefore, F is w-supplemented. The in particular statement is clear. □

Definition 3.6. A lattice L is called a *semilocal lattice* if $\frac{F}{\mathrm{rad}(F)}$ is semisimple for every filter F of L.

Corollary 3.7. *If L is a semilocal lattice. Then the following hold:*

(1) *Every filter of L is (amply) w-supplemented.*

(2) *Every filter of L is totally w-supplemented.*

Proof. Apply Theorem 3.5. □

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