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Biquasigroups linear over a group

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Abstract. We determine the structure of biquasigroups $(Q,\circ,*)$ satisfying variations of Polonijo's Ward double quasigroup identity $(x\circ z)*(y\circ z)=x*y$, including those that are linear over a group.

1. Introduction

J.M. Cardoso and C.P. da Silva, inspired by Ward's paper [11] on postulating the inverse operations in groups, introduced in [1] the notion of *Ward quasigroups* as quasigroups $(Q,\circ)$ containing an element e such that $x\circ x=e$ for all $x\in Q$, and satisfying the identity $(x\circ y)\circ z=x\circ(z\circ(e\circ y))$. Polonijo [8] proved that these two conditions can be replaced by the identity:

$$(x\circ z)\circ(y\circ z)=x\circ y. \tag{1}$$

In [1] it is proved that if $(Q,\circ)$ is a Ward quasigroup, then $(Q,\cdot)$, where $x\cdot y=x\circ(e\circ y)$, is a group in which $e=x\circ x$ and $x^{-1}=e\circ x$ for all $x\in Q$. Also, $x\circ e=x$, $e\circ(e\circ x)=x$ and $e\circ(x\circ y)=y\circ x$. Conversely, if $(Q,\cdot)$ is a group, then Q with the operation $x\circ y=x\cdot y^{-1}$ is a Ward quasigroup (cf. [11]). Other characterizations of Ward quasigroups can be found in [2] and [10], some applications in [5]. Note that the Ward quasigroups corresponding to commutative groups sometimes are called *subtractive quasigroups* (cf. [6] and [12]). A Ward quasigroup $(Q,\circ)$ is subtractive if and only if it is medial (that is, it satisfies the identity $(x\circ y)\circ(z\circ w)=(x\circ z)\circ(y\circ w)$) if and only if it is left modular (that is, it satisfies the identity $x\circ(y\circ z)=z\circ(y\circ x)$) (cf. Lemma 2.4, [3]).

A *biquasigroup*, i.e. an algebra of the form $(Q,\circ,*)$ where $(Q,\circ)$ and $(Q,*)$ are quasigroups, is called a *Ward double quasigroup* if it satisfies the identity

$$(x\circ z)*(y\circ z)=x*y. \tag{2}$$

Obviously each Ward quasigroup $(Q,\circ)$ can be considered as a Ward double quasigroup of the form $(Q,\circ,\circ)$. Ward double quasigroups have a similar characterization as Ward quasigroups.

Theorem 1.1. (cf. [7]) *A biquasigroup $(Q,\circ,*)$ is a Ward double quasigroup if and only if there is a group $(Q,+)$ and bijections α,β on Q such that $x\circ y=x-\beta y$ and $x*y=\alpha(x-y)$.*

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Note that Ward double quasigroups are distinct from the double Ward quasigroups considered by Fiala (cf. [4]).

Let us consider the identity (2). Keeping the variables x, y and z the same and varying only the quasigroup operations $\circ$ and $*$, there are sixteen possible identities. Eight of these have reversible versions obtained by replacing the operation $\circ$ with the operation $*$ and, simultaneously, replacing the operation $*$ with the operation $\circ$.

For example, the identity $(x \circ z) * (y * z) = x \circ y$ has the reversible version $(x * z) \circ (y \circ z) = x * y$. So, if we are to consider all possible versions of Theorem 1.1, we need to explore the following identities:

$$(x \circ z) \circ (y \circ z) = x \circ y, \quad (3)$$
$$(x \circ z) \circ (y \circ z) = x * y, \quad (4)$$
$$(x \circ z) \circ (y * z) = x \circ y, \quad (5)$$
$$(x \circ z) * (y \circ z) = x \circ y, \quad (6)$$
$$(x \circ z) \circ (y * z) = x * y, \quad (7)$$
$$(x \circ z) * (y * z) = x \circ y, \quad (8)$$
$$(x \circ z) * (y * z) = x * y. \quad (9)$$

The biquasigroup $(Q, \circ, \circ)$ satisfies identity (3) if and only if $(Q, \circ)$ is a Ward quasigroup. Our interest is in finding non-trivial models of the other six identities, where 'non-trivial' means that the set Q has more than one element. In particular, since Ward quasigroups are unipotent, we will be interested in biquasigroups $(Q, \circ, *)$ where $(Q, \circ)$ or $(Q, *)$ is unipotent, both are unipotent or when one or both are Ward quasigroups.

Note that a biquasigroup $(Q, \circ, *)$, where $(Q, *)$ is a commutative group and $x \circ y = x * y^{-1}$ satisfies identities (1) through (9) if and only if $(Q, *)$ is a Boolean group.

2. Main Results

We will now characterize the biquasigroups satisfying the identities (2) to (9). First we will describe their general properties then we will characterize biquasigroups linear over a group and satisfying identities (2) to (9).

1. Recall that a quasigroup $(Q, \cdot)$ is *linear* over a group (cf. [9]) if there exists a group $(Q, +)$, its automorphisms φ, ψ and $a \in Q$ such that $x \cdot y = \varphi x + a + \psi y$ for all $x, y \in Q$. Consequently, a biquasigroup $(Q, \circ, *)$ will be called *linear* over a group if both its quasigroups $(Q, \circ)$ and $(Q, *)$ are linear over the same group, i.e. if there is a group $(Q, +)$, its automorphisms $\varphi, \psi, \alpha, \beta$ and elements $a, b \in Q$ such that

$$x \circ y = \varphi x + a + \psi y \qquad \text{and} \qquad x * y = \alpha x + b + \beta y.$$

According to the Toyoda Theorem (cf. [9]), a quasigroup $(Q,\cdot)$ is medial if and only if it is linear over a commutative group with commuting automorphisms φ,ψ. In an analogous way we can shows that a quasigroup $(Q,\cdot)$ is *paramedial*, that is it satisfies the identity $(x\cdot y)\cdot(z\cdot u)=(u\cdot y)\cdot(z\cdot x)$ if and only if it is linear over a commutative group with automorphisms φ, ψ such that $\varphi^2=\psi^2$. Based on these facts we say that a biquasigroup $(Q,\circ,*)$ is *medial* (*paramedial*) if both its quasigroups $(Q,\circ)$ and $(Q,*)$ are medial (paramedial) and linear over the same commutative group.

A biquasigroup $(Q,\circ,*)$ is *unipotent* if there is $q\in Q$ such that $x\circ x=q=x*x$ for and all $x\in Q$. If both quasigroups $(Q,\circ)$ and $(Q,*)$ are idempotent then we say that $(Q,\circ,*)$ is an *idempotent biquasigroup.*

2. We will start with biquasigroups satisfying the identity (2).

A general characterization of such biquasigroups is given by Theorem 1.1. Now we describe a biquasigroup linear over a group $(Q,+)$ and satisfying identity (2).

From (2) for $x=y=z=0$ we obtain $\alpha a+b+\beta a=b$, This together with (2) implies $\varphi=\varepsilon$ (the identity map). Thus

$$\alpha x+\alpha a+\alpha\psi z+b+\beta y+\beta a+\beta\psi z=\alpha x+b+\beta y.$$

This for $z=0$ gives

$$\alpha a+b+\beta y+\beta a=b+\beta y=\alpha a+b+\beta a+\beta y.$$

So $\beta y+\beta a=\beta a+\beta y$, i.e. a is in the center $Z(Q,+)$ of the group $(Q,+)$. Thus using (2) and the above facts we obtain $\alpha\psi z+b+\beta y+\beta\psi z=b+\beta y$. Hence $\alpha v+u+\beta v=u$ for all $u,v\in Q$. Thus $\beta=-\alpha$ and consequently $\alpha v+u=u+\alpha v$ for all $u,v\in Q$, which means that $(Q,+)$ is a commutative group.

In this way we have proved the "only if" part of the following Theorem. The second part is trivial.

Theorem 2.1. *A biquasigroup* $(Q,\circ,*)$ *linear over a group* $(Q,+)$ *is a Ward double quasigroup* (*that is, it satisfies* (2)) *if and only if* $(Q,+)$ *is a commutative group,* $x\circ y=x+\psi y+a$ *and* $x*y=\alpha x-\alpha y+b$.

Obviously such a biquasigroup is medial. The quasigroup $(Q,\circ)$ has a right neutral element and the quasigroup $(Q,*)$ is unipotent. Moreover, a biquasigroup $(Q,\circ,*)$ satisfying (2) is paramedial if and only if $\psi^2=\varepsilon$.

3. Now consider biquasigroups satisfying the identity (3).

Since this identity contains only one operation, it is enough to examine the quasigroup $(Q,\circ)$. Quasigroups satisfying (3) were characterized at the beginning of this paper. If a quasigroup $(Q,\circ)$ linear over a group $(Q,+)$ satisfies (3), then $\varphi=\varepsilon$ and $a+\psi a=0$. So (3) for $y=0$, can be reduced to $\psi z+\psi^2 z=0$. This means that $\psi z=-z$ and $(Q,+)$ is a commutative group. Consequently $x\circ y=x-y+a$.

Theorem 2.2. *A quasigroup $(Q, \circ)$ linear over a group $(Q, +)$ satisfies* (3) *if and only if $(Q, +)$ is a commutative group and $x \circ y = x - y + a$ for some fixed $a \in Q$.*

This quasigroup is medial, paramedial, unipotent and has a right neutral element.

Note that $(Q, \circ)$ is a Ward quasigroup if and only if there is a group $(Q, +)$ and an element $a \in Q$ such that $x \circ y = x - y + a$. The group $(Q, +)$ need not be commutative.

4. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (4), i.e.

$$(x \circ z) \circ (y \circ z) = x * y.$$

Theorem 2.3. *If a biquasigroup $(Q, \circ, *)$ satisfies the identity* (4)*, then both quasigroups $(Q, \circ)$ and $(Q, *)$ are unipotent with $q \in Q$ such that $x \circ x = q = x * x$ and $x * y = (x \circ y) \circ q = q \circ (y \circ x)$ for all $x, y \in Q$.*

Proof. If $(Q, \circ)$ is idempotent, then $x = (x \circ x) \circ (x \circ x) = x * x$. So $(Q, *)$ is idempotent too. If $(Q, *)$ is idempotent, then $x = x * x = (x \circ z) \circ (x \circ z)$ for all $x, z \in Q$. In particular, for $z = x' \in Q$ such that $x = x \circ x'$ we obtain $x = x \circ x$. This shows that both these quasigroups are idempotent or none of them are idempotent.

If both are idempotent, then $x \circ z = (x \circ z) \circ (x \circ z) = x * x = x = x \circ x$ for all $x, z \in Q$, which implies $x = z$. Hence Q has only one element. So it is unipotent.

Now suppose both quasigroups $(Q, \circ)$ and $(Q, *)$ are not idempotent. Then there exists $b \in Q$ such that $b * b = q \neq b$ and for any $x \in Q$ there exist $x', x'' \in Q$ such that $b \circ x' = x$ and $x \circ x'' = x$. Then $x \circ x = (b \circ x') \circ (b \circ x') = b * b = q$ and $x * x = (x \circ x'') \circ (x \circ x'') = x \circ x = q$. Hence, $(Q, \circ)$ and $(Q, *)$ are unipotent, with $q = x \circ x = x * x$ for all $x \in Q$. Also, $x * y = (x \circ x) \circ (y \circ x) = q \circ (y \circ x)$ and $x * y = (x \circ y) \circ (y \circ y) = (x \circ y) \circ q$. □

Corollary 2.4. *If a biquasigroup $(Q, \circ, *)$ satisfies* (4) *and $(Q, \circ)$ has a right neutral element, then $(Q, \circ) = (Q, *)$ is a Ward quasigroup. If $(Q, \circ)$ has a left neutral element, then $x * y = y \circ x$. If $(Q, \circ)$ has a neutral element, then $(Q, \circ) = (Q, *)$ is a commutative Ward quasigroup.*

Any medial unipotent quasigroup $(Q, \circ)$ can be 'extended' to a medial unipotent biquasigroup $(Q, \circ, *)$ satisfying the identity (4), as follows.

Proposition 2.5. *If $(Q, \circ)$ is a medial unipotent quasigroup, then $(Q, \circ, *)$, where $x \circ x = q$ and $x * y = (x \circ y) \circ q$ for all $x, y \in Q$, is a biquasigroup satisfying* (4)*. Moreover, if q is a left neutral element of $(Q, \circ)$, then $x * y = y \circ x$.*

Proof. Indeed, $(Q, *)$ is a quasigroup and $x * y = (x \circ y) \circ q = (x \circ y) \circ (z \circ z) = (x \circ z) \circ (y \circ z)$. Also, if q is a left neutral element of $(Q, \circ)$, then $x * y = (x \circ y) \circ q = (x \circ y) \circ (x \circ x) = (x \circ x) \circ (y \circ x) = y \circ x$. □

Let $(Q,\circ,*)$ be a biquasigroup linear over a group $(Q,+)$. If it satisfies (4), then $\varphi a+a+\psi a=b$ and $\alpha=\varphi^2$. So (4) for $x=y=0$ and $\psi z=a$ gives $2\varphi a+a+2\psi a=b=\varphi a+a+\psi a$ which implies $a=b$. Consequently $a\circ a=a$. Thus, by Theorem 2.3, $a=q$ and $x*y=a\circ(y\circ x)$. Hence

$$x*0=a\circ(0\circ x)=\alpha x+a=\varphi a+a+\psi a+\psi^2x=a+\psi^2x$$

and

$$a=x\circ x=\alpha x+a+\beta x=a+\psi^2x+\beta x.$$

This gives $\psi^2x+\beta x=0$, i.e. $\beta=-\psi^2$. Hence $x*y=\varphi^2x+a-\psi^2y$. Since $x\circ x=a=z*z$ we also have $\varphi x+a=a-\psi x$ and $\varphi^2z+a=a+\psi^2z$. This for $x=\varphi z$ gives $\varphi^2z+a=a-\psi\varphi z$. Hence $a+\psi^2z=a-\psi\varphi z$. Consequently, $\psi=-\varphi$ and $x\circ y=\varphi x+a-\varphi y$. So $a=x\circ x=\varphi x+a-\varphi x$. Thus $a\in Z(Q,+)$. Also $\varphi^2=\psi^2$.

Therefore, $x\circ y=\varphi x+a-\varphi y$ and $x*y=\varphi^2x+a-\varphi^2y$. Inserting these operations to (4) we obtain $-\varphi^2z-\varphi^2y+\varphi^2z=-\varphi^2y$ for all $y,z\in Q$. Hence $(Q,+)$ is a commutative group. Consequently $(Q,\circ,*)$ is medial and unipotent. This proves the "only if" part of the Theorem 2.6 below. The proof of the "if" part follows from a direct calculation and is omitted

Theorem 2.6. *A biquasigroup $(Q,\circ,*)$ linear over a group $(Q,+)$ satisfies the identity (4) if and only if $(Q,+)$ is a commutative group, $x\circ y=\varphi x+a-\varphi y$ and $x*y=\varphi^2x+a-\varphi^2y$.*

It is clear that such a biquasigroup is medial and paramedial. If $a=0$ then it is unipotent.

5. Will now consider a biquasigroup $(Q,\circ,*)$ satisfying the identity (5), i.e.

$$(x\circ z)\circ(y*z)=x\circ y.$$

Theorem 2.7. *If a biquasigroup $(Q,\circ,*)$ satisfies the identity (5), then both quasigroups $(Q,\circ)$ and $(Q,*)$ have only one idempotent. This idempotent is a right neutral element of these quasigroups. Moreover, $(Q,*)$ is unipotent.*

Proof. For each $x\in Q$ there is uniquely determined $\overline{x}\in Q$ such that $x\circ\overline{x}=x$. Then for $x,y\in Q$, by (5), we have

$$x\circ y=(x\circ\overline{x})\circ(y*\overline{x})=x\circ(y*\overline{x}).$$

So, $y=y*\overline{x}$ for each $y\in Q$. Also $y\circ y=(y\circ\overline{x})\circ(y*\overline{x})=(y\circ\overline{x})\circ y$, hence $y=y\circ\overline{x}$ for all $y\in Q$. Thus $e=\overline{x}$ is a right neutral element of $(Q,\circ)$ and $(Q,*)$. There are no other idempotents in $(Q,\circ)$ and $(Q,*)$. Indeed, if $a*a=a$, then for each $x\in Q$

$$x\circ a=(x\circ a)\circ(a*a)=(x\circ a)\circ a,$$

so $x \circ a = x = x \circ e$. Hence $a = e$. Similarly for $a \circ a = a$ we have

$$a \circ x = (a \circ a) \circ (x * a) = a \circ (x * a),$$

which implies $x * a = x = x * e$, so also in this case $a = e$.

For each $x \in Q$ there exists $x' \in Q$ such that $x * x = x \circ x'$. Thus, by (5),

$$(x \circ x) \circ e = x \circ x = (x \circ x) \circ (x * x) = (x \circ x) \circ (x \circ x'),$$

which implies $e = x \circ x' = x * x$. So, $(Q, *)$ is unipotent. □

The following example shows that $(Q, \circ)$ may not be unipotent.

Example 2.8. Let $(Q, \circ)$ be a group. Then $(Q, \circ, *)$, where $x * y = y^{-1} \circ x$ is an example of a biquasigroup satisfying (5) in which only one of quasigroups $(Q, \circ)$ and $(Q, *)$ has a left neutral element. Moreover, $(Q, *)$ is unipotent but $(Q, \circ)$ is unipotent only in the case when it is a Boolean group.

Corollary 2.9. *If in a biquasigroup $(Q, \circ, *)$ satisfying (5) one of quasigroups $(Q, \circ)$ or $(Q, *)$ is idempotent, then Q has only one element.*

Proposition 2.10. *Let $(Q, \circ, *)$ be a biquasigroup satisfying (5). If $(Q, \circ)$ is a Ward quasigroup, then $(Q, \circ) = (Q, *)$.*

Proof. Since $(Q, \circ)$ is a Ward quasigroup, there exists a group $(Q, \cdot)$ such that $x \circ y = x \cdot y^{-1}$ and $e \circ (x \circ y) = y \circ x$, where e is the neutral element of the group $(Q, \cdot)$ (cf. [1]). Then $x \circ y = (x \circ x) \circ (y * x) = e \circ (y * x)$ and so $x \circ y = e \circ (y \circ x) = e \circ (e \circ (x * y)) = x * y$. Hence $(Q, \circ) = (Q, *)$. □

Proposition 2.11. *Let $(Q, \circ, *)$ be a biquasigroup satisfying (5). If $(Q, \circ)$ is medial and unipotent, then $(Q, \circ) = (Q, *)$.*

Proof. For every $x, y \in Q$ there exists $z \in Q$ such that $x * y = x \circ z$. Since $(Q, \circ)$ is medial,

$$(x \circ x) \circ e = x \circ x = (x \circ y) \circ (x * y) = (x \circ y) \circ (x \circ z) = (x \circ x) \circ (y \circ z).$$

Thus, $y \circ z = e = y \circ y$, where e is the right neutral element of $(Q, \circ)$. Therefore $y = z$ and consequently, $x * y = x \circ y$. □

Proposition 2.12. *A biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies (5) if and only if a group $(Q, +)$ is a commutative group, $x \circ y = x + \psi y - \psi b$ and $x * y = x - y + b$.*

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies (5), then $\varphi a + a + \psi b = a$ and $\varphi = \varepsilon$. Thus $a + \psi b = 0$. This together with (5) for $y = 0$ gives $\psi z + \psi \beta z = 0$. So, $\beta z = -z$ for all $z \in Q$. Thus $(Q, +)$ is a commutative group. Consequently, $\alpha = \varepsilon$. Therefore, $x \circ y = x + \psi y - \psi b$, $x * y = x - y + b$.

The proof of the converse follows from a direct calculation and is omitted. □

A biquasigroup $(Q, \circ, *)$ linear over a group and satisfying (5) is medial and both its quasigroups $(Q, \circ)$ and $(Q, *)$ have the same right neutral element. If $\psi^2 = \varepsilon$ then this biquasigroup is also paramedial.

6. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (6), i.e.

$$(x \circ z) * (y \circ z) = x \circ y.$$

Theorem 2.13. *A biquasigroup $(Q, \circ, *)$ satisfies the identity (6) if and only if there is a group $(G, \cdot)$ and a bijection α on Q such that $x \circ y = (\alpha x)^{-1} \cdot (\alpha y)$ and $x * y = x \cdot y^{-1}$.*

Proof. $\Rightarrow$: Let $x, y, z \in Q$. Then for fixed $q \in Q$ there are $x', y', z' \in Q$ such that and $x = x' \circ q$, $y = y' \circ q$ and $z = z' \circ q$. Then, $x * z = (x' \circ q) * (z' \circ q) = x' \circ z'$ and $y * z = (y' \circ q) * (z' \circ q) = y' \circ z'$. So,

$$(x * z) * (y * z) = (x' \circ z') * (y' \circ z') = x' \circ y' = (x' \circ q) * (y' \circ q) = x * y.$$

Therefore, $(Q, *)$ is a Ward quasigroup and there exists a group $(Q, \cdot)$ such that $x * y = x \cdot y^{-1}$ and $x \cdot y = x * (e * y)$, where $e = w * w$ for any $w \in Q$ and $x^{-1} = e * x$.

Let $x \in Q$. Then, $e = (x \circ z) * (x \circ z) = x \circ x$. So, $e * (x \circ y) = (y \circ y) * (x \circ y) = y \circ x$, for any $y \in Q$. Let $\alpha x = e \circ x$. Then, $(\alpha x)^{-1} = e * (e \circ x) = x \circ e$. Thus, $(\alpha x)^{-1} \cdot (\alpha y) = (x \circ e) \cdot (e \circ y) = (x \circ e) * (e * (e \circ y)) = (x \circ e) * (y \circ e) = x \circ y$.

$\Leftarrow$: Let $x, y, z \in Q$. Then, $(x \circ z) * (y \circ z) = [(\alpha x)^{-1} \cdot (\alpha z)] * [(\alpha y)^{-1} \cdot (\alpha z)] = (\alpha x)^{-1} \cdot (\alpha z) \cdot (\alpha z)^{-1} \cdot (\alpha y) = (\alpha x)^{-1} \cdot (\alpha y) = x \circ y$. □

Corollary 2.14. *If a biquasigroup $(Q, \circ, *)$ satisfies the identity (6), then it is unipotent.*

Corollary 2.15. *If in a biquasigroup $(Q, \circ, *)$ satisfying the identity (6) one of quasigroups $(Q, \circ)$ and $(Q, *)$ is commutative, then also the second is commutative. In this case both quasigroups are induced by the same Boolean group.*

Proposition 2.16. *A biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies the identity (6) if and only if a group $(Q, +)$ is commutative, $x \circ y = \varphi x - \varphi y + a$ and $x * y = x - y + a$.*

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies the identity (6) then $\alpha a + b + \beta a = a$ and $\alpha = \varepsilon$. So, $b + \beta a = 0$. Thus (6) for $x = y = 0$ gives $\psi z + \beta \psi z = 0$ which means that $\beta v = -v$ for each $v \in Q$. Hence $(Q, +)$ is commutative and $a = b$. Therefore $x * y = x - y + a$. Substituting this operation to (6) we obtain $x \circ y = \varphi x - \varphi y + a$.

The converse statement is obvious. □

Corollary 2.17. *A linear biquasigroup satisfying the identity (6) is unipotent.*

7. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (7), i.e.

$$(x \circ z) \circ (y * z) = x * y.$$

A simple example of a biquasigroup $(Q, \circ, *)$ satisfying the identity (7) is a commutative group $(Q, +)$ with the operations $x \circ y = y - x$ and $x * y = x + y$. This biquasigroup is medial, both quasigroups $(Q, \circ)$ and $(Q, *)$ have left neutral element but only the first is unipotent.

Suppose now that in a biquasigroup $(Q, \circ, *)$ satisfying the identity (7) the first quasigroup is medial and the second is idempotent. Then, by Toyoda Theorem (cf. [9]), there exists a commutative group $(Q, +)$ and its commuting automorphisms φ, ψ such that $x \circ y = \varphi x + \psi y + a$ for some fixed $a \in Q$. Then $x * y = (x \circ y) \circ (y * y) = (x \circ y) \circ y = \varphi^2 x + \varphi\psi y + \psi y + \varphi a + a$. This, by (7), implies $\varphi^2 - \varphi = \varepsilon$, $\varphi + \varphi\psi + \psi = 0$ and $\varphi a = -a$. Thus $a = 0$ and $x * y = \varphi^2 x + \varphi\psi y + \psi y$. Since $(Q, *)$ is idempotent, $\varphi^2 + \varphi\psi + \psi = \varepsilon$. Hence $x * y = \varphi^2 x + y - \varphi^2 y = \varphi^2 x - \varphi y$. Consequently $(Q, *)$ is medial. Therefore $(Q, \circ, *)$ is medial too.

In this way we have proved

Proposition 2.18. *If in a biquasigroup* $(Q, \circ, *)$ *satisfying the identity* (7) *the first quasigroup is medial and the second is idempotent, then the second is medial too and there exists a commutative group* $(Q, +)$ *and its commuting automorphisms* φ, ψ *such that* $\varphi + \varphi\psi + \psi = 0$, $\varphi^2 = \varphi + \varepsilon$, $x \circ y = \varphi x + \psi y$ *and* $x * y = \varphi^2 x - \varphi y$.

Conversely we have:

Proposition 2.19. *Let* $(Q, +)$ *be a commutative group and* φ, ψ *be its commuting automorphisms such that* $\varphi + \varphi\psi + \psi = 0$ *and* $\varphi^2 = \varphi + \varepsilon$. *Then* $(Q, \circ, *)$, *where* $x \circ y = \varphi x + \psi y$ *and* $x * y = \varphi^2 x - \varphi y$, *is a medial biquasigroup satisfying the identity* (7).

Proof. This is a straightforward calculation. □

As a consequence of the above results we obtain

Corollary 2.20. *An idempotent medial biquasigroup* $(Q, \circ, *)$ *satisfies the identity* (7) *if and only if there exist a commutative group* $(Q, +)$ *and its automorphism* φ *such that* $x \circ y = \varphi x + y - \varphi y$, $x * y = \varphi^2 x - \varphi y$ *and* $\varphi^2 = \varphi + \varepsilon$.

In the case of quasigroups induced by the group $\mathbb{Z}_n$ we have stronger result. For simplicity the value of the integer $t \geqslant 0$ modulo n will be denoted by $[t]_n$.

Corollary 2.21. *An idempotent medial biquasigroup induced by the group* $\mathbb{Z}_n$ *satisfies the identity* (7) *if and only if has the form* $(\mathbb{Z}_n, \circ, *)$, *where* $x \circ y = [ax + (1 - a)y]_n$, $x * y = [a^2 x + (1 - a^2)y]_n$ *and* $[a^2 - a]_n = 1$.

Corollary 2.22. *For every* $a \geqslant 3$ *there is an idempotent medial biquasigroup of order* $n = a^2 - a - 1$ *satisfying* (7). *It has the form* $(\mathbb{Z}_n, \circ, *)$, *where* $x \circ y = [ax + (1 - a)y]_n$ *and* $x * y = [(a + 1)x - ay]_n$, *or* $x \circ y = [(1 - a)x + ay]_n$ *and* $x * y = [(2 - a)x + (a - 1)y]_n$.

Proposition 2.23. *A medial biquasigroup* $(Q, \circ, *)$ *satisfies the identity* (7) *if and only if there exist a commutative group* $(Q, +)$ *and its commuting automorphisms* φ, ψ *such that* $x \circ y = \varphi x + \psi y + c$, $x * y = \varphi^2 x - \varphi y + d$, $\varphi\psi + \varepsilon = 0$ *and* $\varphi c + \psi d + c = d$ *for some fixed* $c, d \in Q$.

Proposition 2.24. *A medial biquasigroup* $(\mathbb{Z}_n, \circ, *)$ *satisfies the identity* (7) *if and only if there exists* $a, b, c, d \in \mathbb{Z}_n$ *such that* $[ab+1]_n = 0$, $[ac+bd+c]_n = d$, $x \circ y = [ax+by+c]_n$ *and* $x * y = [a^2x - ay + d])_n$.

For linear biquasigroup we have the following result.

Theorem 2.25. *A biquasigroup* $(Q, \circ, *)$ *linear over a group* $(Q, +)$ *satisfies the identity* (7) *if and only if* $(Q, +)$ *is a commutative group,* $x \circ y = \varphi x + \psi y + a$, $x * y = \varphi^2 x + \psi\varphi^2 y + b$, $\varphi\psi + \psi^2\varphi^2 = 0$ *and* $\varphi a + a + \psi b = b$.

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies (7), then $\varphi a + a + \psi b = b$ and $\varphi^2 = \alpha$. Thus (7) can be reduced to

$$\varphi a + \varphi\psi z + a + \psi\alpha y + \psi b + \psi\beta z = b + \beta y,$$

which for $z = 0$ gives $\varphi a + a + \psi\alpha y + \psi b = b + \beta y = (\varphi a + a + \psi b) + \beta y$. Hence $\psi\alpha y + \psi b = \psi b + \beta y$. So $\psi\alpha y = \psi b + \beta y - \psi b$. Therefore the previous identity implies $\varphi\psi z + a + \psi b + \beta y + \psi\beta z = a + \psi b + \beta y$. Since every element $v \in Q$ can be presented in the form $v = a + \psi b + \beta y$, the last identity means that $\varphi\psi z + v + \psi\beta z = v$ for all $v, z \in Q$. This implies $\varphi\psi = -\psi\beta$. Hence $\varphi\psi z + v = v - \psi\beta z = v + \varphi\psi z$. So, $(Q, +)$ is commutative. Applying these facts to (7) we can see that $\beta = \psi\alpha$. Hence $x \circ y = \varphi x + \psi y + a$ and $x * y = \varphi^2 x + \psi\varphi^2 y + b$.

The proof of the converse follows from a direct calculation and is omitted. □

8. Will now consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (8), i.e.

$$(x \circ z) * (y * z) = x \circ y.$$

Proposition 2.26. *If a biquasigroup* $(Q, \circ, *)$ *satisfies* (8)*, then* $(Q, *)$ *has no more than one idempotent. If such idempotent exists then it is a right neutral element of a quasigroup* $(Q, *)$*. Moreover, if* $x \circ x = u$ *for some* $u \in Q$ *and all* $x \in Q$*, then* $x \circ y = x * (y * x)$, $x * x = w$ *and* $w \circ u = w$ *for all* $x, y \in Q$.

Proof. Let $e * e = e$. Since $(Q, \circ)$ is a quasigroup, each $z \in Q$ can be expressed in the form $z = x \circ e$. Thus $z = x \circ e = (x \circ e) * (e * e) = z * e$, so e is a right neutral element of $(Q, *)$. If $\bar{e}$ is the second idempotent of $(Q, *)$, then $\bar{e} * e = \bar{e} = \bar{e} * \bar{e}$. Therefore $\bar{e} = e$, so $(Q, *)$ has no more than one idempotent. If $x \circ x = u$ for all $x \in Q$, then, by (8), $u * (x * x) = (x \circ x) * (x * x) = x \circ x = u$. Analogously $u * (y * y) = u$. Thus, $x * x = y * y = w$ for some $w \in Q$, i.e. $(Q, *)$ is unipotent and w is its right neutral element. Then, $x \circ y = (x \circ x) * (y * x) = u * (y * x)$ and $w \circ u = (w \circ w) * (u * w) = u * u = w$. □

Let $(Q, \circ, *)$ be linear over a group $(Q, +)$. If it satisfies (8), then $\alpha a+b+\beta b = a$, $\alpha = \varepsilon$ and $\beta b = -b$. Thus (8) can be reduced to

$$\psi z + b + \beta y + \beta b + \beta^2 z = \psi y. \tag{10}$$

This for $y = 0$ gives $\psi z + \beta^2 z = 0$. So, $\psi = -\beta^2$ and $\psi b = -b = \beta b$.

Now putting $y = b$ in (10) we obtain $\psi z + b + \beta b + \beta b + \beta^2 z = \psi b$, i.e. $-\beta^2 z + \beta b + \beta^2 z = \psi b = \beta b$. So, $\beta b + \beta^2 z = \beta^2 z + \beta b$. This means that b is in the center of $(Q, +)$. Thus putting $z = 0$ in (10) and using the above facts, we obtain $\beta = \psi = -\beta^2$. Hence $\beta = -\varepsilon$. So $(Q, +)$ is commutative, $x \circ y = \varphi x - y + a$ and $x * y = x - y + b$.

In this way, we have proved the "only if" part of Theorem 2.27 below. The proof of the converse part of Theorem 2.27 follows from a direct calculation and is omitted.

Theorem 2.27. *A biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies* (8) *if and only if $(Q, +)$ is commutative, $x \circ y = \varphi x - y + a$ and $x * y = x - y + b$.*

Corollary 2.28. *A linear biquasigroup satisfying* (8) *is medial.*

Corollary 2.29. *A medial biquasigroup induced by the group $\mathbb{Z}_n$ satisfies* (8) *if and only if $x \circ y = [ax - y + c]_n$ and $x * y = [x - y + d]_n$ for some $a, c, d \in \mathbb{Z}_n$ such that $(a, n) = 1$.*

9. Finally, let us consider a biquasigroup $(Q, \circ, *)$ satisfying the identity (9), i.e.

$$(x \circ z) * (y * z) = x * y.$$

Theorem 2.30. *In a biquasigroup $(Q, \circ, *)$ satisfying the identity* (9) *the quasigroups $(Q, \circ)$ and $(Q, *)$ have no more than one idempotent. If such idempotent exists then it is a common right neutral element of these quasigroups.*

Proof. Assume $(Q, \circ)$ has an idempotent a. Then $a*a = (a\circ a)*(a*a) = a*(a*a)$ and so $a*a = a$. Analogously, for $a*a = a$ we have $a*a = (a\circ a)*(a*a) = (a\circ a)*a$, which implies $a \circ a = a$. So, $(Q, \circ)$ and $(Q, *)$ have the same idempotent. Then for each $x \in Q$ $x * a = (x \circ a) * (a * a) = (x \circ a) * a$, which implies $x = x \circ a$. Thus a is a right neutral element of $(Q, \circ)$. On the other hand, $x \circ a = x$ gives $x * x = (x \circ a) * (x * a) = x * (x * a)$, and consequently $x = x * a$. Thus, a is a right neutral element of $(Q, \circ)$ and $(Q, *)$. □

Corollary 2.31. *If in a biquasigroup $(Q, \circ, *)$ satisfying* (9) *the quasigroup $(Q, *)$ is unipotent, then $(Q, \circ) = (Q, *)$ and $(Q, \circ)$ is a Ward quasigroup.*

Proof. Let $x * x = a$ for all $x \in Q$ and some $a \in Q$. Then $a * a = x * x = (x \circ x) * (x * x) = (x \circ x) * a$. Therefore, $x \circ x = a$, i.e. $(Q, \circ)$ is unipotent. Consequently, $a*(x*y) = (y\circ y)*(x*y) = y*x$, which implies $x\circ y = (x\circ y)*a = (x\circ y)*(y*y) = x*y$. Hence $(Q, \circ) = (Q, *)$ and (9) coincides with (1). This means that $(Q, \circ)$ is a Ward quasigroup. □

Theorem 2.32. *A biquasigroup* $(Q, \circ, *)$ *linear over a group* $(Q, +)$ *satisfying the identity* (9) *is medial and can be presented in the form* $x \circ y = x - \beta^2 y - \beta b$ *and* $x * y = x + \beta y + b$, *where* $(Q, +)$ *is a commutative group,* $\beta \in \mathrm{Aut}(Q, +)$ *and* $b \in Q$. *This biquasigroup has a right neutral element* $e = -\varphi^{-1} b$.

Conversely, if $(Q, +)$ *is a commutative group,* $\beta \in \mathrm{Aut}(Q, +)$, $b \in Q$, $x \circ y = x - \beta^2 y - \beta b$ *and* $x * y = x + \beta y + b$, *then the biquasigroup* $(Q, \circ, *)$ *satisfies* (9).

Proof. If a biquasigroup $(Q, \circ, *)$ linear over a group $(Q, +)$ satisfies the identity (9), then $\alpha a + b + \beta b = b$ and $\varphi = \varepsilon$. So, (9) can be reduced to

$$\alpha a + \alpha\psi z + b + \beta\alpha y + \beta b + \beta^2 z = b + \beta y, \tag{11}$$

which for $z = 0$ gives $\alpha a + b + \beta\alpha y + \beta b = b + \beta y = \alpha a + b + \beta b + \beta y$. Since $\alpha a = b = b - \beta b$, the last implies $\beta\alpha y = \beta b + \beta y - \beta b$. This together with (11) (for $y = v$) gives

$$\alpha a + \alpha\psi z + b + \beta b + \beta v + \beta^2 z = b + \beta v. \tag{12}$$

Now adding βv on the right side to (11) and putting $y = 0$ we get

$$\alpha a + \alpha\psi z + b + \beta b + \beta^2 z + \beta v = b + \beta v.$$

Comparing this identity with (12) we obtain $\beta v + \beta^2 z = \beta^2 z + \beta v$ for all $v, z \in Q$. This shows that $(Q, +)$ is a commutative group. Consequently, $\beta\alpha y = \beta y$, so $\alpha = \varepsilon$. This by $\alpha a + b + \beta b = b$ gives $a = -\beta b$. Again putting $y = 0$ in (11) and using the above facts we obtain $\psi = -\beta^2$. Therefore, $x \circ y = x - \beta^2 y - \beta b$ and $x * y = x + \beta y + b$.

The proof of the converse part of the Theorem follows from a direct calculation and is omitted. □

Proposition 2.33. *A medial biquasigroup* $(\mathbb{Z}_n, \circ, *)$ *satisfies the identity* (9) *if and only if* $x \circ y = [x - a^2 y - ab]_n$, $x * y = [x + ay + b]_n$, *where* $a, b \in \mathbb{Z}_n$ *are fixed and* $(a, n) = 1$.

Example 2.34. Let $n = a^2 + 1 > 4$. Then $(\mathbb{Z}_n, \circ, *)$, where $x \circ y = [x + y]_n$ and $x * y = [x + ay]_n$ is an example of a biquasigroup satisfying (9).

10. Many authors study linear quasigroups of the second type, namely quasigroups $(Q, \cdot)$ where, in the definition of the operation, the constant element is not placed in the middle of the formula but at its end, i.e. $x \cdot y = \varphi x + \psi y + a$.

Biquasigroups of this type satisfying the identities (2) – (9) coincide with the quasigroups of the previous type. Namely, if a biquasigroup $\widehat{Q} = (Q, \circ, *)$ with the operations $x \circ y = \varphi x + \psi y + a$ and $x * y = \alpha x + \beta y + b$, where $\alpha, \beta, \varphi, \psi$ are automorphisms of a group $(Q, +)$, satisfies (2) then $\alpha a + \beta a = 0$ and $\varphi = \varepsilon$. Thus $\alpha\psi z + \alpha a + \beta y + \beta\psi z + \beta a = \beta y$. This for $y = 0$ and $\psi z = v$ gives $\alpha v = -\beta a - \beta v - \alpha a$. Since α and β are automorphisms of $(Q, +)$ the last expression for $v = u + w$ implies $\beta(u + w) = \beta w + \beta u$. Thus $\beta u + \beta w = \beta u + \beta w$ for all $u, w \in Q$. Hence $(Q, +)$ is a commutative group. Such biquasigroups are described in subsection **2**.

If a biquasigroup $(Q, \circ, \circ)$ with $x\circ y = \varphi x+\psi y+a$ satisfies (3), then $\varphi a+\psi a = 0$, $\varphi = \varepsilon$ and $\psi z + a + \psi y + \psi^2 z + \psi a = \psi y$, which for $y = a$ gives $\psi = -\varepsilon$. This shows that $(Q, +)$ is a commutative group and $x \circ y = x - y + a$. Also in the case when $(Q, \circ)$ with $x \circ y = \varphi x + a + \psi y$ satisfies (1), the group $(Q, +)$ must be commutative and $x \circ y = x - y + a$. This means that these two cases coincide.

If a biquasigroup $\widehat{Q}$ satisfies (4), then $\varphi a + \psi a + a = b$ and $\alpha = \varphi^2$. Because by Theorem 2.3 we have $q = \varphi 0 + \psi 0 + a = \alpha 0 + \beta 0 + b$, must be $q = a = b$. Consequently, $a = \varphi x + \psi x + a$. This implies $\varphi = -\psi$, which together with $(x \circ 0) \circ (x \circ 0) = a$ implies $\varphi^2 x + \varphi a - \varphi^2 x - \varphi a = 0$. Hence $\varphi x + a = a + \varphi x$ for all $x \in Q$. So, a is in the center of $(Q, +)$. Therefore this case is reduced to the case described in subsection **4**.

If a biquasigroup $\widehat{Q}$ satisfies (5), then by Theorem 2.7 the quasigroup $(Q, *)$ has a right neutral element e. Thus $x = x * e = \alpha x + \beta e + b$ for all $x \in Q$. In particular $0 = 0 * e = \beta e + b$. Consequently, $x = x * e = \alpha x$ and $x * y = x + \beta y + b$. Applying this formula to (5) we can see that $\varphi = \varepsilon$ and $\varphi b = -a$. Therefore the identity (5) can be written in the form

$$\psi z + a + \psi y + \psi\beta z = \psi y + a.$$

This for $z = 0$ implies $a + \psi y = \psi y + a$. Hence a is in the center of $(Q, +)$. Also b is in the center of $(Q, +)$ because $\varphi b = -a$. Thus this case reduces to the case from subsection **5**.

By Corollary 2.14 any quasigroup satisfying (6) is unipotent. Thus if $\widehat{Q}$ satisfies (6), then $\alpha 0 + \beta 0 + b = b$ implies $b = x * x = \alpha x + \beta x + b$, i.e. $\beta x = -\alpha x$ for all $x \in Q$. From (6) it follows $\alpha = \varepsilon$. Thus $\beta x = -x$. Since β is an automorphism of $(Q, +)$, $(Q, +)$ is commutative. Hence this case reduces to subsection **6**.

If a biquasigroup $\widehat{Q}$ satisfies (7), then $\varphi a + \psi b + a = b$ which together with (7) fort $x = y = 0$ implies

$$\varphi\psi z + \varphi a + \psi\beta z + \psi b + a = b = \varphi a + \psi b + a.$$

Thus $\varphi\psi z = \varphi a - \psi\beta z - \varphi a$. Since $\varphi\psi$ and $\psi\beta$ are automorphisms of $(Q, +)$ the last for $z = u + v$ gives

$$\varphi\psi(u + v) = \varphi a - \psi\beta(u + v) - \varphi a.$$

On the other side,

$$\varphi\psi u + \varphi\psi v = \varphi a - \psi\beta u - \varphi a + \varphi a - \psi\beta v - \varphi a = \varphi a - \psi\beta(v + u) - \varphi a.$$

Comparing these two expression we obtain $\psi\beta(u + v) = \psi\beta(v + u)$. Hence $(Q, +)$ is a commutative group and this case reduces to **7**.

If a biquasigroup $\widehat{Q}$ satisfies (8), then $\alpha a + \beta b + b = a$ and $\alpha = \varepsilon$. This together with (8) for $x = y = 0$ implies $\psi z + a + \beta^2 z = a$. Hence $\beta^2 z = -a - \psi z + a$.

From this for $z = u + v$, in a similar way as in the previous case, we obtain $\psi(u + v) = \psi(v + u)$. Therefore $(Q, +)$ is a commutative group and this case reduces to **8**.

The case when $\widehat{Q}$ satisfies (9) reduces to **9**. Indeed, in this case $\alpha a + \beta b = 0$, which together (9) for $x = y = 0$ shows that $\beta^2 z = -\alpha a - \alpha\psi z - \beta b$. From this we compute $\alpha\psi(u + v) = \alpha\psi(v + u)$. Hence $(Q, +)$ is commutative.

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