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On the component graphs of finitely generated free semimodules

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Abstract. A semiring S is said to have invariant basis number property if any two bases of a finitely generated free semimodule over S have the same cardinality. Here we characterize reduced zero and reduced non-zero component graphs of every finitely generated free semimodule $\mathcal{V}$ over such semirings. It is shown that if $|S| \geqslant \aleph_0$, these two graphs of a semimodule $\mathcal{V}$ over S are isomorphic.

1. Introduction

In the recent years, there has been a flow of various ideas in the study of algebraic structures using graphs defined on themselves. Various algebraic structures like semigroups [5], groups [2], rings [1] and vector spaces [6, 8] have been characterized in this way. In [6], Das introduced non-zero component graph on a finite dimensional vector space.

Recently, semimodules over a semiring have created attention to the researchers for their different interesting uncommon features. Many of the results of vector spaces do not match with the results of semimodules. For example, in a vector space every basis is a free basis and converesely, which does not hold in a semimodule in general [10].

Here we consider both zero and non-zero component graphs of a finitely generated free semimodule. Also we introduce reduced non-zero component graph and reduced zero component graph on a finitely generated free semimodule and prove that they are isomorphic. This isomorphism ensures that studying either of them is sufficient to know about both. Here we study reduced non-zero component graph.

2. Definitions and preliminary results

Let $G = (M, E)$ be a graph. All graphs considered here are simple. A subset I of M is said to be independent if no two elements of I are pairwise adjacent. The maximum number of elements of an independent set is called the independence number of G. A subset D of M is called dominating if each element of

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$M \setminus D$ is adjacent to at least one element of D. If no proper subset of D is a dominating set for G, then D is called a minimal dominating set for G. The least cardinality of a dominating set is called the domination number of G. Two graphs (M, E) and $G' = (M', E')$ are said to be isomorphic if there exists a bijective mapping $\phi : M \to M'$ such that $a \sim b$ in M if and only if $\phi(a) \sim \phi(b)$ in M'. A path of length k in a graph G is an alternating sequence of vertices and edges $a_0, e_0, a_1, e_1, a_2, \dots, a_{k-1}, e_{k-1}, a_k$, where a_i's are distinct (except possibly a_0, a_k) and e_i is the edge joining a_i and a_{i+1}. If there exists a path between any pair of distinct vertices, then it is called connected. The distance between two vertices $a, b \in M$, $d(a, b)$ is defined as the length of the shortest path between a and b. The diameter of a graph G is defined as $diam(G) = max_{a,b\in M} d(a, b)$, if it exists. Otherwise, $diam(G)$ is defined as ∞.

We refer to [3] for further notions on graph theory and [7] for basic notions and results on semirings and semimodules.

A semiring S is an algebraic system $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid, connected by the ring-like distributive laws. Also we assume that $(S, \cdot)$ is commutative, $0 \neq 1$ and the zero element 0 is absorbing, that is $s0 = 0s = 0$ for all $s \in S$. We say that $s \in S$ is invertible if $st = 1$ for some $t \in S$ and denote the set of all invertible elements of S by $U(S)$. If a semiring S is such that $U(S) = S \setminus \{0\}$, then S is said to be a semifield.

Definition 2.1. Let S be a semiring. A *left S-semimodule* is a commutative monoid $(\mathcal{V}, +)$ with additive identity θ for which we have a function $S \times \mathcal{V} \longrightarrow \mathcal{V}$, denoted by $(\lambda, \alpha) \longmapsto \lambda\alpha$ and called as *scalar multiplication*, which satisfies the following conditions for all $a, b \in S$ and $u, v \in \mathcal{V}$:

(i) $a(u + v) = au + av$;
(ii) $(a + b)v = av + bv$;
(iii) $(ab)v = a(bv)$;
(iv) $1v = v$;
(v) $a\theta = \theta = 0v$.

Right S-semimodules are defined analogously. In this paper a semimodule $\mathcal{V}$ over S means left S-semimodule. The elements of $\mathcal{V}$ are called vectors and the elements of S are called scalars.

Let S be a semiring. Then a semimodule $\mathcal{V}$ over S is also known as a semilinear space over S. If a semiring S is a ring, then any semilinear space $\mathcal{V}$ over S is an S-module. In particular, if S is a field then any semilinear space over S is a linear space (or, vector space) over S.

Let B be a non-empty subset of $\mathcal{V}$. Then we denote

$$span(B) = \{\textstyle\sum_{i=1}^{n} c_i x_i : n \in \mathbb{N}, c_i \in S, x_i \in B\}.$$

If $span(B) = \mathcal{V}$, then B is called a generating subset of $\mathcal{V}$. A semimodule $\mathcal{V}$ having a finite generating set B is called finitely generated. A nonempty subset D of vectors in $\mathcal{V}$ is called linearly dependent if there exists $x \in D$ such that

$x \in span(D - \{x\})$; otherwise it is called linearly independent; and free if each element of $\mathcal{V}$ is expressed as a linear combination of the elements of D in at most one way. It is easy to see that every free subset of $\mathcal{V}$ is linearly independent. A linearly independent generating subset of $\mathcal{V}$ is called a basis of $\mathcal{V}$ and a free generating subset of $\mathcal{V}$ is called a free basis of $\mathcal{V}$ [10]. If $\mathcal{V}$ has a free basis then it is called a free semimodule. It is easy to see that every finitely generated semimodule has a basis, and every free basis is a basis [10].

Definition 2.2. A semiring S is said to have *invariant basis number* [IBN] property if any two bases of a finitely generated free semimodule over S have the same cardinality.

A semiring S has the IBN property if and only if for every $s, t \in S$, $s + t = 1$ implies that either $s \in U(S)$ or $t \in U(S)$ [Theorem 4.3; [10]]. Hence every semifield has the IBN property. In particular, the semiring $\mathbb{R}^+ \cup \{0\}$ of all non-negative real numbers, the max-plus semiring $\mathbb{R}_{max}$ and many other tropical semirings are of this type. Apart from the semifields, the semiring $\mathbb{N} \cup \{0\}$ of all non-negative integers also has this property. Thus we see that many useful as well as algebraically important semirings have the IBN property.

Henceforth, unless stated otherwise, S is a semiring having invariant basis number property and $\mathcal{V}$ is a finitely generated free semimodule over S. Let $\mathcal{V}$ be a finitely generated free semimodule over S, then from Corollary 3.1 [10], it follows that every vector of $\mathcal{V}$ can be expressed uniquely in terms of each basis. The cardinality of a basis of $\mathcal{V}$ is denoted by $dim(\mathcal{V})$.

Isomorphism of semimodules is defined similarly to modules. It follows from Corollary 5.2 [9], that semimodules $\mathcal{V}$ and $\mathcal{W}$ are isomorphic if and only if $dim(\mathcal{V}) = dim(\mathcal{W})$.

If $X = \{x_1, x_2, \ldots, x_n\}$ is a basis of a semimodule $\mathcal{V}$, then every vector $v \in \mathcal{V}$ can be expressed uniquely as $v = c_1x_1 + \cdots + c_nx_n$; $c_i \in S$. We call c_i the i-th component of $\mathcal{V}$ and is denoted by v_i.

Definition 2.3. The non-zero component graph of $\mathcal{V}$ relative to the basis X, is defined as $\Gamma_X(\mathcal{V}) = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \mathcal{V} \setminus \{\theta\}$ and $(\alpha, \beta) \in \mathbb{E}$ if there exists i such that α_i, β_i are non-zero.

Note that the vectors of the form $v = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, whose all components are non-zero, adjacent to every other vertex of $\Gamma_X(\mathcal{V})$. These vertices do not have much role on the parameters of $\Gamma_X(\mathcal{V})$. So we propose to consider the graph $\Gamma^*_X(\mathcal{V})$ obtained from $\Gamma_X(\mathcal{V})$ after deletion of such vertices. We call $\Gamma^*_X(\mathcal{V})$ the reduced non-zero component graph of $\mathcal{V}$ with respect to the basis $X = \{x_1, x_2, \ldots, x_n\}$.

Theorem 2.4. *Let $\mathcal{V}$ be a semimodule over a semiring S. Let $\Gamma^*_X(\mathcal{V})$ and $\Gamma^*_Y(\mathcal{V})$ be the reduced non-zero component graphs of $\mathcal{V}$ with respect to the bases $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of $\mathcal{V}$. Then $\Gamma^*_X(\mathcal{V})$ and $\Gamma^*_Y(\mathcal{V})$ are graph isomorphic.*

Proof. Define a mapping $\sigma : \mathcal{V} \longrightarrow \mathcal{V}$ such that

$$\sigma(c_1x_1 + c_2x_2 + \cdots + c_nx_n) = c_1y_1 + c_2y_2 + \cdots + c_ny_n.$$

Then clearly σ is an isomorphism on $\mathcal{V}$ such that $\sigma(x_i) = y_i$ for all $i \in \{1, 2, \ldots, n\}$. We show that the restriction of σ on non-null vectors of $\mathcal{V}$ such that at least one component is zero induces a graph isomorphism $\sigma^* : \Gamma^*_X(\mathcal{V}) \longrightarrow \Gamma^*_Y(\mathcal{V})$. Clearly σ^* is a bijection. Let $\alpha = c_1x_1+c_2x_2+\cdots+c_nx_n$ and $\beta = d_1x_1+d_2x_2+\cdots+d_nx_n$ with $\alpha \sim \beta$ in $\Gamma^*_X(\mathcal{V})$. Then there exists i such that $c_i, d_i \neq 0$. Hence $\sigma^*(\alpha) \sim \sigma^*(\beta)$ in $\Gamma^*_Y(\mathcal{V})$. Similarly it can be shown that if α and β are not adjacent in $\Gamma^*_X(\mathcal{V})$, then $\sigma^*(\alpha)$ and $\sigma^*(\beta)$ are not adjacent in $\Gamma^*_Y(\mathcal{V})$. □

Now we define the zero component graph $\Gamma_{0X}(\mathcal{V})$ and reduced zero component graph $\Gamma^*_{0X}(\mathcal{V})$ of a semimodule $\mathcal{V}$ as follows:

Definition 2.5. Let $\mathcal{V}$ be a semimodule with a basis X. The zero component graph of $\mathcal{V}$ is defined as the graph $\Gamma_{0X}(\mathcal{V}) = (V, \mathcal{E})$, where $V = \mathcal{V} \setminus \{\sum c_ix_i : c_1 \cdot c_2 \cdots c_n \neq 0\}$, that is V consists of the elements whose at least one component is zero and $(\alpha, \beta) \in \mathcal{E}$ if there exists i such that both α_i and β_i are zero. Note that $\theta \sim v$ for every v in $\Gamma_{0X}(\mathcal{V})$. In fact θ is the only vertex having this property. The subgraph $\Gamma^*_{0X}(\mathcal{V})$ obtained by deletion of θ, is called the *reduced zero-component graph* of $\mathcal{V}$.

For any two bases X and Y of $\mathcal{V}$, proceeding similarly as in the proof of Theorem 2.4, we can prove that $\Gamma^*_{0X}(\mathcal{V})$ and $\Gamma^*_{0Y}(\mathcal{V})$ are graph isomorphic.

Since the graphs are independent of the choice of a particular basis (up to isomorphism), so we denote the reduced non-zero component graph of $\mathcal{V}$ by $\Gamma^*(\mathcal{V})$ and the reduced zero component graph of $\mathcal{V}$ by $\Gamma^*_0(\mathcal{V})$.

Notice that the vertex set of both $\Gamma^*(\mathcal{V})$ and $\Gamma^*_0(\mathcal{V})$ is same and for the next sections of this article we denote it by $\mathbb{V}$ and $X = \{x_1, x_2, \ldots, x_n\}$ denotes a basis.

3. Properties of the graph $\Gamma^*(\mathcal{V})$

In this section, we investigate some basic properties like connectedness, completeness, domination number, independence number of the graph $\Gamma^*(\mathcal{V})$. Also we show that two semimodules $\mathcal{V}$ and $\mathcal{W}$ are isomorphic if and only if the graphs $\Gamma^*(\mathcal{V})$ and $\Gamma^*(\mathcal{W})$ are isomorphic.

Since any two elements of a basis are pairwise non-adjacent, $\Gamma^*(\mathcal{V})$ is not complete.

Theorem 3.1. *If* $n \geqslant 3$, $\Gamma^*(\mathcal{V})$ *is connected and* $diam(\Gamma^*(\mathcal{V})) = 2$.

Proof. If $\alpha, \beta \in \mathbb{V}$ are adjacent, then $d(\alpha, \beta) = 1$; otherwise, there exist distinct i, j such that $\alpha_i, \beta_j \neq 0$. Since $n \geqslant 3$, there exists $\gamma \in \mathbb{V}$ such that $\gamma_i, \gamma_j \neq 0$. So $\alpha \sim \gamma$ and $\beta \sim \gamma$ and hence $d(\alpha, \beta) = 2$. Thus $\Gamma^*(\mathcal{V})$ is connected and $diam(\Gamma^*(\mathcal{V})) = 2$. □

Theorem 3.2. *The domination number of* $\Gamma^*(\mathcal{V})$ *is* 2.

Proof. It is easily seen that $\{x_1 + x_2 + \cdots + x_{n-1}, x_2 + x_3 + \cdots x_n\}$ is a minimal dominating subset of $\Gamma^*(\mathcal{V})$. If possible, let $\{\alpha\}$ be a dominating subset of $\Gamma^*(\mathcal{V})$. Then there exists i such that $\alpha_i = 0$. Consider $\beta \in \mathbb{V}$ such that $\beta_i \neq 0$ but $\beta_j = 0$ for all $j \neq i$. Then $\alpha \nsim \beta$. Hence the result. □

Theorem 3.3. *If* $D = \{y_1, y_2, \ldots, y_l\}$ *is a minimal dominating set of* $\Gamma^*_X(\mathcal{V})$*, then* $l \leqslant n$.

Proof. Let $D_i = D \setminus \{y_i\}$ for $i \in \{1, 2, \ldots, n\}$. Since D is a minimal dominating set, for all $i \in \{1, 2, \ldots, l\}$, D_i is not a dominating subset of $\Gamma^*_X(\mathcal{V})$. So, for each $i \in \{1, 2, \ldots, l\}$, there exists $z_i \in \Gamma^*(\mathcal{V})$ such that $z_i \sim y_i$ but $z_i \nsim y_j$ for $j \neq i$. Since $z_i \neq \theta$, there exists t_i such that $(z_i)_{t_i} \neq 0$. So $x_{t_i} \nsim y_j$ for $j \neq i$ but $x_{t_i} \sim y_i$ as D is a minimal dominating set.

Now we show that $x_{t_i} \neq x_{t_j}$ for $i \neq j$. If possible, let $x_{t_i} = x_{t_j}$ for some $i \neq j$. Since $x_{t_i} \sim y_i$ and $x_{t_i} = x_{t_j}$, so $x_{t_j} \sim y_i$ which contradicts that $x_{t_i} \nsim y_j$ for all $i \neq j$. Hence $x_{t_i} \neq x_{t_j}$ for $i \neq j$. Since $x_{t_1}, x_{t_2}, \ldots, x_{t_l}$ are all distinct, it follows that $l \leqslant n$. □

Theorem 3.4. *The independence number of* $\Gamma^*_X(\mathcal{V})$ *is* n.

Proof. It is easy to observe that $\{x_1, x_2, \ldots, x_n\}$ is an independent set of $\Gamma^*_X(\mathcal{V})$. So the independence number of $\Gamma^*_X(\mathcal{V})$ is greater than or equal to n. If possible, let $\{y_1, y_2, \ldots, y_l\}$ be an independent set of $\Gamma^*_X(\mathcal{V})$ such that $l > n$. Since for all $i \in \{1, 2, \ldots, l\}$, $y_i \neq \theta$, there exists t_i such that $(y_i)_{t_i} \neq 0$. We show that $t_i \neq t_j$ when $i \neq j$. If $t_i = t_j = t$ for some $i \neq j$, then t_i th component of both y_i and y_j is non-zero and hence $y_i \sim y_j$, which is a contradiction to the independence of y_i and y_j. Since there are exactly n distinct x_i, the independence number of $\Gamma^*_X(\mathcal{V})$ is n. □

Lemma 3.5. *Let* I *be an independent set in* $\Gamma^*_X(\mathcal{V})$*, then* I *is linearly independent in* $\mathcal{V}$.

Proof. Let $I = \{y_1, y_2, \ldots, y_l\}$ be an independent set of $\Gamma^*_X(\mathcal{V})$. Then by Theorem 3.4, $l \leqslant n$. If possible, let I be linearly dependent in $\mathcal{V}$. Then there exists $i \in \{1, 2, \ldots, l\}$ such that y_i is expressed as a linear combination of $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_l$, i.e., $y_i = c_1y_1 + c_2y_2 + \cdots + c_{i-1}y_{i-1} + c_{i+1}y_{i+1} + \cdots + c_ly_l = \sum_{j=1, j\neq i}^{l} c_jy_j$

Let $y_j = \sum_{t=1}^{n} d_{tj}x_t$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, l$. Thus,

$$y_i = (c_1d_{11} + \cdots + c_{i-1}d_{1,i-1} + c_{i+1}d_{1,i+1} + \cdots + c_ld_{1l})x_1 + (c_1d_{21} + \cdots + c_{i-1}d_{2,i-1} + c_{i+1}d_{2,i+1} + \cdots + c_ld_{2l})x_2 + \cdots + (c_1d_{n1} + \cdots + c_{i-1}d_{n,i-1} + c_{i+1}d_{n,i+1} + \cdots + c_ld_{nl})x_n$$

Since $y_i \neq \theta$, there exists t_0 such that $(y_i)_{t_0} \neq 0$. So, there exists y_k such that $k \neq i$ and $(y_k)_{t_0} \neq 0$, otherwise t_0-th component of y_i will be 0. Which shows that $\{y_1, y_2, \ldots, y_l\}$ is not independent in $\Gamma^*_X(\mathcal{V})$. This contradiction shows that $\{y_1, y_2, \ldots, y_l\}$ is linearly independent in $\mathcal{V}$. □

Remark 3.6. Converse of the Lemma 3.5 is not true, in general, if we consider the subset $\{x_1, x_1 + x_2\}$ of a three dimensional semimodule with respect to the basis $\{x_1, x_2, x_3\}$.

Now, we show that two semimodules are isomorphic if and only if their corresponding reduced non-zero component graphs are isomorphic.

Lemma 3.7. *Two semimodules $\mathcal{V}$ and $\mathcal{W}$ are isomorphic if and only if the reduced non-zero component graphs $\Gamma^*(\mathcal{V})$ and $\Gamma^*(\mathcal{W})$ are isomorphic.*

Proof. Let $\mathcal{V}$ and $\mathcal{W}$ be isomorphic and $\sigma : \mathcal{V} \longrightarrow \mathcal{W}$ be an isomorphism. Then $Y = \{\sigma(x_1), \sigma(x_2), \dots,$
$\sigma(x_n)\}$ is a basis of W. Consider the restriction $\sigma^* : \Gamma^*_X(\mathcal{V}) \longrightarrow \Gamma^*_Y(\mathcal{W})$ given by

$$\sigma^*(c_1x_1 + c_2x_2 + \cdots + c_nx_n) = c_1\sigma(x_1) + c_2\sigma(x_2) + \cdots + c_n\sigma(x_n)$$

where $c_1 \cdot c_2 \cdots c_n = 0$ but $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$. Clearly σ^* is a bijection. Let $\alpha = c_1x_1 + \cdots + c_nx_n$ and $\beta = d_1x_1 + \cdots + d_nx_n$. Then $\alpha \sim \beta$ in $\Gamma^*_X(\mathcal{V})$ if and only if there exists i such that $c_i, d_i \neq 0$ if and only if $\sigma^*(\alpha) \sim \sigma^*(\beta)$ in $\Gamma^*_Y(W)$. Therefore $\Gamma^*(\mathcal{V})$ and $\Gamma^*(W)$ are isomorphic.

Conversely, let $\phi : \Gamma^*(\mathcal{V}) \to \Gamma^*(\mathcal{W})$ be a graph isomorphism. let $dim(\mathcal{V}) = m$ and $dim(W) = n$. Since isomorphism preserves the independence number, the independence number of $\Gamma^*(\mathcal{V})$ equals to the independence number of $\Gamma^*(\mathcal{W})$ and hence $m = n$. So $\mathcal{V}$ and $\mathcal{W}$ are isomorphic. $\square$

Thus we see that a semimodule isomorphism $\sigma : \mathcal{V} \to \mathcal{W}$ is also a graph isomorphism (ignoring the null vector and vectors of the form $c_1x_1 + \cdots + c_nx_n$ such that $c_1 \cdot c_2 \dots c_n \neq 0$), however the converse may not be true which is shown in the following example.

Example 3.8. Consider the semimodule $\mathbb{N}_0^2$ over $\mathbb{N}_0$, the set of all nonnegative integers, with respect to usual addition and multiplication. Then the vertex set $\mathbb{V}$ of $\Gamma^*(\mathbb{N}_0^2)$, is $\{(a, b) \in \mathbb{N}_0^2 : a = 0 \text{ or } b = 0 \text{ and } (a, b) \neq (0, 0)\}$. Define a map $\phi : \mathbb{V} \to \mathbb{V}$ defined by $\phi(1, 0) = (3, 0)$, $\phi(3, 0) = (1, 0)$, $\phi(n, 0) = (n, 0)$ for $n \neq 1, 3$ and $\phi(0, m) = (0, m)$, where $m \in \mathbb{N}_0$. Then ϕ is a graph isomorphism on $\Gamma^*(\mathbb{N}_0^2)$ but it can not be extended to a linear transformation on $\mathbb{N}_0^2$. Otherwise $(1, 0) = \phi(3, 0) = 3\phi(1, 0) = (9, 0)$, which is a contradiction.

Now, we study the form of automorphisms of $\Gamma^*(\mathcal{V})$.

Theorem 3.9. *Let ϕ be a graph automorphism on $\Gamma^*_X(\mathcal{V})$. Then ϕ permutes the elements of $X = \{x_1, x_2, \dots, x_n\}$ of $\mathcal{V}$ with some non-zero scalar multiplication, i.e. there exists a permutation $\sigma \in S_n$ such that $\phi(x_i) = c_ix_{\sigma(i)}$, where c_i's are non-zero.*

Proof. Since ϕ is a graph automorphism on $\Gamma^*_X(\mathcal{V})$ and $\{x_1, x_2, \dots, x_n\}$ is an independent set of vertices in $\Gamma^*_X(\mathcal{V})$, therefore $\{\phi(x_i) : i = 1, 2, \dots, n\}$ is also an independent set of vertices in $\Gamma^*_X(\mathcal{V})$. Let

$$\begin{aligned}
\phi(x_1) &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n\\
\phi(x_2) &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\phi(x_n) &= c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n
\end{aligned}$$

Since $\phi(x_1) \neq 0$, there exists $j_1 \in \{1, 2, \ldots, n\}$ such that $c_{1j_1} \neq 0$. Therefore $c_{ij_1} = 0$ for all $i \neq 1$ because $\phi(x_i)$ is not adjacent to $\phi(x_1)$ for all $i \neq 1$. Similarly, for $\phi(x_2)$, there exists $j_2 \in \{1, 2, \ldots, n\}$ such that $c_{2j_2} \neq 0$ and $c_{ij_2} = 0$ for all $i \neq 2$. Moreover, $j_1 \neq j_2$ as $\phi(x_1)$ and $\phi(x_2)$ are not adjacent. Proceeding in this way, we see that for $\phi(x_n)$, there exists $j_n \in \{1, 2, \ldots, n\}$ such that $c_{nj_n} \neq 0$ and $c_{ij_n} = 0$ for all $i \neq n$ and $j_1, j_2, \ldots, j_n \in \{1, 2, \ldots, n\}$ are all distinct numbers.

Thus we see that $c_{kj_l} = 0$ for all $k \neq l$ and $c_{kj_k} \neq 0$, where $k, l \in \{1, 2, \ldots, n\}$. Let $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix}$. Then σ is a permutation on $\{1, 2, \ldots, n\}$ and $\phi(x_i) = c_{ij_i}x_{j_i} = c_{ij_i}x_{\sigma(i)}$, where $c_{ij_i} \neq 0$ and hence the result follows. □

Theorem 3.10. *Let ϕ be a graph automorphism on $\Gamma^*_X(\mathcal{V})$ such that ϕ maps x_i into $c_{ij_i}x_{\sigma(i)}$ for some $\sigma \in S_n$, where $c_{ij_i} \neq 0$. Then, for $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, if c_i's are non-zero, then $\phi(c_1x_{i_1} + \cdots + c_kx_{i_k}) = d_1x_{\sigma(i_1)} + \cdots + d_kx_{\sigma(i_k)}$, where d_i's are non-zero.*

Proof. Since $cx_i \sim x_i$ in $\Gamma^*_X(\mathcal{V})$ and ϕ is an automorphism on $\Gamma^*_X(\mathcal{V})$, $\phi(cx_i) \sim \phi(x_i)$ i.e., $\phi(cx_i) \sim c_{ij}x_{\sigma(i)}$. So, $\sigma(i)$-th component of $\phi(cx_i)$ is non-zero. If possible, let $\sigma(j)$-th component of $\phi(cx_i)$ is non-zero for $j \neq i$. Then $\phi(cx_i) \sim x_{\sigma(j)}$ i.e. $\phi(cx_i) \sim \phi(x_j)$. Which in turn implies that $cx_i \sim x_j$ for $i \neq j$, which is a contradiction since $\{x_1, x_2, \ldots, x_n\}$ is an independent set. Therefore, $\phi(cx_i) = dx_{\sigma(i)}$ for some $d \neq 0$.

Now, for all $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,

$$\begin{aligned}
&c_1x_{i_1} + c_2x_{i_2} + \cdots + c_kx_{i_k} \sim x_{i_1}\\
\Rightarrow &\phi(c_1x_{i_1} + c_2x_{i_2} + \cdots + c_kx_{i_k}) \sim \phi(x_{i_1}) = cx_{\sigma(i_1)} \quad \text{for some } \; c \neq 0.
\end{aligned}$$

Which implies that $\phi(c_1x_{i_1} + c_2x_{i_2} + \cdots + c_kx_{i_k}) \sim x_{\sigma(i_1)}$. Similarly, $\phi(c_1x_{i_1} + c_2x_{i_2} + \cdots + c_kx_{i_k}) \sim x_{\sigma(i_2)}$ and so on. Therefore, $\phi(c_1x_{i_1} + c_2x_{i_2} + \cdots + c_kx_{i_k}) = d_1x_{\sigma(i_1)} + d_2x_{\sigma(i_2)} + \cdots + d_kx_{\sigma(i_k)}$, where d_i's are non-zero □

Corollary 3.11. *If $n \geqslant 3$, then $\Gamma^*_X(\mathcal{V})$ is not vertex transitive.*

Proof. If $n \geqslant 3$, then by Theorem 3.10, there does not exist any automorphism which maps x_1 to $x_1 + x_2$. □

4. Graph isomorphism of $\Gamma^*(\mathcal{V})$ and $\Gamma^*_0(\mathcal{V})$

In this section we show that for a semimodule $\mathcal{V}$ over S, if $|S| \geqslant \aleph_0$, then the reduced non-zero component graph and reduced zero component graph of $\mathcal{V}$ are isomorphic. Now we show that if $|S| < \aleph_0$, then the two graphs $\Gamma^*(\mathcal{V})$ and $\Gamma_0^*(\mathcal{V})$ may not be isomorphic.

Example 4.12. Let $S = \{0, 1, a\}$ be the chain $0 < a < 1$. Consider the semimodule S^3 over S and a basis $E = \{(1,0,0), (0,1,0), (0,0,1)\}$. Then the vertex set of both reduced zero and reduced non-zero component graphs of S^3 is $\{(1,0,0), (1,0,a), (1,0,1), (1,1,0), (1,a,0), (a,0,0), (a,0,a), (a,0,1), (a,a,0), (a,1,0), (0,a,a), (0,a,0), (0,a,1), (0,0,a), (0,0,1), (0,1,0), (0,1,1), (0,1,a)\}$. The degree of $(1,0,0)$ in $\Gamma_E^*(S^3)$ is 9. But, there does not exist any element of degree 9 in $\Gamma_{0E}^*(S^3)$. Therefore $\Gamma_E^*(S^3)$ is not isomorphic to $\Gamma_{0E}^*(S^3)$.

Theorem 4.13. *Let $\mathcal{V}$ be a semimodule over S and X be a basis of $\mathcal{V}$. If $|S| \geqslant \aleph_0$, then $\Gamma_X^*(\mathcal{V})$ and $\Gamma_{0X}^*(\mathcal{V})$ are isomorphic.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$. Then the two graphs $\Gamma_X^*(\mathcal{V})$ and $\Gamma_{0X}^*(\mathcal{V})$ have the same set of vertices $\mathbb{V} = \{\sum_{i=1}^{n} a_i x_i : \exists i, j \text{ such that } a_i = 0 \text{ and } a_j \neq 0\}$. For $A = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, denote $Z_A = \{\sum a_i x_i \in \mathcal{V} : a_{i_1} = \ldots = a_{i_k} = 0$ and $a_i \neq 0$ otherwise $\}$. Then $\mathbb{V}$ is a disjoint union of the sets Z_A, where A is a non-empty proper subset of $\{1, 2, \ldots, n\}$, i.e. $\mathbb{V} = \bigcup Z_A$. Now, since $|S| \geqslant \aleph_0$, $|Z_A| = |Z_{A^c}| = |\mathbb{V}|$, which implies that there exists a bijection $\phi_A : Z_A \to Z_{A^c}$.

Thus we get a bijection $\phi = \cup \phi_A : \mathbb{V} \longrightarrow \mathbb{V}$ such that $a \sim b$ in $\Gamma_{0X}^*(\mathcal{V})$ if and only if $\phi(a) \sim \phi(b)$ in $\Gamma_X^*(\mathcal{V})$. Hence $\Gamma_X^*(\mathcal{V})$ and $\Gamma_{0X}^*(\mathcal{V})$ are isomorphic as graphs. □

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