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Some properties of i-quasigroups

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Abstract. We describe the relationship between some type of quasigroups containing left neutral element with Bol and Moufang quasigroups (in the sense of Belousov) and characterize certain pseudo-automorphisms of these quasigroups.

1. Introduction

A quasigroup $(Q,\cdot)$ is

- an *LIP-quasigroup* (has the *left inverse-property*), if there exists a bijection $x \to \lambda x$ $(x \to I_l x)$ of the set Q such that
$$\lambda x \cdot (x \cdot y) = y \tag{1}$$
for all $x, y \in Q$;
- a *RIP-quasigroup* (has the *right inverse-property*), if there exists a bijection $x \to \rho x$ $(x \to I_r x)$ of the set Q such that
$$(y \cdot x) \cdot \rho x = y \tag{2}$$
for all $x, y \in Q$;
- an *IP-quasigroup* (has the *inverse property*), if satisfies (1) and (2);
- a *left Bol quasigroup*, if $x(y \cdot xz) = R_{e_x}^{-1}(x \cdot yx) \cdot z$ for all $x, y, z \in Q$, where $x \cdot e_x = x$ and $R_{e_x} y = ye_x$;
- a *right Bol quasigroup*, if $(yx \cdot z)x = yL_{f_x}^{-1}(xz \cdot x)$ for all $x, y, z \in Q$;
- a *Belousov-Moufang quasigroup*, if $x(y \cdot xz) = ((x \cdot yf_x)x)z$ for all $x, y, z \in Q$, where $f_x \cdot x = x$.

Such quasigroups (under name *Moufang quasigroups*) were described by V. D. Belousov in his book [1]. Since these quasigroups are not Moufang quasigroups in the classical sense we will call them Belousov-Moufang quasigroups (cf. [4]). Note that a qasigroup with a neutral element is a Belousov-Moufang quasigroup if and only if it satisfies the identity $x(y \cdot xz) = (xy \cdot x)z$ (cf. [3]). So, the concept of Belousov-Moufang loops coincides with the concept of Moufang loops.

Other undefined concepts can be found in [1], [8] and [9].

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2. i-quasigroups with non-empty distributant

According to [6], by the *distributant* of a quasigroup $(Q, \cdot)$ we mean the set D containing of all elements $d \in Q$ such that $(x \cdot y) \cdot d = (x \cdot d) \cdot (y \cdot d)$, $d \cdot (x \cdot y) = (d \cdot x) \cdot (d \cdot y)$ for all $x, y \in Q$.

A quasigroup $(Q, \cdot)$ satisfying the identity:

$$x\,(xy \cdot z) = y\,(zx \cdot x) \tag{3}$$

is called an *i-quasigrup*.

Examples. Examples of i-quasigroups.

A. The set $\mathbb{C}$ of all complex numbers with the operation $x \circ y = ix - y$ is an i-quasigrup.

B. Every group $(G, \cdot)$ in which of elements of the form x^2 are in the center is an *i-quasigrup*.

C. A commutative Moufang loop is an i-quasigroup. Also a left Bol quasigroup is an i-quasigroup.

D. There is four i-quasigroups induced by the group $\mathbb{Z}_5$:

$$x \cdot_1 y = (x + y)(\operatorname{mod} 5), \qquad x \cdot_2 y = (2x + 4y)(\operatorname{mod} 5),$$
$$x \cdot_3 y = (3x + 4y)(\operatorname{mod} 5), \qquad x \cdot_4 y = (4x + y)(\operatorname{mod} 5)$$

and five i-quasigroups with neutral element that are not induced by $\mathbb{Z}_5$:

	1	2	3	4	5
1	1	2	3	4	5
2	2	3	5	1	4
3	3	5	4	2	1
4	4	1	2	5	3
5	5	4	1	3	2

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	1	5	3
3	3	1	5	2	4
4	4	5	2	3	1
5	5	3	4	1	2

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	5	3	1
3	3	5	2	1	4
4	4	3	1	5	2
5	5	1	4	2	3

	1	2	3	4	5
1	1	2	3	4	5
2	2	5	1	3	4
3	3	1	4	5	2
4	4	3	5	2	1
5	5	4	2	1	3

	1	2	3	4	5
1	1	2	3	4	5
2	2	5	4	1	3
3	3	4	2	5	1
4	4	1	5	3	2
5	5	3	1	2	4

Remark 2.1. The translation R_f, where f is a left neutral element of a quasigroup $(Q, \cdot)$, will be denoted by R. In an i-quasigroup with a left neutral element, $R^2 = \varepsilon$ (the identity translation) and $R^{-1} = R$.

Theorem 2.2. *If an i-quasigroup $(Q, \cdot)$ is a RIP-quasigroup, then it is a Belousov-Moufang quasigroup with a left neutral element f and the distributant $D = \{f\}$.*

Proof. From (3), for $y = x$, we have

$$x^2 z = zx \cdot x \tag{4}$$

for all $x, z \in Q$. From $yx \cdot x^{-1} = y$ and (4) we have $\left(x^2 z\right) x^{-1} \cdot x^{-1} = z$. Then

$$x^{-2}\left(x^2 z\right) = z, \tag{5}$$

for all $x, z \in Q$.

From (3), (5), (4) we obtain $x\left(x^{-2}\right)\cdot z=x^{-2}\left(x^{2}z\right)=z$. Consequently,

$$x\left(xx^{-2}\cdot z\right)=x\left(I_{l}x\cdot z\right)=z, \tag{6}$$

where $I_{l}x=x\cdot x^{-2}$. From (6), by replacing z with xz, we deduce $I_{l}x\cdot xz=z$. So, $(Q,\cdot)$ is an IP-quasigroup, where $I_{l}x\cdot xy=y$ and $yx\cdot I_{r}x=y$.

We will now prove the existence of a left neutral element. We have $ye_{y}\cdot e_{y}^{-1}=y$, $ye_{y}^{-1}=y$, $e_{y}^{-1}=e_{y}$. Hence, $ze_{y}\cdot e_{y}^{-1}=ze_{y}\cdot e_{y}=e_{y}^{2}z=z$, $e_{y}^{2}=f$, $fz=z$ for any $z\in Q$. So, f is a left neutral element.

From $fy\cdot y^{-1}=f$, we obtain $yy^{-1}=f$, $y^{-1}={}^{-1}yf$, $I_{r}=RI_{l}$. Therefore, $I_{r}I_{l}=R$, $(I_{r}I_{l})^{-1}=R^{-1}=R$ and $I_{r}I_{l}=I_{l}I_{r}$.

In every IP-quasigroup ${}^{-1}((xy)^{-1})={}^{-1}({}^{-1}y\cdot{}^{-1}x)=({}^{-1}x)^{-1}\cdot({}^{-1}y)^{-1}$, $I_{l}I_{r}(xy)=I_{r}I_{l}x\cdot I_{r}I_{l}y$, so we have the autotopy $T=(I_{r}I_{l},I_{r}I_{l},I_{l}I_{r})=(R,R,R)$. Hence R is an automorphism of $(Q,\cdot)$. Thus $(xy)\,f=xf\cdot yf$ and $D=\{f\}$.

We must prove that $(Q,\cdot)$ is a Belousov-Moufang quasigroup, i.e. we must prove that the identity $x\,(y\cdot xz)=((x\cdot yf)\,x)\,z$ is satisfied. It is sufficient to prove that in $(Q,\cdot)$ there exists the autotopy $T=\left(R_{x}L_{x}R_{f},L_{x}^{-1},L_{x}\right)$.

From $x\,(xy\cdot z)=y\,(zx\cdot x)$ we obtain the autotopy $T_{1}=\left(L_{x}^{-1},R_{x}^{2},L_{x}\right)$ and $(x(xy\cdot z))^{-1}=(y(zx\cdot x))^{-1}$, ${}^{-1}(xy\cdot z)\cdot{}^{-1}x={}^{-1}(zx\cdot x)\cdot{}^{-1}y$, $(z^{-1}(xy)^{-1})\cdot{}^{-1}x=(x^{-1}(zx)^{-1})^{-1}\cdot{}^{-1}y$.

Further, we have

$$z^{-1}({}^{-1}y\cdot{}^{-1}x)\cdot{}^{-1}x=(x^{-1}\cdot({}^{-1}x\cdot{}^{-1}z))\cdot{}^{-1}y. \tag{7}$$

Applying to (7) the equalities $I_{l}^{2}=I_{r}^{2}=\varepsilon$, $R^{2}=\varepsilon$, $R=I_{l}I_{r}=I_{r}I_{l}$ and the substitutions $z\to z^{-1}$, $y\to y^{-1}$, $x\to x^{-1}$ we obtain $(z\cdot yx)\,x=(Rx\cdot(x\cdot Rz))\cdot y$. Thus, we have the autotopy $T_{2}=\left(L_{xf}L_{x}R,R_{x}^{-1},R_{x}\right)$. Then $T_{3}=T_{2}T_{1}=(\alpha,R_{x},R_{x}L_{x})$, where $\alpha=L_{xf}L_{x}R_{f}L_{x}^{-1}$. Since $(Q,\cdot)$ is an IP-quasigrup, we have the autotopy $T_{4}=(R_{x}L_{x},I_{r}R_{x}I_{r},\alpha)$, where $I_{r}R_{x}I_{r}y=(y^{-1}\cdot x)^{-1}={}^{-1}x\cdot{}^{-1}(y^{-1})$ $=L_{x}^{-1}Ry$, $I_{r}R_{x}I_{r}=L_{x}^{-1}R=L_{{}^{-1}x}R$, $T_{4}=\left(R_{x}L_{x},L_{{}^{-1}x}R,\alpha\right)$.

Therefore, $\alpha\,(yz)=(xy\cdot x)\cdot{}^{-1}x\,(zf)$. If $y=x^{-1}$, where $xx^{-1}=f$, then $\alpha(x^{-1}z)=zf$, $\alpha L_{x^{-1}}=R$, $\alpha=RL_{x^{-1}}^{-1}=RL_{{}^{-1}(x^{-1})}=RL_{xf}$, where $RL_{xf}z=R\,(xf\cdot z)=xRz=L_{x}Rz$, $\alpha=L_{x}R$, $T_{4}=\left(R_{x}L_{x},L_{x}^{-1}RL_{x}R\right)$. Thus R is an automorphism of $(Q,\cdot)$. Finally, we have the autotopy

$$T=T_{4}\cdot(R,R,R)=\left(R_{x}L_{x}R,L_{x}^{-1},L_{x}\right).$$

This completes the proof.

Theorem 2.3. *If an i-quasigroup $(Q,\cdot)$ with a left neutral element f is isotopic to an abelian group, then it is a medial Belousov-Moufang quasigroup with the distributant $D=\{f\}$.*

Proof. Obviously, the isotope $(Q,\circ)$, where $x\circ y=R^{-1}x\cdot y=Rx\cdot y=xf\cdot y$, is an abelian group with the neutral element f.

Then $Rx \circ (R(Rx \circ y) \circ z) = Ry \circ (R(Rz \circ x) \circ x)$ by $x(xy \cdot z) = y(zx \cdot x)$, which for $z = f$ gives $Rx \circ R(Rx \circ y) = Ry \circ (Rx \circ x)$, $R(Rx \circ y) = Ry \circ x$, $R(x \circ y) = Rx \circ Ry$. Hence R is an automorphism of the group $(Q, \circ)$ and the quasigroup $(Q, \cdot)$. Then the distributant $D = \{f\}$.

Further, from $(y \circ x) \circ x^{-1} = y$ it follows $R(Ry \cdot x) \cdot x^{-1} = y$ and $(y \cdot Rx) \cdot x^{-1} = y$. Thus, $(Q, \cdot)$ is a RIP-quasigroup and, by Theorem 2.2, a Belousov-Moufang quasigroup.

It is medial because from $xy \cdot uv = xu \cdot yv'$ we obtain $R(Rx \circ y) \circ (Ru \circ v) = R(Rx \circ u) \circ (Ry \circ v')$ and $x \circ Ry \circ Ru \circ v = x \circ Ru \circ Ry \circ v'$, which implies $v = v'$. □

Theorem 2.4. *If an i-quasigroup $(Q, \cdot)$ with the non-empty distributant D is isotopic to a left Bol loop, then it is a left Bol quasigroup.*

Proof. Let $a \in D$. Then $a \cdot xy = ax \cdot ay$, $xy \cdot a = xa \cdot ya$ for all $x, y \in Q$. So, L_a and R_a are automorphisms of $(Q, \cdot)$ and $a^2 = a$, $L_a R_a = R_a L_a$, $R_a L_a^{-1} = L_a^{-1} R_a$, $R_a^{-1} L_a = L_a R_a^{-1}$. Further, $a(ay \cdot a) = y(aa \circ a)$, $a(a \circ ya) = ya$ for any $y \in Q$. Thus, $a \cdot az = z$ for $z = ya$. Consequently, $L_a^2 = \varepsilon$, $L_a = L_a^{-1}$.

The isotope $(Q, \circ)$, where $x \circ y = R_a^{-1} x \cdot L_a^{-1} y$, is a right Bol loop with neutral element $e = a$. From ${}^{-1}x \circ (x \circ y) = y$ we obtain $R_a^{-1}({}^{-1}x) \cdot L_a^{-1}(R_a^{-1} x \cdot L_a^{-1} y) = y$ and $R_a^{-1} Ix \cdot L_a^{-1}(R_a^{-1} x \cdot L_a^{-1} y) = R_a^{-1} Ix \cdot (L_a R_a^{-1} x \cdot L_a^2 y) = y$, $R_a^{-1} I R_a L_a x(xy) = y$, $I_l x(xy) = y$, where $I_l = R_a^{-1} I R_a L_a$.

Hence, an i-quasigroup $(Q, \cdot)$ is an LIP-quasigroup and an isotope of a left Bol loop, so it is a left Bol quasigroup. □

Theorem 2.5. *An i-quasigroup $(Q, \cdot)$ with the non-empty distributant D is a left Bol quasigroup if and only if*

$$xa \cdot xy = xx \cdot ay \tag{8}$$

holds for all $x, y \in Q$ and fixed $a \in D$.

Proof. From $a \cdot xy = ax \cdot ay$ and $xy \cdot a = xa \cdot ya$ it follows $ax \cdot a = a \cdot xa$, $R_a L_a = L_a R_a$ and $a^2 = a$. Hence, L_a and R_a are automorphisms of the quasigroup $(Q, \cdot)$ and the loop $(Q, \circ)$, where $x \circ y = R_a^{-1} x \cdot L_a^{-1} y$.

The identity (3) from the definition the i-quasigroup, for $x = z = a$ ant $t = ya$, gives $a \cdot at = t$. So, $L_a^2 = \varepsilon$. Thus, $a(aa \cdot z) = a(za \cdot a)$ implies $L_a = R_a^2$.

Let $L = L_a$ and $R = R_a$. Then $x \circ y = R^{-1} x \cdot L^{-1} y$, and consequently $xy = Rx \circ Ly$. Hence, $Rx \circ L(R(Rx \circ Ly) \circ Lz) = Ry \circ L(R(Rz \circ Lx) \circ Lx)$, $Rx \circ \big((LR^2 x \circ LRLy) \circ L^2 z\big) = Ry \circ \big((LR^2 z \circ LRLx) \circ L^2 x\big)$. But $L = R^2$, $L^2 = \varepsilon$, so $Rx \circ ((x \circ Ry) \circ z) = Ry \circ ((z \circ Rx) \circ x)$, whence, replacing y with $R^{-1} y$, we obtain

$$Rx \circ ((x \circ y) \circ z) = y \circ ((z \circ Rx) \circ x) . \tag{9}$$

This for $y = Rx$, gives $(x \circ Rx) \circ z = (z \circ Rx) \circ x$, $Rx \circ ((x \circ y) \circ z) = y \circ ((x \circ Rx) \circ z)$. If $y = x^{-1}$, where $x \circ x^{-1} = e = a$, and e is the identity of the loop $(Q, \circ)$, then

$$Rx \circ z = x^{-1} \circ ((x \circ Rx) \circ z) . \tag{10}$$

From (8) we have $R^2x\circ L(Rx\circ Ly)=R(Rx\circ Lx)\circ L^2y,\ \ Lx\circ(LRx\circ L^2y)=(R^2x\circ RLx)\circ L^2y$. This for $x:=Lx$ gives

$$x\circ(Rx\circ y)=(x\circ Rx)\circ y. \tag{11}$$

From (10) and (11) we have

$$Rx\circ z=x^{-1}\circ(x\circ(Rx\circ z)),\qquad t=x^{-1}\circ(x\circ t), \tag{12}$$

for $t=Rx\circ z$, $x,z\in Q$.

So, $t=R^{-1}x^{-1}\cdot L^{-1}(R^{-1}x\cdot L^{-1}t)=R^{-1}x^{-1}\cdot(L^{-1}R^{-1}x\cdot t)=R^{-1}IRLx\cdot xt=I_lx(xt)$, where $I_l=R^{-1}IRL$. Thus, $(Q,\cdot)$ is an LIP-quasigroup.

It is an IP-loop, too. Indeed, (9) for $y=e$, where e is the identity of $(Q,\circ)$, gives

$$Rx\circ(x\circ z)=(z\circ Rx)\circ x. \tag{13}$$

From (9), putting $z=e$, we also obtain

$$Rx\circ(x\circ y)=y\circ(Rx\circ x). \tag{14}$$

Comparing these two identities, we get

$$(z\circ Rx)\circ x=z\circ(Rx\circ x), \tag{15}$$

which together with (9) implies $Rx\circ((x\circ y)\circ z)=y\circ(z\circ(Rx\circ x))$. This for $z=y^{-1}$ gives $(x\circ y)\circ y^{-1}=x$. So, $(Q,\circ)$ is an IP-loop.

To prove that $(Q,\circ)$ is a Moufang loop satisfying the identity $(Rx\circ x)\circ y=y\circ(Rx\circ x)$, observe that from $Rx\circ((x\circ y)\circ z)=y\circ((z\circ Rx)\circ x)$ we obtain the autotopy $T_1=\big(L_x^{-1},R_xR_{Rx},L_{Rx}\big)$ of $(Q,\circ)$. Further, we have $(Rx\circ((x\circ y)\circ z))^{-1}=(y\circ((z\circ Rx)\circ x))^{-1}$ and $((x\circ y)\circ z)^{-1}\circ(Rx)^{-1}=((z\circ Rx)\circ x)^{-1}\circ y^{-1}$. Consequently, $(z^{-1}\circ(x\circ y)^{-1})\circ(Rx)^{-1}=(x^{-1}\circ(z\circ Rx)^{-1})\circ y^{-1}$ and

$$\big(z^{-1}\circ\big(y^{-1}\circ x^{-1}\big)\big)\circ(Rx)^{-1}=\Big(x^{-1}\circ\Big((Rx)^{-1}\circ z^{-1}\Big)\Big)\circ y^{-1}. \tag{16}$$

Since $x\circ x^{-1}=e$, $Rx\circ Rx^{-1}=Re=e$, $Rx^{-1}=(Rx)^{-1}$, the condition (16) for $x:=x^{-1}$, $y:=y^{-1}$, $z:=z^{-1}$ gives $(z\circ(y\circ x))\circ Rx=(x\circ(Rx\circ z))\circ y$. Thus $T_2=\big(L_xL_{Rx},R_x^{-1},R_{Rx}\big)$ is the autotopy of $(Q,\circ)$. Also $T_3=T_1T_2=\big(L_{Rx},R_xR_{Rx}R_x^{-1},L_{Rx}R_{Rx}\big)=(L_{Rx},\alpha,L_{Rx}R_{Rx})$, where $\alpha=R_xR_{Rx}R_x^{-1}$, is the autotopy of $(Q,\circ)$. Thus, $L_{Rx}R_{Rx}(y\circ z)=L_{Rx}y\circ\alpha z$.

If $y=e$, then $\alpha=R_{Rx}$ and consequently, $T_3=(L_{Rx},R_{Rx},L_{Rx}R_{Rx})$. For $x:=R^{-1}x$ we obtain $T_3=(L_x,R_x,L_xR_x)$. Since $(Q,\circ)$ is an IP-loop, $T_4=(L_xR_x,IR_xI,L_x)=\big(L_xR_x,L_x^{-1},L_x\big)$ is the autotopy of $(Q,\circ)$. This implies the left Bol identity $x\circ(y\circ(x\circ z))=(x\circ(y\circ x))\circ z$. Since $(Q,\circ)$ is an IP-loop, it is a Moufang loop.

From (9), for $y=Rx$, we obtain $(x\circ Rx)\circ z=z\circ(Rx\circ x)$. This for $z=e$ gives $x\circ Rx=Rx\circ x$. Thus, $(Rx\circ x)\circ z=z\circ(Rx\circ x)$. Hence $(Q,\cdot)$ is invertible from the left and it is an isotope of a Moufang loop $(Q,\circ)$.

From results obtained in [5] it follows that $(Q,\cdot)$ is a left Bol quasigroup.

To prove the converse statement assume that $(Q,\cdot)$ is a left Bol quasigrup. Then $(Q,\cdot)$ is invertible from the left and $I_l x \cdot xy = y$, $RI_l x \circ L(Rx \circ Ly) = y$, $RI_l x \circ (LRx \circ y) = y$, $RI_l R^{-1}L^{-1}x \circ (x \circ y) = y$. Hence $(Q,\circ)$ is left invertible. Using (10), we obtain $Rx \circ z = x^{-1} \circ ((x \circ Rx) \circ z) = x^{-1} \circ (x \circ (Rx \circ z))$, $(x \circ Rx) \circ z = x \circ (Rx \circ z)$, $R^{-1}(R^{-1}x \cdot L^{-1}Rx) \cdot L^{-1}z = R^{-1}x \cdot L^{-1}(x \cdot L^{-1}z)$, $(R^{-2}x \cdot L^{-1}x) \cdot z = R^{-1}x \cdot (L^{-1}x \cdot L^{-1}z)$, $(Lx \cdot Lx) \cdot Lz = R^{-1}x \cdot (Lx \cdot z)$, $xx \cdot az = R^{-1}Lx \cdot xz$, $xx \cdot az = xa \cdot xz$. □

3. Connections with Bol and Moufang quasigroups

Proposition 3.1. *An idempotent i-quasigroup $(Q,\cdot)$ is a left Bol quasigroup.*

Proof. Since $(Q,\cdot)$ is idempotent, (3) implies $xz = zx \cdot x$. Multiplying this identity by x we obtain $xz \cdot x = (zx \cdot x)\,x = x \cdot zx$. So, $xz \cdot x = x \cdot zx$. Thus, $x\,(xy \cdot x) = y\left(x^2 \cdot x\right) = yx$ and $x\,(x \cdot yx) = yx$. The last, for $yx = z$, gives $x \cdot xz = z$, so $(Q,\cdot)$ is an LIP-quasigroup and $L_x^2 = \varepsilon$. From $x(xy \cdot z) = y \cdot x^2 z = y \cdot xz$, for $y := xy$, we deduce $x((x \cdot xy)z) = xy \cdot xz$, which implies $x \cdot yz = xy \cdot xz$. Therefore, $(Q,\cdot)$ is a left distributive quasigroup and L_x is its automorphism.

Since

$$x(y \cdot xz) = xy \cdot (x \cdot xz) = xy \cdot z = R_x^{-1}(xy \cdot x) \cdot z = R_{e_x}^{-1}(x \cdot yx) \cdot z$$

$(Q,\cdot)$ is a left Bol quasigroup too. □

Remark 3.2. An idempotent i-quasigroup $(Q,\cdot)$ isotopic to a commutative loop is left distributive.

Remark 3.3. Every loop isotopic to an idempotent i-quasigroup is a left Bol loop (cf. [5]).

Proposition 3.4. *An i-quasigroup $(Q,\cdot)$ with a right neutral element is a Moufang loop in which $x^2y = yx^2$ for all $x, y \in Q$.*

Proof. Let e by the right neutral element of an i-quasigroup $(Q,\cdot)$. Then $e \cdot ez = e(ee \cdot z) = e(ze \cdot e) = ez$, i.e. $ez = z$. Thus $(Q,\cdot)$ is a loop with $x \cdot xz = zx \cdot x$.

Since ${}^{-1}yy = e$ for every $y \in Q$, for every $z \in Q$ we have ${}^{-1}yz = {}^{-1}y({}^{-1}yy \cdot z) = y(z\,{}^{-1}y \cdot {}^{-1}y) = y({}^{-1}y \cdot {}^{-1}yz)$, i.e. $t = y \cdot {}^{-1}yt$ for $t = {}^{-1}yz$. Hence $(Q,\cdot)$ is an LIP-loop. $x(xx \cdot y) = x(yx \cdot x)$ implies $x^2y = yx \cdot x$. Also, $yx^2 = y(ex \cdot x) = x(xy \cdot e) = x \cdot xy = yx \cdot x$. Hence $x^2y = yx^2$. Then $x(xz \cdot y) = z(yx \cdot x) = z \cdot yx^2$, which implies $xz \cdot y = {}^{-1}x(z \cdot yx^2)$. This for $y = {}^{-1}z$ gives $xz \cdot {}^{-1}z = {}^{-1}x(z \cdot {}^{-1}zx^2) = {}^{-1}xx^2 = x$. So, $(Q,\cdot)$ is a RIP-loop too. By Theorem 2.2 it is a Moufang loop. □

Proposition 3.5. *Every i-quasigroup $(Q,\cdot)$ with a left neutral element f is an LIP-quasigroup isotopic to the LIP-loop $(Q,\circ)$, where $x \circ y = R_f^{-1}x \cdot y$.*

Proof. From $x\,(xx\cdot z)=x\,(zx\cdot x)$ we have obtain $x^2z=zx\cdot x$, which for $z=f$ gives $x^2f=x^2$. Thus $x^2\left(x^2y\cdot z\right)=y\left(zx^2\cdot x^2\right)$, for $y=f$ implies $x^2\cdot x^2z=zx^2\cdot x^2$. Hence, $x^2(x^2y\cdot z)=y(x^2\cdot x^2z)$.

Let $x^2{(x^2)}^{-1}=f$. Then $x^2z=x^2(x^2{(x^2)}^{-1}\cdot z)={(x^2)}^{-1}(zx^2\cdot x^2)={(x^2)}^{-1}(x^2\cdot x^2z)$, i.e. $t={(x^2)}^{-1}\cdot x^2t$ for $t=x^2z$. Therefore, $x(x{(x^2)}^{-1}\cdot z)={(x^2)}^{-1}(x^2\cdot z)=z$, so $x\,(I_lx\cdot z)=z$, where $I_lx=x\cdot{(x^2)}^{-1}$. Hence $(Q,\cdot)$ is an LIP-quasigroup.

The isotope $(Q,\circ)$, $x\circ y=Rx\cdot y$, of $(Q,\cdot)$ is a loop with the neutral element f. From $I_lx\cdot xy=y$ it follows $RI_lx\circ(Rx\circ y)=y,\ RI_lR^{-1}x\circ(x\circ y)=y$. Then $(Q,\circ)$ is an LIP-loop. □

Proposition 3.6. *A unipotent i-quasigroup is a Belousov-Moufang quasigroup with a left neutral element.*

Proof. Let $x^2=y^2=f$. Then from $x^2z=zx\cdot x$ we obtain $fz=zx\cdot x$. In particular, $xx=f=ff=fx\cdot x$. Thus $x=fx$ and $z=fz=zx\cdot x$ for all $x,z\in Q$. Hence $(Q,\cdot)$ is a RIP-quasigroup with a left neutral element f. By Theorem 2.2, is a Belousov-Moufang quasigroup. □

Proposition 3.7. *A unipotent i-quasigroup is isotopic to an abelian group.*

Proof. Let $(Q,\cdot)$ and f be as in the previous proposition. Then $x\,(xy\cdot z)=y\,(zx\cdot x)=yz$ and $x\,(xf\cdot z)=z$. Hence $x(R_x\cdot z)=z=x(I_lx\cdot z)$, so $R=I_l$.

Multiplying $x(xy\cdot z)=y(zx\cdot x)=yz$ by xf and using fact that $R_f=I_l$, we obtain $xy\cdot z=xf\cdot yz$, which for $z=f$ gives $xy\cdot f=xf\cdot yf$. So, $D=\{f\}$ and $R=I_l$ is an automorphism of $(Q,\cdot)$ and $(Q,\circ)$, where $x\circ y=Rx\cdot y,\ xy=Rx\circ y$ (Proposition 3.5). Now $xy\cdot z=R(Rx\circ y)\circ z=(R^2x\circ Ry)\circ z=(x\circ Ry)\circ z$ and $xf\cdot yz=R(xf)\circ(Ry\circ z)=R^2x\circ(Ry\circ z)=x\circ(Ry\circ z)$. But $xy\cdot z=xf\cdot yz$, so $(x\circ Ry)\circ z=x\circ(Ry\circ z)$. Hence the operation $\circ$ is associative. Consequently, $(Q,\circ)$ is a group. Since $I_lx\cdot xy=Rx\cdot xy=y$, $x\circ(Rx\circ y)=y$. This for $y=f$ gives $x\circ Rx=f$. Hence $Rx=x^{-1}$. This shows that $(Q,\circ)$ is an abelian group. □

Proposition 3.8. *An i-quasigroup $(Q,\cdot)$ with a left neutral element f is a Belousov-Moufang quasigroup if and only if $R=R_f$ is its automorphism.*

Proof. Let $(Q,\cdot)$ be a Belousov-Moufang quasigroup with a left neutral element, then it is a right Bol quasigroup, (cf. [3]). Hence $(zx\cdot y)\,x=z\cdot L_{f_x}^{-1}(xy\cdot x)=z\cdot L_f^{-1}(xy\cdot x)=z(xy\cdot x)$. This for $x=f$ and $z=vf$ gives

$$xy\cdot f=xf\cdot yf. \tag{17}$$

So, R_f is an automorphism of $(Q,\cdot)$.

Conversely, let R_f be an automorphism of an i-quasigroup $(Q,\cdot)$. Since $yy^{-1}=f$, by Proposition 3.5, $(Q,\cdot)$ is an LIP-quasigroup. Therefore, ${}^{-1}x\cdot xy=y$. This together with ${}^{-1}y\cdot({}^{-1}y)^{-1}=f$ gives $({}^{-1}y)^{-1}=yf$. Thus, $I_rI_l=R_f$. But $R_f^2=\varepsilon=I_l^2$, so $I_r=R_fI_l$ and $y^{-1}={}^{-1}yf$. Consequently, from (17), we obtain

$x({}^{-1}y) \cdot f = xf \cdot {}^{-1}yf = xf \cdot y^{-1}$, which by (3), gives ${}^{-1}y\,x^2 = {}^{-1}y(fx \cdot x) = x(x({}^{-1}y) \cdot f) = x(xf \cdot y^{-1}) = f(y^{-1}x \cdot x) = f(x^2 \cdot y^{-1}) = x^2y^{-1}$. So, ${}^{-1}yx^2 = x^2y^{-1}$, which implies $x^2 = y \cdot x^2y^{-1}$. Consequently, $x = {}^{-1}x(y \cdot x^2y^{-1})$.

Since $(Q, \cdot)$ is an i-quasigroup, $x(xy \cdot y^{-1}) = y(x^2 \cdot y^{-1})$. Hence, $xy \cdot y^{-1} = {}^{-1}x(y(x^2 \cdot y^{-1})) = x$. Thus $xy \cdot y^{-1} = x$. Therefore $(Q, \cdot)$ is a RIP-quasigroup. By Theorem 2.2 it is a Belousov-Moufang quasigroup. □

4. Alternative and elastic i-quasigroups

Proposition 4.1. *A left (right) alternative i-quasigroup $(Q, \cdot)$ is a Moufang loop in which $x^2y = yx^2$ for all $x, y \in Q$.*

Proof. Let $(Q, \cdot)$ be a left alternative i-quasigroup. Then $x \cdot xy = x^2y$ and for every $y \in Q$ there is f_y such that $y = f_yy$. Thus $y = f_y \cdot f_yy = f_y^2y$ implies $f_y^2 = f_y$. Hence, $f_y \cdot f_yz = f_y^2z = f_yz$, i.e. $f_yz = z$ for all $y, z \in Q$. So, $f = f_y$ is a left neutral element of $(Q, \cdot)$. Moreover, $x(xf \cdot y) = f(x^2 \cdot y) = x^2 \cdot y = x \cdot xy$, i.e. $xf = x$ for any $x \in Q$. This means that $(Q, \cdot)$ is a loop.

Now let $(Q, \cdot)$ be a right alternative i-quasigroup. Then $yx \cdot x = yx^2$ and for every $y \in Q$ there is e_y such that $y = ye_y$. Thus $y = ye_y \cdot e_y = y \cdot e_y^2$ implies $e_y^2 = e_y$. Hence, $ze_y \cdot e_y = ze_y^2 = ze_y$, i.e. $ze_y = z$ for all $y, z \in Q$. So, $e = e_y$ is a right neutral element of $(Q, \cdot)$. Also, $x^2 = x(xe \cdot e) = e(x^2 \cdot e) = ex^2$. Consequently, $ex \cdot x = ex^2 = x^2$, which implies $ex = x$. Thus, as in the previous case, $(Q, \cdot)$ is a loop.

Proposition 3.4 completes the proof. □

Proposition 4.2. *The set of all local right neutral elements of an elastic i-quasigroup forms a left Bol quasigroup.*

Proof. Let $(Q, \cdot)$ be an elastic i-quasigroup. Then $xy \cdot x = x \cdot yx$ for $x, y \in Q$ and the set of all its local neutral elements has the form $E = \{e_x \mid xe_x = x,\ x \in Q\}$.

From (3) and $xy \cdot x = x \cdot yx$ we obtain $xx \cdot z = zx \cdot x$. Thus $e_ze_z \cdot z = ze_z \cdot e_z = z$ which implies $e_z^2 = e_z$ for every $z \in Q$. Hence

$$e_z \cdot e_zy = e_z(e_z \cdot ye_z) = y(e_ze_z \cdot e_z) = y.$$

So, $e_z \cdot e_zy = y$ for all $y, z \in Q$. Therefore, $e_x(e_xy \cdot z) = y(e_x^2 \cdot z) = y \cdot e_xz$. This for $y := e_xy$ gives $e_x \cdot yz = e_xy \cdot e_xz$. In particular, $e_xy = e_x \cdot ye_y = e_xy \cdot e_xe_y$. So, $e_xe_y = e_{e_xy}$. This means that the set E is closed under the quasigroup operation.

For $e_a, e_b \in E$, the equation $e_ax = e_b$ is solved by $x = e_ae_b \in E$. The equation $ye_a = e_b$ is solved by $d = e_a \cdot e_be_a$. Indeed, since $e_z^2 = e_z$, $e_z^2y = ye_z \cdot e_z$ and $e_z \cdot e_zy = y$ for all $y, z \in Q$, for $d = e_ae_b \cdot e_a$ we have $de_a = (e_ae_b \cdot e_a)e_a = e_a^2 \cdot e_ae_b = e_a \cdot e_ae_b = e_b$. This shows that $(E, \cdot)$ is a subquasigroup of $(Q, \cdot)$.

To show that $(E, \cdot)$ is a left Bol quasigroup observe that from the above for all $x, y, z \in E$ we have $x^2 = x$, $e_x = x$, $x \cdot xy = y$ and $x \cdot yz = xy \cdot xz$. Thus

$$x(y \cdot xz) = xy \cdot (x \cdot xz) = xy \cdot z = R_x^{-1}(xy \cdot x) \cdot z = R_{e_x}^{-1}(xy \cdot x) \cdot z.$$

Thus, $(E,\cdot)$ is a left Bol quasigroup. □

Proposition 4.3. *An i-quasigroup $(Q,\cdot)$ with a left neutral element f is a Belousov-Moufang quasigroup if and only if it satisfies the identity:*

$$zx\cdot x = zf\cdot xx. \tag{18}$$

Proof. Let an i-quasigroup $(Q,\cdot)$ with a left neutral element f be a Belousov-Moufang quasigroup. Then, as in the proof of Proposition 3.8, it is a right Bol quasigroup and $(zx\cdot y)x = z\cdot L_{f_x}^{-1}(xy\cdot x) = z(xy\cdot x)$.

This for $x=f$ and $z=vf$ gives $vy\cdot f = vf\cdot yf$. Thus, $R=R_f$ is an automorphism of $(Q,\cdot)$ and the loop $(Q,\circ)$, where $x\circ y = Rx\cdot y$.

From $x(xy\cdot z) = y(zx\cdot x)$ we obtain

$$Rx\circ((x\circ y)\circ z) = y\circ((z\circ Rx)\circ x)\,. \tag{19}$$

This for $y=Rx$ gives

$$(x\circ Rx)\circ z = (z\circ Rx)\circ x, \tag{20}$$

and

$$z\circ(Rx\circ x) = Rx\circ(x\circ z)\,, \tag{21}$$

for $z=f$, and

$$(z\circ Rx)\circ x = Rx\circ(x\circ z)\,, \tag{22}$$

for $y=f$.

Thus, $z\circ(Rx\circ x) = Rx\circ(x\circ z) = (z\circ Rx)\circ x = (x\circ Rx)\circ z$. Hence,

$$z\circ(Rx\circ x) = (x\circ Rx)\circ z \tag{23}$$

and $x\circ Rx = Rx\circ x$. Therefore $Rz\cdot\big(R^2x\cdot x\big) = R\big(R^2x\cdot x\big)\cdot z$. Consequently, $zf\cdot xx = x(xf\cdot z) = f(zx\cdot x) = zx\cdot x$.

Conversely, let $(Q,\cdot)$ be an i-quasigroup with a left neutral element f. Then from (3), for $x=y$, we obtain $xx\cdot z = zx\cdot x$, which for $x=f$ gives $z=zf\cdot f$. So, $R_f^2=\varepsilon$ and $R_f=R_f^{-1}=R$. By putting $x=y$, $z=f$ in (3) we get $x^2f=x^2$.

The quasigroup $(Q,\circ)$, where $x\circ y = R_f^{-1}x\cdot y = Rx\cdot y$, is a loop and f is its neutral element. Since $xy = Rx\circ y$, from $xx\cdot z = zx\cdot x$ and (18), we obtain $zf\cdot xx = xx\cdot z$, and consequently, $z\circ(Rx\circ x) = R\,(xx)\circ z = xx\circ z = (Rx\circ x)\circ z$. So,

$$z\circ(Rx\circ x) = (Rx\circ x)\circ z.$$

Also $x^2\left(x^2y\cdot z\right) = y\left(x^2x^2\cdot z\right)$. From this, putting $y=f$, we get

$$x^2\cdot x^2z = x^2x^2\cdot z.$$

Therefore $x^2z = x^2(x^2(x^2)^{-1}\cdot z) = (x^2)^{-1}(x^2x^2\cdot z) = (x^2)^{-1}(x^2\cdot x^2z)$, which gives $x^2z = (x^2)^{-1}(x^2\cdot x^2z)$. So,

$$t = (x^2)^{-1}(x^2\cdot t)$$

for all $x, t \in Q$, $t = x^2 z$.

On the other hand, $x(x(x^2)^{-1} \cdot z) = (x^2)^{-1}(x^2 \cdot z) = z$, for $z = xv$ implies $x(x^2)^{-1} \cdot xv = v$. Thus, $I_l x \cdot xv = v$ for $I_l x = x(x^2)^{-1}$. This shows that $(Q, \cdot)$ is an LIP-quasigroup. Also $(Q, \circ)$ is an LIP-quasigroup (LIP-loop) because $I_l x \cdot xy = y$ means that $RI_l x \circ (Rx \circ y) = y$. Hence, $RI_l R^{-1} x \circ (x \circ y) = y$.

Now from (3), we obtain

$$Rx \circ (R(Rx \circ y) \circ z) = Ry \circ \left(Rx^2 \circ z\right) = Ry \circ \left(x^2 \circ z\right) = Ry \circ ((Rx \circ x) \circ z) = Ry \circ (z \circ (Rx \circ x)).$$

Therefore

$$Rx \circ (R(Rx \circ y) \circ z) = Ry \circ (z \circ (Rx \circ x)), \tag{24}$$

which for $z = f$ gives

$$Rx \circ R(Rx \circ y) = Ry \circ (Rx \circ x).$$

From this, applying (23), (20) and (22), we obtain

$$Rx \circ R(Rx \circ y) = Ry \circ (Rx \circ x) = (Rx \circ x) \circ Ry = Rx \circ (x \circ Ry).$$

So, $R(Rx \circ y) = x \circ Ry$ and $R(x \circ y) = Rx \circ Ry$. Hence R is an automorphism of $(Q, \circ)$ and $(Q, \cdot)$.

Then, using (21), we can rewrite (24) in the form

$$Rx \circ ((x \circ y) \circ z) = y \circ (Rx \circ (x \circ z)). \tag{25}$$

By substituting $z = y^{-1}$, where $y^{-1} \circ (y \circ z) = z$, we can see that $(Q, \circ)$ is a RIP-loop. Consequently, $(Q, \circ)$ is an IP-loop.

It is a Moufang loop too. Indeed, (25) can be written as

$$L_{Rx}(L_x y \circ z) = y \circ R_x R_{Rx} z,$$

whence, replacing y with $L_x^{-1} y$, we get $L_{Rx}(y \circ z) = L_x^{-1} y \circ R_x R_{Rx} z$. This shows that $T_1 = \left(L_x^{-1}, R_x R_{Rx}, L_{Rx}\right)$ is an autotopy of $(Q, \circ)$.

Moreover, from (19) we have $(Rx \circ ((x \circ y) \circ z))^{-1} = (y \circ ((z \circ Rx) \circ x))^{-1}$, which gives

$$(z^{-1} \circ (y^{-1} \circ x^{-1})) \circ (Rx)^{-1} = (x^{-1} \circ ((Rx)^{-1} \circ z^{-1})) \circ y^{-1}.$$

Thus also $T_2 = (L_x^{-1} L_{Rx}^{-1},\ R_x, R_{Rx}^{-1})$ and $T_3 = T_2^{-1} T_1 = (L_{Rx}, R_{Rx}, R_{Rx} L_{Rx}) = (L_a, R_a, R_a L_a)$, where $a = Rx$, are autotopies of $(Q, \circ)$. Thus $(Q, \cdot)$ is an LIP-quasigroup, and consequently IP-quasigroup isotopic to a Moufang loop.

Since $(Q, \circ)$ is an IP-loop, then $T_5 = (R_x L_x, IR_x I, L_x) = (R_x L_x, L_x^{-1}, L_x)$ is an autotopy too. Thus $L_x(y \circ z) = R_x L_x y \circ L_x^{-1} z$ for all $x, y, z \in Q$. This gives the identity $x \circ (y \circ (x \circ z)) = ((x \circ y) \circ x) \circ z$ and shows that $(Q, \circ)$ is a Moufang loop. Moreover, from $x^{-1} \circ (x \circ y) = (y \circ x) \circ x^{-1} = y$ it follows $R(R \cdot x) \cdot x^{-1} = (R^2 y \cdot Rx) \cdot x^{-1} = (y \cdot Rx)x^{-1} = y$. Thus $(Q, \cdot)$ is a RIP-quasigroup, and consequently, IP-quasigroup isotopic to a Moufang loop. Hence, by results proved in [3], it is a Belousov-Moufang quasigroup. □

5. Pseudo-automorphisms of i-quasigroups

A bijection θ of Q is called a *left pseudo-automorphism* of a quasigroup $(Q,\cdot)$ if there exists at least one element $k \in Q$ such that $k\cdot\theta(xy) = (k\cdot\theta x)\cdot\theta y$ for all $x, y \in Q$, i.e. if $T = (L_k\theta, \theta, L_k\theta)$ is an autotopy of a quasigroup $(Q,\cdot)$. The element k is called a *companion* of θ (cf. [1] or [9]).

Proposition 5.1. *If a bijection α of an i-quasigroup $(Q,\cdot)$ with a left neutral element f is its left pseudo-automorphism with the companion k, then k is a left Bol element, i.e. $k(x\cdot ky) = (k\cdot xk)y$ holds for all $x, y \in Q$.*

Proof. Let e_k be such that $ke_k = k$. Then $k\cdot\theta(xy) = (k\cdot\theta x)\cdot\theta y$ for $\theta x = e_k$ gives $\theta(\theta^{-1}e_k\cdot y) = \theta y$. Thus, $\theta^{-1}e_k\cdot y = y$. Hence, $e_k = \theta f$. So, $T_1 = (L_k\theta, \theta, L_k\theta)$ is an autotopy of $(Q,\cdot)$. Since, by Proposition 3.5, $(Q,\cdot)$ is an LIP-quasigroup, we have the autotopy $T_2 = (I_lL_k\theta I_l, L_k\theta, \theta)$. Then $T_3 = T_2T_1^{-1} = (\gamma, L_k, L_k^{-1})$ also is an autotopy, i.e. $L_k^{-1}(yz) = \gamma y\cdot L_kz$ for some γ. By putting $z = e_k$ we can see that $\gamma = R_k^{-1}L_k^{-1}R_{e_k}$. But $T_3^{-1} = (R_k^{-1}L_k^{-1}R_{e_k}, L_k, L_k^{-1})^{-1} = (R_{e_k}^{-1}L_kR_k, L_k^{-1}, L_k)$ also is an autotopy of $(Q,\cdot)$. Thus, $k(x\cdot ky) = R_{e_k}^{-1}(k\cdot xk)\cdot y$ for all $x, y \in Q$. □

Example. The set Q of all rational numbers with the operation $x\circ y = y - x$ is an i-quasigroup with $f = 0$ as a left neutral element. The left translation L_a of $(Q,\circ)$ is a left pseudo-automorphism with the companion $k = -\frac{a}{2}$. Hence $(Q,\circ)$ is a left Bol quasigroup.

Proposition 5.2. *If in an i-quasigroup $(Q,\cdot)$ with a left neutral element f the translations L_a and R_b are left pseudo-automorphisms with the companion k, then $a = e_kf$ and $b = e_k$, where $ke_k = k$.*

Proof. From $k\cdot L_a(xy) = (k\cdot L_ax)\cdot L_ay$ we obtain $k(a\cdot xy) = (k\cdot ax)\cdot ay$. This for $x = f$, gives $k = k\cdot af$. Thus $af = e_k$, hence $af\cdot f = e_kf$ and $a = e_kf$.

In the case of R_b the proof is very similar. □

Proposition 5.3. *If an i-quasigroup $(Q,\cdot)$ with a left neutral element f is isotopic to an abelian group, then L_a and R_b, where $a = e_kf$, $b = e_k$, are its left pseudo-automorphisms with the companion k.*

Proof. Consider the isotope $(Q,\circ)$, where $x\circ y = Rx\cdot y$. Then $(Q,\circ)$ is a loop and f is its neutral element. Thus $(Q,\circ)$, as a loop isotopic to an abelian group, is an abelian group (cf. [1] or [9]).

We will now show that L_{e_kf} is a left pseudo-automorphism of $(Q,\cdot)$ with the companion k. To this aim, note that

$$A = k\cdot L_{e_kf}(xy) = k(e_kf\cdot xy) = Rk\circ(R^2e_k\circ(Rx\circ y)) = kf\circ(e_k\circ(Rx\circ y)),$$

and

$$B = (k\cdot L_{e_kf}x)\cdot L_{e_kf}y = (k\cdot(e_kf\cdot x))\cdot(e_kf\cdot y) = R(Rk\circ(e_k\circ x))\circ(e_k\circ y)$$
$$= k\circ e_kf\circ xf\circ e_k\circ y = k\circ e_kf\circ(e_k\circ Rx\circ y).$$

Since R is an automorphism of $(Q, \circ)$ and $R^2 = \varepsilon$, we have $R(k \circ e_k f) = Rk \circ R^2 e_k = k \cdot e_k = k = R(kf)$, which implies $kf = k \circ e_k f$. Thus $A = B$. This shows that $L_{e_k f}$ is a left pseudo-automorphism of $(Q, \cdot)$ with the companion k.

Analogously, we can show that $C = D$, where $C = k \cdot R_{e_k}(xy) = k\,(xy \cdot e_k) = kf \circ (R\,(Rx \circ y) \circ e_k) = kf \circ x \circ Ry \circ e_k$ and $D = (k \cdot R_{e_k} x) \cdot R_{e_k} y = (k \cdot xe_k) \cdot (ye_k) = R(Rk \circ Rx \circ e_k) \circ Ry \circ e_k = k \circ x \circ e_k f \circ Ry \circ e_k$. □

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