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Table of marks and markaracter table of certain finite groups

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Abstract. Let G be a finite group and $C(G)$ be a family of representatives of the conjugacy classes of subgroups in G. The table of marks of G is a matrix $TM(G) = (a_{HK})$, where $H, K \in C(G)$ and a_{HK} is the number of fixed points of the right cosets of H in G under the action of K. The markaracter table of G is a matrix obtained from the table of marks of G by selecting rows and columns corresponding to cyclic subgroups of G. In this paper, the table of marks and markaracter table of some classes of finite groups are computed.

1. Introduction

Throughout this paper all groups and sets are assumed to be finite. Our calculations are done with the aid of Gap [10] and we refer to the books [5, 6] for notions and notations not presented here.

Suppose G is a finite group containing subgroups H and K. Define $C(H)$ to be the set of all conjugates of H in G and $\mathcal{K}(G) = \{C(H_1), C(H_2), \ldots, C(H_s)\}$ to be a complete set of representatives of the conjugacy classes of subgroups in G. The right cosets of H in H is denoted by $G\backslash H$. It is well-known that the action of G on $G\backslash H$ is transitive and all transitive actions have such a form up to isomorphism. The mark $\beta_H(K) = \beta_{G\backslash H}(K)$ is defined as $|Fix_{G\backslash H}(K)| = |\{Hx \in G\backslash H \mid Hxk = Hx, \forall k \in K\}$. The table of marks of G, Table 1, is the square matrix $MT(G) = (\beta_{G\backslash G_i}(G_j))$, where $G_i, G_j \in \mathcal{X}$. The table $MT(G)$ was introduced in the second edition of the famous book of W. Burnside [2]. We refer the interested reader to consult an old but interesting paper by Pfeiffer [7] for more information on this topic.

The markaracter table of a finite group was introduced by a Japanese chemist Shinsaku Fujita in the context of stereochemistry and enumeration of molecules [3]. This table can be obtained from the table of marks by removing all rows and columns corresponding to non-cyclic subgroups. The markaracter table of dihedral, generalized quaternion and groups of order pqr, p, q, r are distinct primes, were computed in some earlier paper [1, 4, 8]. The aim of this paper is to continue these works by computing the table of marks and markaracter table of certain classes of groups.

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Table 1. The table of marks of group G

$*$	$C(H_1)$	$C(H_2)$	$\cdots$	$C(H_s)$
G/H_1	$\beta_{H_1}(K_1)$	$\beta_{H_1}(K_2)$	$\cdots$	$\beta_{H_1}(K_s)$
G/H_2	$\beta_{H_2}(K_1)$	$\beta_{H_2}(K_2)$	$\cdots$	$\beta_{H_2}(K_s)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
G/H_s	$\beta_{H_s}(K_1)$	$\beta_{H_s}(K_2)$	$\cdots$	$\beta_{H_s}(K_s)$

where $K_i \in C(H_i)$ for all i.

2. Main Results

The aim of this section is to calculate the table of marks and markaracter table of the dicyclic group T_{4n}, the semi-dihedral group SD_{2^n}, and the group $H(n)$ that will be defined later. For the sake of completeness we mention here a known result about table of marks. The interested readers can be consulted an interesting paper of G. Pfeiffer [7].

Theorem 2.1. *Let G be a finite group, $\mathcal{K}(G) = \{C(H_1), C(H_2), \cdots, C(H_s)\}$ and $MT(G) = (m_{ij})$ in which $|K_i| \leqslant |K_j|$, when $K_i \in C(H_i), K_j \in C(H_j)$ and $i \leqslant j$. Then,*

1. *The matrix $M(G)$ is a lower triangular matrix,*
2. *m_{ij} divides m_{i1}, for all $1 \leqslant i, j \leqslant r$,*
3. *$m_{i1} = [G : H_i]$, for all $1 \leqslant i \leqslant s$,*
4. *$m_{ii} = [N_G(H_i) : H_i]$,*
5. *If $H_i \trianglelefteq G$, then $m_{ij} = m_{i1}$ whenever $K_j \nleq H_i$ and zero otherwise.*

2.1. Dicyclic group T_{4n}

The dicyclic group T_{4n} can be presented as $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$. The present authors [9], obtained the structure and the number of all subgroups of the dicyclic group T_{4n}. Based on the given information on subgroup lattice of dicyclic group, we know that it has two types of subgroups. The first type is cyclic subgroups of $\langle a\rangle$ and the second type is a subgroup H of index $2^l d$ conjugate to $C_{\frac{m}{d}} : Q_{\frac{2^{r+2}}{2^l}}$, where $n = 2^r m$. It is clear that $H = \langle a^n, a^j b\rangle$, $1 \leqslant j \leqslant n$, is a cyclic subgroup of order four. Thus, we have $\tau(2n)$ subgroups of the first type and the second type subgroups can be partitioned into two parts. The first part are subgroups in the form of $\langle a^d, a^j b\rangle$, where d is odd. These subgroups are all conjugate. If d is even then all subgroups in the form $\langle a^d, a^j b\rangle$, $2 \mid j$, are in a conjugacy class of subgroups and all subgroups in the form $\langle a^d, a^j b\rangle$, $2 \nmid j$, are in another conjugacy classes of subgroups. In Table 2 the table of marks are computed in two different cases that n is a prime number greater than or equal to five or $n = 3$.

Table 2. Table of marks when $n = p$ is odd prime.

$n=3$	e	$\langle x^3\rangle$	$\langle x^2\rangle$	$\langle x^3, ab\rangle$	$\langle x\rangle$	G
G/e	12	0	0	0	0	0
$G/\langle x^3\rangle$	6	6	0	0	0	0
$G/\langle x^2\rangle$	4	0	4	0	0	0
$G/\langle x^3, ab\rangle$	3	3	0	1	0	0
$G/\langle x\rangle$	2	2	2	0	2	0
e	1	1	1	1	1	1

$n\geqslant 5$	e	$\langle x^p\rangle$	$\langle x^p, ab\rangle$	$\langle x^2\rangle$	$\langle x\rangle$	G
G/e	$4p$	0	0	0	0	0
$G/\langle x^p\rangle$	$2p$	$2p$	0	0	0	0
$G/\langle x^p, ab\rangle$	p	p	1	0	0	0
$G/\langle x^2\rangle$	4	0	0	4	0	0
$G/\langle x\rangle$	2	2	0	2	2	0
e	1	1	1	1	1	1

From calculations given in [9, Section 2.2], one can see that this group has exactly $\mid \mathcal{K}(G) \mid = \tau(2n) + 2r\tau(m) + \tau(m) = \tau(2n) + \tau(m)(r+1) + r\tau(m) = \tau(2n) + \tau(n) + r\tau(m)$ subgroups. This shows that we have the following lemma:

Lemma 2.2. *The order of the table of marks of the dicyclic group T_{4n}, $n = 2^r m$ and m is odd is $\tau(2n) + \tau(n) + r\tau(m)$.*

Proposition 2.3. *In the dicyclic group T_{4n}, $m_{i2} = [G : H_i]$, for any subgroup H_i if $\langle a^n\rangle \leqslant H$. In other case, $m_{i2} = 0$.*

Proof. To prove $m_{i2} = [G : H_i]$, we put $C_2 = \langle a^n\rangle$. If $C_2 \leqslant H_i$, then by definition

$$m_{i2} = [N_G(H_i) : H_i]\cdot \mid \{H^g \mid \langle a^n\rangle \leqslant H^g \ \&\ g \in T_{4n}\}|.$$

If H is a normal subgroup then $m_{i2} = [G : H]$. Suppose $H = \langle a^d, a^j b\rangle$, $1 \leqslant j \leqslant d$ and d is even. Then $H \cong T_{4\frac{n}{d}}$ and $N_G(\langle a^d, a^j b\rangle) = \langle a^{\frac{d}{2}}, a^j b\rangle$ which implies that $[N_G(\langle a^d, a^j b\rangle) : \langle a^d, a^j b\rangle] = 2$. On the other hand, $|\{(\langle a^d, a^j b\rangle)^g \mid \langle a^n\rangle \leqslant (\langle a^d, a^j b\rangle)^g \ \&\ g \in T_{4n}\}| = \frac{d}{2}$. Now, since $[T_{4n} : \langle a^d, a^j b\rangle] = d$, we have that $m_{i2} = [T_{4n} : \langle a^d, a^j b\rangle]$. Next we assume that d is odd which shows that $\langle a^d, a^j b\rangle$ is self-normalizer. Therefore, $[N_{T_{4n}}(\langle a^d, a^j b\rangle) : \langle a^d, a^j b\rangle] = 1$. This proves that the number $H-$conjugate classes is d. □

In [6, Lemma 3.5.3(a)], it is proved that if $M(G) = [m_{ij}]$ is the table of marks of G then $m_{ij} = [N_G(H_i) : H_i]\cdot b_{ij}$, where b_{ij} is the number of subgroups conjugate to H_i which contain H_j. In particular, $m_{ii} = [N_G(H_i) : H_i]$. By this result, one can easily seen that if H_i is normal then $\beta_{G/H}(K) = [G : H]$.

Proposition 2.4. *Let d is an odd positive divisor and $H = \langle a^d, a^j b\rangle$. Then*

$$\beta_{T_{4n}/H}(K) = [T_{4n} : H] = d \text{ or } 1.$$

Proof. Since d is odd, H is a self-normalizing subgroup of T_{4n}. We first assume that $K \leqslant T_{4n}$ is normal. Then $\beta_{T_{4n}/H}(K) = |\{H^g \mid K \leqslant H^g \ \& \ g \in T_{4n}\}| = |\{H^g \mid K \leqslant H^g, g \in T_{4n}\}| = |\{H^g \mid K \leqslant H\}| = [T_{4n} : N_G(H)] = [T_{4n} : H]$. But $H \cong T_{4\frac{n}{d}}$ and so $\beta_{T_{4n}/H}(K) = d$, as desired. If K is not normal in T_{4n}, then $K = \langle a^h, a^j b\rangle$, where $h < d$. Thus $\beta_{T_{4n}/H}(K) = |\{H^g \mid \langle a^h, a^j b\rangle \leqslant H^g \ \& \ g \in T_{4n}\}| = 1$. □

By Lemma 2.2 and Propositions 2.3, 2.4 we have the following theorem:

Theorem 2.5. *The table of marks of the dicyclic group T_{4n} is given in Tables 3 and 4.*

Table 3. Table of marks of the dicyclic group T_{4n}, when $n = 2^r m$ and $3 \mid m$.

$*$	e	$\langle a^n\rangle$	$\langle a^{\frac{2n}{p_1}}\rangle$	$\langle a^{\frac{n}{2}}\rangle$	$\langle a^n, b\rangle$	$\langle a^n, ab\rangle$	$K_{j,\,7\leqslant j\leqslant s}$
G/e	$4n$	0	0	0	0	0	$\cdots$
$G/\langle a^n\rangle$	$2n$	$2n$	0	0	0	0	$\cdots$
$G/\langle a^{\frac{2n}{p_1}}\rangle$	$\frac{4n}{p_1}$	-	$\frac{4n}{p_1}$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}}\rangle$	n	n	0	n	0	0	$\cdots$
$G/\langle a^n, b\rangle$	n	n	0	0	2	0	$\cdots$
$G/\langle a^n, ab\rangle$	n	n	0	0	0	2	$\cdots$
$G/H_i, 7 \leqslant i \leqslant s$				δ_{ij}			

Table 4. Table of marks of the dicyclic group T_{4n}, when $n = 2^r m$ and $3 \nmid m$.

$*$	e	$\langle a^n\rangle$	$\langle a^{\frac{n}{2}}\rangle$	$\langle a^n, b\rangle$	$\langle a^n, ab\rangle$	$K_{j,\,6\leqslant j\leqslant s}$
G/e	$4n$	0	0	0	0	$\cdots$
$G/\langle a^n\rangle$	$2n$	$2n$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}}\rangle$	n	n	n	0	0	$\cdots$
$G/\langle a^n, b\rangle$	n	n	0	2	0	$\cdots$
$G/\langle a^n, ab\rangle$	n	n	0	0	2	$\cdots$
$G/H_i, 6 \leqslant i \leqslant s$				δ_{ij}		

In Tables 3 and 4, the quantity δ_{ij} can be computed by the following formula:

$$\delta_{ij} = \begin{cases} m_{i1} & \text{if } K_j \leqslant H_i \trianglelefteq T_{4n} \\ 2 & \text{if } K_j \leqslant H_i \leqslant T_{4n} \\ 1 & \text{if } K_j \leqslant N_{T_{4n}}(H_i) = H_i \\ 0 & \text{otherwise.} \end{cases}.$$

Suppose $\mathcal{K}(G)$ denotes the set of all conjugacy classes of a given group G. By definition of the markaracter table, one can easily seen that the markaracter table of G has exactly $\mathcal{K}(G)$ rows and columns.

We are now ready to calculate the markaracter table of the dicyclic group T_{4n}. The matrix $MC(T_{4n})$ can be obtained from $MT(T_{4n})$ in which we select rows and columns corresponding to cyclic subgroups of T_{4n}. By Lemma 2.2, the dicyclic group T_{4n}, $n = 2^r m$ and m is odd is $\tau(2n) + \tau(n) + r\tau(m)$.

Lemma 2.6. *The number of conjugacy classes of dicyclic group T_{4n} can be computed by the following formula:*

$$|\mathcal{K}(T_{4n})| = \begin{cases} \tau(2n) + 2 & 2 \mid n, \\ \tau(2n) + 1 & 2 \nmid n. \end{cases}$$

Proof. It is easy to see that for each i, i $2n$, $\langle a^i \rangle$ is a normal subgroup of T_{4n} and so there are $\tau(2n)$ conjugacy classes of cyclic subgroups of this type. Suppose n is even. Among two generator subgroups $\langle a^i, a^j b\rangle$ of T_{4n}, $\langle a^n, a^j b\rangle$ is a cyclic subgroup of order 4 and other subgroups of this form are not cyclic. On the other hand, all subgroup of the form $\langle a^n, a^j b\rangle$, j is odd, are conjugate in T_{4n}, and all subgroups of the form $\langle a^n, a^j b\rangle$, j is even, are conjugate in T_{4n}. This shows that in the case that n is even, we have exactly $\tau(2n) + 2$ conjugacy classes of cyclic subgroups. If n is even then all subgroups of the form $\langle a^n, a^j b\rangle$ (j can be odd or even) are conjugate in T_{4n} and so we have exactly $\tau(n) + 1$ conjugacy classes of cyclic subgroups in T_{4n}. This completes our argument. □

By previous lemma the non-conjugate subgroups of T_{4n} are as follows:

- $C(H_1) = \langle e \rangle$,
- $C(H_2) = \langle a^n \rangle$,
- $C(H_3) = \langle a^{\frac{2n}{3}} \rangle$,
- $C(H_4) = \langle a^{\frac{n}{2}} \rangle$,
- $C(H_5) = \langle a^n, a^j b\rangle, 2 \mid j$,
- $C(H_6) = \langle a^n, a^j b\rangle, 2 \nmid j$,
- $C(H_i)_{7 \leqslant i \leqslant s} = \langle a^{\frac{2n}{d}} \rangle, d \neq 2, 3$, where $|\mathcal{K}(T_{4n})| = s$.

By Lemma 2.6, the markaracter table of the dicyclic group T_{4n} are recorded in Tables 5 and 6 in which

$$\delta_{ij} = \begin{cases} m_{i1} & \textit{if } K_j \leqslant H_i \trianglelefteq T_{4n} \\ 2 & \textit{if } K_j \leqslant H_i \leqslant T_{4n} \\ 1 & \textit{if } K_j \leqslant N_{T_{4n}}(H_i) = H_i \\ 0 & \textit{otherwise.} \end{cases}$$

Table 5. The markaracter table of T_{4n}, when $n = 2^r m$ and $3 \mid m$.

$*$	e	$\langle a^n \rangle$	$\langle a^{\frac{2n}{p_1}} \rangle$	$\langle a^{\frac{n}{2}} \rangle$	$\langle a^n, b \rangle$	$\langle a^n, ab \rangle$	$K_{j,\, 7\leqslant i\leqslant s}$
G/e	$4n$	0	0	0	0	0	$\cdots$
$G/\langle a^n \rangle$	$2n$	$2n$	0	0	0	0	$\cdots$
$G/\langle a^{\frac{2n}{3}} \rangle$	$\frac{4n}{3}$	0	$\frac{4n}{3}$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}} \rangle$	n	n	0	n	0	0	$\cdots$
$G/\langle a^n, b \rangle$	n	n	0	0	**2**	0	$\cdots$
$G/\langle a^n, ab \rangle$	n	n	0	0	0	**2**	$\cdots$
$G/H_{i_{7\leqslant i\leqslant s}}$				δ_{ij}			

Table 6. The Markaracter Table of T_{4n}, when $n = 2^r m$ and $3 \nmid m$.

$*$	e	$\langle a^n \rangle$	$\langle a^{\frac{n}{2}} \rangle$	$\langle a^n, b \rangle$	$\langle a^n, ab \rangle$	$K_{j,\, 5\leqslant j\leqslant s}$
G/e	$4n$	0	0	0	0	$\cdots$
$G/\langle a^n \rangle$	$2n$	$2n$	0	0	0	$\cdots$
$G/\langle a^{\frac{n}{2}} \rangle$	n	n	n	0	0	$\cdots$
$G/\langle a^n, b \rangle$	n	n	0	**2**	0	$\cdots$
$G/\langle a^n, ab \rangle$	n	n	0	0	**2**	$\cdots$
$G/H_{i_{6\leqslant i\leqslant s}}$				δ_{ij}		

2.2. Table of marks of the semi-dihedral group SD_{2^n}

In [9, Section 2.5], the present authors studied the structure of subgroups of the group SD_{2^n}. From the results given the mentioned paper, we can see that we have two types of cyclic subgroups in SD_{2^n}. The first type subgroups are in the form $\langle a^d \rangle$ of order $\frac{2^{n-1}}{d}$, where $d \mid 2^{n-1}$. The second type of subgroups have the form $\langle a^d, a^k b \rangle$, where $1 \leqslant k \leqslant d$. If $2 \mid k$ then $\langle a^d, a^k b \rangle \cong D_{\frac{2n}{d}}$, and if $2 \nmid k$ then $\langle a^d, a^k b \rangle \cong Q_{\frac{2n+1}{d}}$.

Since all subgroups of the first type are normal, there are $\tau(2^{n-1}) = n$ conjugacy classes of cyclic subgroups. Among subgroups of the second time, it is easy to see that all subgroups of the form $\langle a^j b \rangle$, $1 \leqslant j \leqslant 2^{n-1}$ and $2 \mid j$, are conjugate and so these subgroups constitute a conjugacy class of subgroups in SD_{2^n}. Choose the subgroups $\langle a^{2^k}, a^j b \rangle$, $1 \leqslant j \leqslant k$ and $k \mid 2^{n-3}$. Fix a positive integer k. Then all subgroups of the form $\langle a^{2^k}, a^j b \rangle$ with even positive integer j are conjugate and so we have $2(n-2)$ conjugacy classes of subgroups of this form. The same will be happened when j varies on the set of all odd integers with condition $1 \leqslant j \leqslant k$. Hence there are $2(n-2) + n + 2 = 3n - 2$ conjugacy classes of subgroups in SD_{2^n}. Therefore, the non-conjugate subgroups of SD_{2^n} are as follows:

- $C(H_1) = \{\langle e \rangle\}$;
- $C(H_2) = \{\langle a^{2^{n-2}} \rangle\}$;

- $C(H_3) = \{\langle a^{2^{n-1}}, a^j b\rangle \mid j \text{ is even}\}$;
- $C(H_{4+3i}) = \{\langle a^{2^{n-3-i}}\rangle\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{5+3i}) = \{\langle a^{2^{n-2-i}}, a^j b\rangle, j \text{ is even}\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{6+3i}) = \{\langle a^{2^{n-1-i}}, a^j b\rangle, j \text{ is odd}\}$, $0 \leqslant i \leqslant n-3$;
- $C(H_{3n-2}) = \{\langle a, b\rangle\}$.

Therefore, we proved the following proposition:

Proposition 2.7. *The semi-dihedral group SD_{2^n} has exactly $3n-2$ conjugacy classes of subgroups.*

Theorem 2.8. *The table of marks of the semi-dihedral group SD_{2^n} is given in Table 7.*

Table 7. Table of marks of the dicyclic group SD_{2^n}.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	$\cdots$	K_s
G/H_2	2^{n-1}	2^{n-1}	0	0	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_3	2^{n-1}	0	**2**	0	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_4	2^{n-2}	2^{n-2}	0	2^{n-2}	0	0	0	0	0	0	0	0	$\cdots$	0
G/H_5	2^{n-2}	2^{n-2}	0	0	**2**	0	0	0	0	0	0	0	$\cdots$	0
G/H_6	2^{n-2}	2^{n-2}	**2**	0	0	**2**	0	0	0	0	0	0	$\cdots$	0
G/H_7	2^{n-3}	2^{n-3}	0	2^{n-3}	0	0	2^{n-3}	0	0	0	0	0	$\cdots$	0
G/H_8	2^{n-3}	2^{n-3}	0	2^{n-3}	**2**	0	0	**2**	0	0	0	0	$\cdots$	0
G/H_9	2^{n-3}	2^{n-3}	**2**	2^{n-3}	0	**2**	0	0	**2**	0	0	0	$\cdots$	0
G/H_{10}	2^{n-4}	2^{n-4}	0	2^{n-4}	0	0	2^{n-4}	0	0	2^{n-4}	0	0	$\cdots$	0
G/H_{11}	2^{n-4}	2^{n-4}	0	2^{n-4}	**2**	0	2^{n-4}	**2**	0	0	**2**	0	$\cdots$	0
G/H_{12}	2^{n-4}	2^{n-4}	**2**	2^{n-4}	0	**2**	2^{n-4}	0	**2**	0	0	**2**	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_s	1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$	1

where $s = 3n-2$.

Proof. We first calculate the entry m_{ij} in table of marks of semi-dihedral group SD_{2^n}. We claim that

$$m_{ij} = \beta_{(SD_{2^n}/H_i)}(k_j) = \begin{cases} [SD_{2^n} : H_i] & \textit{if } \; K_j \trianglelefteq H_i \trianglelefteq SD_{2^n} \text{or} K_j \leqslant H_i \trianglelefteq SD_{2^n} \\ 2 & \textit{if } \; K_j \leqslant H_i \leqslant SD_{2^n} \\ 0 & \textit{if } \; K_j \nleqslant H_i \end{cases}.$$

To prove, we assume that $K_j \trianglelefteq H_i \trianglelefteq SD_{2^n}$. Thus

$$\begin{aligned} [N_{SD_{2^n}}(H) : H] &= [SD_{2^n} : H] \\ |\{H^g \mid K \leqslant H^g \;\&\; g \in SD_{2^n}\}| &= 1. \end{aligned}$$

Since H_i is normal, $m_{ij} = \beta_{(SD_{2^n}/H_i)}(K_j) = [SD_{2^n} : H_i]$. Next we assume that $K_j \leqslant H_i \leqslant SD_{2^n}$ and H_i is not normal in SD_{2^n}. Then $[N_{SD_{2^n}}(H) : H] = 2$. We write $K_j = \langle a^r, a^j b\rangle$ and $H_i = \langle a^d, a^j b\rangle$. If $r \mid d$, then it easy see to that K_j is contained in a unique conjugate of H_i.

Since $H_i \ntrianglelefteq SD_{2^n}$ and $K_j \leqslant SD_{2^n}$,

$$\begin{aligned} N_{SD_{2^n}}(\langle a^d, a^j b\rangle) &= \langle a^{\frac{d}{2}}, a^j b\rangle, \\ |\{K_j \leqslant H_i^g \ \&\ g \in SD_{2^n}\}| &= 1. \end{aligned}$$

Finally, if $K_j \nleq H_i$ then $|\{H_i^g \mid K_j \leqslant H_i^g \ \&\ g \in SD_{2^n}\}| = 0$ and so $\beta_{SD_{2^n}/H_i}(K_j) = 0$. □

By the proof of the previous theorem, one can see that the number of cyclic subgroups of the semi-dihedral group SD_{2^n} are $n + 2^{n-3} + 2^{n-2}$. There are two conjugacy classes of subgroups of index 2^{n-1} with representatives $C_2 = \langle a^{2^{n-2}}\rangle$ and $D_2 = \langle a^2 b\rangle$. There are also two conjugacy classes of subgroup of index 2^{n-2} with representatives $C_4 = \langle a^{2^{n-3}}\rangle$ and $Q_4 = \langle a^{2^{n-2}}, ab\rangle$. For all other integers appeared as the index of a subgroup in SD_{2^n}, there exists a unique conjugacy classes of cyclic subgroups. In an exact phrase, there exists a unique subgroup of index 2^{n-3-k}, $0 \leqslant k \leqslant n-3$, generated by $a^{2^{n-4-k}}$. Therefore, there are $n+2$ conjugacy classes of cyclic subgroups. Hence we proved the following proposition:

Corollary 2.9. *The order of markaracter table in the group SD_{2^n} is equal to $s = n + 2$.*

Theorem 2.10. *The markaracter table of semi-dihedral group SD_{2^n} is given by Table 8.*

Table 8. Markaracter table of the semi-dihedral group SD_{2^n}.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	$\cdots$	K_s
G/H_1	2^n	0	0	0	0	0	0	0	$\cdots$	0
G/H_2	2^{n-1}	2^{n-1}	0	0	0	0	0	0	$\cdots$	0
G/H_3	2^{n-1}	0	**2**	0	0	0	0	0	$\cdots$	0
G/H_4	2^{n-2}	2^{n-2}	0	2^{n-2}	0	0	0	0	$\cdots$	0
G/H_5	2^{n-2}	2^{n-2}	0	0	**2**	0	0	0	$\cdots$	0
G/H_6	2^{n-3}	2^{n-3}	0	2^{n-3}	0	2^{n-3}	0	0	$\cdots$	0
G/H_7	2^{n-4}	2^{n-4}	0	2^{n-4}	0	2^{n-4}	2^{n-4}	0	$\cdots$	0
G/H_8	2^{n-5}	2^{n-5}	0	2^{n-5}	0	2^{n-5}	2^{n-5}	2^{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_s	2	2	0	2	0	2	2	2	$\cdots$	2

Proof. Apply Theorem 2.8. □

2.3. The group $H(n)$

Define $H(n) = \langle x, y, z \mid x^{2^{n-2}} = y^2 = z^2 = e, [x,y] = [y,z] = e, xz = xy\rangle$. The aim of this section is to calculate the table of marks and markaracter table of the group

$H(n)$. In [9, Section 2.6], the present authors studied the structure of subgroups of this group and proved that the normal subgroups of $H(n)$ have the following forms:

- $G_1 = \langle a^d \rangle$, where $d \mid 2^{n-2}$ and $d \neq 1$;
- $G_2 = \langle a^d, b \rangle$, where $d \mid 2^{n-2}$;
- $G_3 = \langle a^d b \rangle$, where $d \mid 2^{n-3}$ and $d \neq 1$;
- $G_4 = \langle a^d c, a^d bc \rangle$, where $d \mid 2^{n-3}$;
- $G_5 = \langle a^d, b, c \rangle$, where $d \mid 2^{n-2}$.

We now consider non-normal subgroups of $H(n)$. Suppose $d \mid 2^{n-2}$. Since $a^{-1}\langle a^d, c\rangle a = \langle a^d, bc\rangle$ and $a^{-1}\langle a^d b, a^d c\rangle a = \langle a^d b, a^d bc\rangle$, $\langle a^d, c\rangle$, $\langle a^d, bc\rangle$ and also $\langle a^d b, a^d c\rangle$, $\langle a^d b, a^d bc\rangle$ are conjugate subgroups of $H(n)$. Moreover, $c^{-1}\langle a\rangle c = \langle ab\rangle$ and so $\langle a\rangle$ and $\langle ab\rangle$ are conjugate. In what follows, we record the representatives of conjugacy classes of subgroups of $H(n)$. In the case that the conjugacy class has one or two elements, the complete conjugacy class of those subgroups are recorded.

1. $C(H_1) = \{\langle e\rangle\}$, $C(H_2) = \{\langle a^{2^{n-3}}\rangle\}$, $C(H_3) = \{\langle b\rangle\}$, $C(H_4) = \{\langle a^{2^{n-3}} b\rangle\}$, $C(H_5) = \{\langle c\rangle, \langle bc\rangle\}$, $C(H_6) = \{\langle a^{2^{n-3}} c\rangle, \langle a^{2^{n-3}} bc\rangle\}$;
2. $C(H_{7+8j}) = \{\langle a^{2^{n-4-j}}\rangle\}$, $0 \leqslant j \leqslant n-5$;
3. $C(H_{8+8j}) = \{\langle a^{2^{n-3-j}}, b\rangle\}$, $0 \leqslant j \leqslant n-5$;
4. $C(H_{9+8j}) = \{\langle a^{2^{n-4-j}} b\rangle\}$, $0 \leqslant j \leqslant n-5$;
5. $C(H_{10+8j}) = \{\langle a^{2^{n-3-j}} b, a^{2^{n-3-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
6. $C(H_{11+8j}) = \{\langle a^{2^{n-2-j}}, b, c\rangle\}$, $0 \leqslant j \leqslant n-5$;
7. $C(H_{12+8j}) = \{\langle a^{2^{n-4-j}} c\rangle, \langle a^{2^{n-4-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
8. $C(H_{13+8j}) = \{\langle a^{2^{n-3-j}}, c\rangle, \langle a^{2^{n-3-j}}, bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
9. $C(H_{14+8j}) = \{\langle a^{2^{n-3-j}} c, a^{2^{n-3-j}} b\rangle, \langle a^{2^{n-3-j}} c, a^{2^{n-3-j}} bc\rangle\}$, $0 \leqslant j \leqslant n-5$;
10. $C(H_{8n-25}) = \{\langle a^2, b\rangle\}$, $C(H_{8n-24}) = \{\langle a^2 c, a^2 bc\rangle\}$, $C(H_{8n-23}) = \{\langle a^4, b, c\rangle\}$;
11. $C(H_{8n-22}) = \{\langle a\rangle, \langle ab\rangle\}$, $C(H_{8n-21}) = \{\langle ac\rangle, \langle abc\rangle\}$;
12. $C(H_{8n-20}) = \{\langle a^2 b, a^2 bc\rangle, \langle a^2 b, a^2 c\rangle\}$, $C(H_{8n-19}) = \{\langle a^2, c\rangle, \langle a^2, bc\rangle\}$, $C(H_{8n-18}) = \{\langle a, b\rangle\}$, $C(H_{8n-17}) = \{\langle a, c\rangle\}$, $C(H_{8n-16}) = \{\langle a^2, b, c\rangle\}$, $C(H_{8n-15}) = \{\langle a, b, c\rangle\}$.

Among these classes of subgroups, conjugacy classes recorded in the cases $1, 2, 4, 7$ and 11 are related to cyclic subgroups. We now record our calculations in the following lemma:

Lemma 2.11. *There are $8n-15$ conjugacy classes of subgroups in the group $H(n)$ and among them there are $3n-4$ conjugacy classes of cyclic subgroups. In particular, the order of table of marks and markaracter table of $H(n)$ are $8n-15$ and $3n-4$, respectively.*

To calculate the table of marks of $H(n)$, we have to calculate the values $m_{ij}(H(n))$.

Proposition 2.12.

$$\delta_{ij} = \beta_{H(n)/H_i}(K_j) = \begin{cases} [H(n):H_i] & K_j \trianglelefteq H_i \trianglelefteq H(n) \text{ or } K_j \leqslant H_i \trianglelefteq H(n), \\ [N_{H(n)}(H_i):H_i] & K_j \leqslant H_i \leqslant H(n), \\ 0 & K_j \nleq H_i. \end{cases}$$

Proof. Suppose $K_j \leqslant H_i$. It is easy to see that $|N_{H(n)}(H_i)| = 2^{n-1}$, when H_i is a non-normal subgroup of $H(n)$. On the other hand,

$$\begin{aligned} \beta_{H(n)/H_i}(K_j) &= [N_{H(n)}(H_i):H_i]|\{H_i^g \mid K_j \leqslant H_i^g \ \&\ g \in H(n)\}| \\ &= [N_{H(n)}(H_i):H_i], \end{aligned}$$

proving the result. □

Theorem 2.13. *The table of marks and markaracter table of the group $H(n)$ are given in Tables 9 and 10, respectively.*

Table 9. Table of marks of the group $H(n)$.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	$K_{j,\ 7\leqslant j\leqslant 8n-15}$
$H(n)/e$	2^n	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}\rangle$	2^{n-1}	2^{n-1}	0	0	0	0	$\cdots$
$H(n)/\langle b\rangle$	2^{n-1}	0	2^{n-1}	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}b\rangle$	2^{n-1}	0	0	2^{n-1}	0	0	$\cdots$
$H(n)/\langle bc\rangle$	2^{n-1}	0	0	0	$\mathbf{2^{n-2}}$	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}bc\rangle$	2^{n-1}	0	0	0	0	$\mathbf{2^{n-2}}$	$\cdots$
$H(n)/(H_i)_{7\leqslant i\leqslant 8n-15}$					δ_{ij}		

Table 10. The markaracter table of the group $H(n)$.

$*$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
$H(n)/e$	2^n	0	0	0	0	0	0
$\mathbf{H(n)/\langle a^{2^{n-3}}\rangle}$	2^{n-1}	$\mathbf{2^{n-1}}$	0	0	0	0	0
$H(n)/\langle b\rangle$	2^{n-1}	0	2^{n-1}	0	0	0	0
$H(n)/\langle a^{2^{n-3}}b\rangle$	2^{n-1}	0	0	2^{n-1}	0	0	0
$H(n)/\langle bc\rangle$	2^{n-1}	0	0	0	2^{n-2}	0	0
$H(n)/\langle a^{2^{n-3}}bc\rangle$	2^{n-1}	0	0	0	0	2^{n-2}	0
$\mathbf{H(n)/\langle a^{2^{n-4}}\rangle}$	2^{n-2}	2^{n-2}	0	0	0	0	$\mathbf{2^{n-2}}$
$H(n)/\langle a^{2^{n-4}}b\rangle$	2^{n-2}	2^{n-2}	0	0	0	0	0
$H(n)/\langle a^{2^{n-4}}bc\rangle$	2^{n-2}	2^{n-2}	0	0	0	0	0
$\mathbf{H(n)/\langle a^{2^{n-5}}\rangle}$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/\langle a^{2^{n-5}}b\rangle$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/\langle a^{2^{n-5}}bc\rangle$	2^{n-3}	2^{n-3}	0	0	0	0	$\mathbf{2^{n-3}}$
$H(n)/H_{i,\ 13\leqslant i\leqslant 3n-4}$							δ_{ij}

$*$	K_8	K_9	K_{10}	K_{11}	K_{12}	$K_{i,\ 13\leqslant i\leqslant 3n-4}$
$H(n)/e$	0	0	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-3}}\rangle}$	0	0	0	0	0	$\cdots$
$H(n)/\langle b\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}b\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle bc\rangle$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-3}}bc\rangle$	0	0	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-4}}\rangle}$	0	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-4}}b\rangle$	2^{n-2}	0	0	0	0	$\cdots$
$H(n)/\langle a^{2^{n-4}}bc\rangle$	0	2^{n-3}	0	0	0	$\cdots$
$\mathbf{H(n)/\langle a^{2^{n-5}}\rangle}$	0	0	$\mathbf{2^{n-3}}$	0	0	$\cdots$
$H(n)/\langle a^{2^{n-5}}b\rangle$	0	0	0	2^{n-3}	0	$\cdots$
$H(n)/\langle a^{2^{n-5}}bc\rangle$	0	0	0	0	2^{n-4}	$\cdots$
$H(n)/H_{i_{13\leqslant i\leqslant 3n-4}}$						

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