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Characterization of inverse ordered semigroups by their ordered idempotents and bi-ideals

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Abstract. We prove that an ordered semigroup is complete semilattice of group-like ordered semigroups if and only if it is completely regular and inverse. The relation between principal bi-ideals generated by two inverses of an element in an inverse ordered semigroup has been presented here. Furthermore we bring the opportunity to study complete regularity on an inverse ordered semigroups by their bi-ideals.

1. Introduction

Inverse semigroups have a natural ordering which has deep impact on their structure. The study of behavior of inverses of an element in ordered semigroups had been an area of interest among the semigroup theorists since last fifty years. Bhuniya and Hansda [1] have deal with ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. Class of these ordered semigroups are natural generalization of class of inverse semigroups (without order). We call these ordered semigroups as inverse ordered semigroups.

We characterize inverse ordered semigroups by their ordered idempotents. We study complete regularity in an inverse ordered semigroup and explore the look of resulting ordered semigroup. Keeping in mind that bi-ideals have been studied more, we give several characterizations of inverse ordered semigroups by their bi-ideals.

2. Preliminaries

An ordered semigroup is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$ $a \leqslant b$ implies $xa \leqslant xb$ and $ax \leqslant bx$. It is denoted by $(S, \cdot, \leqslant)$.

For every $H \subseteq S$, we define $(H] = \{t \in S : t \leqslant h, \text{ for some } h \in H\}$.

Throughout this paper unless otherwise stated S stands for an ordered semigroup. An equivalence relation ρ is called a *left* (*right*) *congruence* on S if for $a, b, c \in S$ $a\rho b$ implies $ca\rho cb$ ($ac\rho bc$). By a *congruence* we mean both left and

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right congruence. A congruence ρ is called a *semilattice congruence* on S if for all $a, b \in S$ $a\rho a^2$ and $ab\rho ba$. By a *complete semilattice congruence* on S we mean a semilattice congruence σ on S such that for $a, b \in S$ $a \leqslant b$ implies that $a\sigma ab$. An ordered semigroup S is called a *complete semilattice of subsemigroups* of type τ if there exists a complete semilattice congruence ρ such that $(x)_\rho$ is a type τ subsemigroup of S.

Let I be a nonempty subset of an ordered semigroup S. I is a *left* (*right*) *ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is an *ideal* of S if it is both a left and a right ideal of S.

Following Kehayopulu [4], a nonempty subset B of an ordered semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$ and $(B] = B$. Here our aim is to study completely regular and inverse ordered semigroups by their bi-ideals.

The principal [5] left ideal, right ideal, ideal and bi-ideal [4] generated by $a \in S$ are denoted by $L(a)$, $R(a)$, $I(a)$and $B(a)$ respectively and have form

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS],\ I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } B(a) = (a \cup aSa].$$

Kehayopulu [5] defined Greens relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows:

$$a\mathcal{L}b \text{ if } L(a) = L(b),$$
$$a\mathcal{R}b \text{ if } R(a) = R(b),$$
$$a\mathcal{J}b \text{ if } I(a) = I(b),$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

A regular ordered semigroup S is said to be a *group-like* (resp. *left group-like*) [1] *ordered semigroup* if for every $a, b \in S$, $a \in (Sb]$ and $b \in (aS]$ (resp. $a \in (Sb]$). A right group-like ordered semigroup can be defined dually. Two elements $a, b \in S$ are said to *$\mathcal{H}$-related* if $a\mathcal{H}b$. An ordered semigroup S is called an *regular* (*completely regular*) [3] if for every $a \in S$, $a \in (aSa]$ ($a \in (a^2Sa^2]$). An element $b \in S$ is *inverse* of a if $a \leqslant aba$ and $b \leqslant bab$. The set of all inverses of an element $a \in S$ is denoted by $V_\leqslant(a)$. Two elements $a, b \in S$ are said to *$\mathcal{H}$-commutative* [1] if $ab \leqslant bxa$ for some $x \in S$. A regular ordered semigroup S is called *inverse* [1] if for every $a \in S$ and $a', a'' \in V_\leqslant(a)$, $a'\mathcal{H}a''$, that is, any two inverses of a are $\mathcal{H}-$related.

By an *ordered idempotent* [1] in an ordered semigroup S, we shall mean an element $e \in S$ such that $e \leqslant e^2$. We denote the set of all ordered idempotents of S by $E_\leqslant(S)$.

For the convenience of readers we state the following three results from [1].

Lemma 2.1. *Let S be completely regular ordered semigroup. Then for every $a \in S$ there is $x \in S$ such that $a \leqslant axa^2$ and $a \leqslant a^2xa$.*

Theorem 2.2. *An ordered semigroup S is completely regular if and only if for all $a \in S$ there exists $a' \in V_\leqslant(a)$ such that $aa' \leqslant a'ua$ and $a'a \leqslant ava'$ for some $u, v \in S$.*

Lemma 2.3. *Let S be a completely regular ordered semigroup. Then following statements hold in S:*

1. *$\mathcal{J}$ is the least complete semilattice congruence on S.*

2. *S is a complete semilattice of completely simple ordered semigroups.*

3. Inverse ordered semigroup

Let S be an ordered semigroup and ρ be an equivalence on S. We say thatl an ideal I of S is generated by a ρ-unique element $b \in S$ if $b\rho x$ for any generator x of I.

Definition 3.1. A regular ordered semigroup S is called *inverse* if for every $a \in S$, any two inverses of a are $\mathcal{H}$-related.

Example 3.2. The ordered semigroup $S = \{a, e, f\}$ with the multiplication defined below and with the discrete order is an inverse ordered semigroup.

$\cdot$	a	e	f
a	a	e	f
e	f	e	a
f	e	a	f

We present a role of ordered idempotents in an inverse ordered semigroup in the next theorem.

Theorem 3.3. *An ordered semigroup S is inverse if and only if every principal left ideal and every principal right ideal of S are generated by an $\mathcal{H}$-unique ordered idempotent.*

Proof. Suppose that S is inverse. Let I be a principal left ideal of S. Then there exists $e \in E_{\leqslant}(S)$ such that $I = (Se]$. If possible let $I = (Sf]$ for some $f \in E_{\leqslant}(S)$. Then $e\mathcal{L}f$ and thus $e \leqslant xf$ and $f \leqslant ye$ for some $x, y \in S$. Now $e \leqslant ee \leqslant eee \leqslant exfe$. Therefore $exf \leqslant exfexf$ so that $exf \in E_{\leqslant}(S)$. Also $exf \leqslant exfexf \leqslant exf(fe)exf$ and $fe \leqslant feee \leqslant fexfe \leqslant fe(exf)fe$. Therefore $fe \in V_{\leqslant}(exf)$. Also $exf \in V_{\leqslant}(exf)$. Since S is inverse, we have $fe\mathcal{H}exf$. Then $e \leqslant ee \leqslant exffe \leqslant fezexf$ for some $z \in S$, and so $e \leqslant fz_1$, where $z_1 = ezexf$. Similarly $f \leqslant ez_2$ for some $z_2 \in S$. So $e\mathcal{R}f$. Hence $e\mathcal{H}f$. Likewise every principal right ideal of S generated by certain $\mathcal{H}$-unique ordered idempotent.

Conversely assume that given condition holds in S. Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Clearly $(Sa] = (Sa'a] = (Sa''a]$. Since $a'a, a''a \in E_{\leqslant}(S)$ we have that $a'a\mathcal{H}a''a$, by given condition. Then there are $s, t \in S$ such that $a' \leqslant a''asa'$ and $a'' \leqslant a'ata''$. Thus $a'\mathcal{R}a''$. Likewise $a'\mathcal{L}a''$, that is $a'\mathcal{H}a''$. Hence S is an inverse ordered semigroup. □

In the following we show that an ordered semigroup S is inverse if and only if any two ordered idempotents of S are $\mathcal{H}$-commutative.

Theorem 3.4. *The following conditions are equivalent on an ordered semigroup S.*

1. *S is an inverse semigroup;*
2. *S is regular and its idempotents are $\mathcal{H}$-commutative;*
3. *For every $e, f \in E_{\leqslant}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$.*

Proof. (1) $\Rightarrow$ (2): Obviously S is regular. Let us assume that $a \in S$ and $a', a'' \in V_{\leqslant}(a)$.

Consider $e, f \in E_{\leqslant}(S)$. Since S is regular, so there is $x \in S$ such that $x \in V_{\leqslant}(ef)$. Now $x \leqslant xefx$ implies that $fxe \leqslant fxe(ef)fxe$ and $ef \leqslant efxef$ implies $ef \leqslant ef(fxe)ef$. Thus $ef \in V_{\leqslant}(fxe)$. Also $fxe \leqslant fxefxe$ that is $fxe \in E_{\leqslant}(S)$. So $fxe \in V_{\leqslant}(fxe)$. Since S is inverse, so $fxe\mathcal{H}ef$. Then there are $s_1, s_2 \in S$ such that $ef \leqslant fxes_1$ and $ef \leqslant s_2fxe$. Now $ef \leqslant efxef$ implies that $ef \leqslant f(xes_1xs_2fx)e$. Therefore $ef \leqslant fye$, where $y = xes_1xs_2fx$. Similarly there is $z \in S$ such that $fe \leqslant ezf$. Hence any two idempotents are $\mathcal{H}$-commutative.

(2) $\Rightarrow$ (3): Let $e, f \in E_{\leqslant}(S)$ be such that $e\mathcal{L}f$. Then $e \leqslant xf$ and $f \leqslant ye$ for some $x, y \in S$. Now $e \leqslant xf$ implies $e \leqslant exf$, and so $e \leqslant ee \leqslant exfe$ which implies that $exf \leqslant exfexf$. So $exf \in E_{\leqslant}(S)$. Similarly $fye \in E_{\leqslant}(S)$. Now $e \leqslant exf \leqslant exff \leqslant exffye$. Since $exf, fye \in E_{\leqslant}(S)$, by condition (2) we have $exffye \leqslant (fye)z(exf)$ for some $z \in S$. Hence $e \leqslant ft$, where $t = yezexf$. Similarly $f \leqslant ew$ for some $w \in S$, so that $e\mathcal{R}f$. Hence $e\mathcal{H}f$. If $e\mathcal{R}f$ then $e\mathcal{H}f$ can be done dually.

(3) $\Rightarrow$ (1): Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Now $aa' \leqslant aa''aa'$ and $aa'' \leqslant aa'aa''$. So $aa'\mathcal{R}aa''$ which implies that $aa'\mathcal{H}aa''$, by the condition (3). Also $a'a\mathcal{H}a''a$. Then $a' \leqslant a'aa'$ gives that $a' \leqslant a''axa$ for some $x \in S$. Therefore $a' \leqslant a''t$ where $t = axa$. In similar way it is possible to obtained $u, v, w \in S$ such that $a' \leqslant ua''$, $a'' \leqslant a'v$ and $a'' \leqslant wa'$. So $a'\mathcal{H}a''$. Hence S is an inverse ordered semigroup.

□

Lemma 3.5. *Let S be an inverse ordered semigroup. Then following statements hold in S.*

1. *$a\mathcal{L}b$ if and only if $a'a\mathcal{H}b'b$ for some $a, b \in S$ and $a' \in V_{\leqslant}(a)$ $b' \in V_{\leqslant}(b)$;*
2. *$a\mathcal{R}b$ if and only if $aa'\mathcal{H}bb'$ for some $a, b \in S$ and $a' \in V_{\leqslant}(a)$ $b' \in V_{\leqslant}(b)$;*
3. *for any $a \in S$ and $e \in E_{\leqslant}(S)$ there are $x, y \in S$ such that $aexa', a'eya \in E_{\leqslant}(S)$; where $a' \in V_{\leqslant}(a)$.*
4. *for any $a, b \in S$ there are $x, y \in S$ such that $ab \leqslant abb'xa'ab$ and $b'a' \leqslant b'a'aybb'a'$, where $a' \in V_{\leqslant}(a)$ and $b' \in V_{\leqslant}(b)$.*

Proof. (1): Let $a, b \in S$ be such that $a\mathcal{L}b$. Let $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$. Since $a \leqslant aa'a$ and $a'a \leqslant a'aa'a$, we have $a\mathcal{L}a'a$ which implies that $b\mathcal{L}a'a$. Also $b\mathcal{L}b'b$. Hence $a'a\mathcal{L}b'b$. Since $a'a, b'b \in E_{\leqslant}(S)$ and S is inverse we have $a'a\mathcal{H}b'b$, by Theorem 3.4(3).

Conversely suppose that given condition holds in S. Let $a, b \in S$ with $a' \in V_{\leqslant}(a)$ and $b' \in V_{\leqslant}(b)$. Then by given condition $aa'\mathcal{H}bb'$. Also we have $a\mathcal{L}a'a$ and $b\mathcal{L}b'b$ so that $a\mathcal{L}b$.

(2): This is similar to (1).

(3): Let $a \in S$ and $e \in E_{\leqslant}(S)$. Also $a'a \in E_{\leqslant}(S)$. Since S is an inverse, there is an $x \in S$ such that $a'ae \leqslant exa'a$, by Theorem 3.4(2). Now $aexa' \leqslant aa'aeexa' \leqslant aexa'aexa'$. So $aexa' \in E_{\leqslant}(S)$. Likewise $a'eya \in E_{\leqslant}(S)$; for some $y \in S$.

(4): Let $a, b \in S$ with $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$. So $a'a, b'b \in E_{\leqslant}(S)$. Now $ab \leqslant aa'abb'b \leqslant$ and $a'abb' \leqslant b'bxa'a$, by Theorem 3.4(2). Thus $ab \leqslant abb'xa'ab$. Likewise $b'a' \leqslant b'a'aybb'a'$; for some $y \in S$. □

In the following theorem an inverse ordered semigroup has been characterized by the inverse of an element of the set $(eSf]$.

Theorem 3.6. *Let S be an ordered semigroup and $e, f \in E_{\leqslant}(S)$. Then S is inverse if and only if for every $x \in (eSf]$ implies $x' \in (fSe]$, where $x' \in V_{\leqslant}(x)$.*

Proof. First suppose that S is an inverse ordered semigroup and $x \in (eSf]$. Then $x \leqslant es_1f$ for some $s_1 \in S$. Let $x' \in V_{\leqslant}(x)$. Now $x' \leqslant x'xx' \leqslant x'es_1fx'$, and so $es_1fx' \leqslant es_1fx'es_1fx'$. Hence $es_1fx' \in E_{\leqslant}(S)$. Similarly $x'es_1f \in E_{\leqslant}(S)$. Now there is $s_2 \in S$ such that $x'es_1fx' \leqslant x'es_1ffx' \leqslant fs_2x'es_1fx'$, by Theorem 3.4(2) . Also $fs_2x'es_1fx' \leqslant fs_2x'ees_1fx' \leqslant fs_2x'es_1fx's_3e$, for some $s_3 \in S$. Then $x' \leqslant x'xx'$ implies that $x' \leqslant fs_2x'es_1fx' \leqslant fs_2x'es_1fx's_3e$. Hence $x' \in (fSe]$.

Conversely assume that the given conditions hold in S. First consider a left ideal L of S such that $L = (Se] = (Sf]$ for $e, f \in E_{\leqslant}(S)$. Then $e\mathcal{L}f$, so that $e \leqslant ee \leqslant ezf$ for some $z \in S$. Therefore $e \in (eSf]$. Since $e \in V_{\leqslant}(e)$ we have $e \in (fSe]$, by given condition. Likewise $f \in (eSf]$. This implies that $e\mathcal{R}f$ and so $e\mathcal{H}f$. Similarly it can be shown that every principal right ideal of S generated by $\mathcal{H}$-unique ordered idempotent. Thus by Theorem 3.3, S is an inverse ordered semigroup. □

Corollary 3.7. *The following conditions are equivalent on a regular ordered semigroup S.*

(1) *S is an inverse ordered semigroup;*

(2) *for any $a \in S$ and for any $a' \in V_{\leqslant}(a)$, $aa', a'a$ are $\mathcal{H}$-commutative;*

(3) *for any $e \in E_{\leqslant}(S)$, any two inverses of e are $\mathcal{H}$-related;*

(4) *for any $e \in E_{\leqslant}(S)$ and all its inverses are $\mathcal{H}$-commutative;*

5) *for any* $e \in E_{\leqslant}(S)$ *and* $e' \in V_{\leqslant}(e)$, ee' *and* $e'e$ *are* $\mathcal{H}$*-commutative.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (5): These are obvious.

(5) $\Rightarrow$ (1): Let $e, f \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(ef)$. So $ef \leqslant efxef \leqslant effxeef$ and $x \leqslant xefx$ implies that $fxe \leqslant fxeeffxe$. So $ef \in V_{\leqslant}(fxe)$. Also $fxe \in E_{\leqslant}(S)$. Now $ef \leqslant efxef \leqslant effxeef \leqslant effxefxeef \leqslant fxez_1efz_2fxe$, for some $z_1, z_2 \in S$, by the given condition. So $ef \leqslant fz_3e$ where $z_3 = xemefnfx$. Similarly $fe \leqslant ez_4f$, for some $z_4 \in S$. So e, f are $\mathcal{H}$-commutative. Hence by Theorem 3.4 S is inverse ordered semigroup. □

We study inverse ordered semigroup together with complete regularity in the following theorem.

Theorem 3.8. *The following conditions are equivalent on a regular ordered semigroup* S.

(1) S *is inverse and completely regular;*

(2) S *is a complete semilattice of group like ordered semigroups;*

(3) $ab\mathcal{H}ba$ *whenever* $ab, ba \in E_{\leqslant}(S)$*;*

(4) *any ordered idempotent of* S *is* $\mathcal{H}$*-commutative to any element of* S*;*

(5) *for any* $e, f \in E_{\leqslant}(S)$ $e\mathcal{J}f$ *implies* $e\mathcal{H}f$*;*

(6) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{J}$.

Proof. (1) $\Rightarrow$ (2): Let S be a completely regular and inverse ordered semigroup. Then by Lemma 2.3, $\mathcal{J}$ is the complete semilattice congruence on S and every $\mathcal{H}$-class is a group-like ordered semigroup. We now prove $\mathcal{H} = \mathcal{J}$. Let $a, b \in S$ be such that $a\mathcal{J}b$. So there are $x, y, u, v \in S$ such that $a \leqslant xby$ and $b \leqslant uav$. Since S is completely regular, so there are $h, g, f \in S$ such that $x \leqslant x^2hx$, $b \leqslant b^2gb$, $b \leqslant bgb^2$, $y \leqslant yfy^2$, by Lemma 2.3. Now $a \leqslant x^2hxb^2gbyfy^2 \leqslant x^2hxb^2gbgb^2yfy^2$.

Let $p \in V_{\leqslant}(x^2hxb^2g)$. So

$$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2g(b^2gpx^2h)x^2hxb^2g$$

and

$$b^2gpx^2h \leqslant b^2gpx^2hxb^2gpx^2h \leqslant b^2gpx^2h(x^2hxb^2g)b^2gpx^2h.$$

This shows that $b^2gpx^2h \in V_{\leqslant}(x^2hxb^2g)$. Also

$$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2g(x^2hxb^2gp^2)x^2hxb^2g$$

and

$$x^2hxb^2gp^2 \leqslant x^2hxb^2gpx^2hxb^2gp^2 \leqslant x^2hxb^2gp^2(x^2hxb^2g)x^2hxb^2gp^2,$$

which implies that $x^2hxb^2gp^2 \in V_{\leqslant}(x^2hxb^2g)$. Similarly $p^2x^2hxb^2g \in V_{\leqslant}(x^2hxb^2g)$. Since $b^2gpx^2h, x^2hxb^2gp^2 \in V_{\leqslant}(x^2hxb^2g)$ and S is inverse, so there is $t \in S$ such that $x^2hxb^2gp^2 \leqslant b^2gpx^2ht$. Thus

$x^2hxb^2g \leqslant x^2hxb^2gpx^2hxb^2g \leqslant x^2hxb^2gp^2(x^2hxb^2g)^2$

implies that $x^2hxb^2g \leqslant b^2gpx^2hxt(x^2hxb^2g)^2 = bs$ where $s = bgpx^2ht(x^2hxb^2g)^2$.

Similarly there is $s_1 \in S$ such that $b^2gyfy^2 \leqslant s_1b$. Hence $a \leqslant x^2hxb^2gbyfy^2 \leqslant bsbyfy^2 = bs_2$, where $s_2 = sbyfy^2$. Similarly $a \leqslant s_3b$ for some $s_3 \in S$. Likewise $b \leqslant s_4a$ and $b \leqslant as_5$, for some $s_4, s_5 \in S$. So $a\mathcal{H}b$. Thus $\mathcal{J} \subseteq \mathcal{H}$. Also $\mathcal{H} \subseteq \mathcal{J}$, and Hence $\mathcal{J} = \mathcal{H}$. Therefore S is complete semilattice of group-like ordered semigroups.

(2) $\Rightarrow$ (3): Suppose that S is a complete semilattice Y of group like ordered semigroups $\{S_\alpha\}_{\alpha\in Y}$. Let $a, b \in S$ such that $ab, ba \in E_\leqslant(S)$. Let ρ be the corresponding semilattice congruence on S. Then there is $\alpha \in Y$ such that $ab, ba \in S_\alpha$. Since S_α is group like ordered semigroups, so $ab\mathcal{H}ba$.

(3) $\Rightarrow$ (4): Let $a \in S$ and $e \in E_\leqslant(S)$. Since S is regular there is an $x \in S$ such that $a \leqslant axa$. Clearly $ax, xa \in E_\leqslant(S)$. Thus by condition (3) $ax\mathcal{H}xa$. So $xa \leqslant axu$ and $ax \leqslant vxa$, for some $u, v \in S$. Then we have $a \leqslant axa \leqslant axaxa \leqslant axaxaxa \leqslant aaxuxvxaa = a^2ta^2$, where $t = xuxvx$. Now $a \leqslant a^2ta^2 \leqslant a(a^2ta^2ta^2ta^2)a \leqslant a^2(a^2ta^2ta^2ta^2ta^2)a$, that is $a \leqslant a^2ya$, where $y = a^2ta^2ta^2ta^2ta^2$. Similarly $a \leqslant aya^2$. Clearly $a^2y, ya^2 \in E_\leqslant(S)$.

Let $e, f \in E_\leqslant(S)$ and $x \in V_\leqslant(ef)$. Then we have $x \leqslant xefx$. So $fxe \leqslant fxefxe \leqslant fxeeffxe$ and $ef \leqslant efxef \leqslant effxeef$. So $ef \in V_\leqslant(fxe)$. Also $ef \leqslant effxeef$ implies that $effxe \leqslant effxeeffxe$, and $fxeef \leqslant fxeeffxeef$. So $effxe, fxeef \in E_\leqslant(S)$ and thus $effxe\mathcal{H}fxeef$, by the condition (3). Then there are $u, v \in S$ such that $effxe \leqslant fxeefu$ and $fxeef \leqslant veffxe$. Now $ef \leqslant effxefxeef \leqslant fxeefuveffxe = fce$; where $c = xe^2fuvef^2x$. Likewise $fe \leqslant edf$, for some $d \in S$.

Now $ae \leqslant a^2yae$. Let $z \in V_\leqslant(a^2yae)$. So

$$a^2yae \leqslant a^2yaeza^2yae \leqslant a^2yae(eza^2y)a^2yae.$$

Clearly $a^2yaeeza^2y, eza^2ya^2yae \in E_\leqslant(S)$ and thus $a^2yaeeza^2y\mathcal{H}eza^2ya^2yae$, by condition (3). Now $ae \leqslant a^2yae \leqslant a^2yaeeza^2ya^2yae \leqslant eza^2ys_1a^2yaea^2yae$, for some $s_1 \in S$. So $ae \leqslant es_2ae$, where $s_2 = za^2ys_1a^2yaea^2y$. Again $ae \leqslant es_2aya^2e \leqslant es_2aes_3ya^2$, for some $s_3 \in S$, since $ya^2, e \in E_\leqslant(S)$. That is $ae \leqslant es_4a$, for some $s_4 \in S$. Similarly $ea \leqslant as_5e$, for some $s_5 \in S$. So a, e are $\mathcal{H}$-commutative.

(4) $\Rightarrow$ (5): Let $e, f \in E_\leqslant(S)$ such that $e\mathcal{J}f$. Then there are $x, y, z, u \in S$ such that $e \leqslant xfy$ and $f \leqslant zeu$. Now $e \leqslant xfy$ implies that $e \leqslant fhxy$ and $e \leqslant xykf$ by the given condition for some $h, k \in S$. Similarly $f \leqslant zeu$ gives $f \leqslant es_1zu$ and $f \leqslant zus_2e$ for some $s_1, s_2 \in S$. Hence $e\mathcal{H}f$.

(5) $\Rightarrow$ (6): Let $a, b \in S$ such that $a\mathcal{J}b$. Then there are $s, t, u, v \in S$ such that $a \leqslant sbt$ and $b \leqslant uav$. Since S is regular so $a \leqslant axa$ and $b \leqslant byb$ for some $x, y \in S$ so that $ax \leqslant axax$ and $by \leqslant byby$. Now $axax \leqslant axsbtx \leqslant axsbybtx$ that is $ax \leqslant axsbybtx$. Likewise $by \leqslant byuaxavy$. Thus $ax\mathcal{J}by$, and so from given condition $ax\mathcal{H}by$. Similarly $xa\mathcal{H}yb$. So there is $c \in S$ such that $ax \leqslant byc$, that is $a \leqslant byca = bd$, for some $d = yca \in S$. Likewise $a \leqslant pb$, $b \leqslant qa$ for some $p, q \in S$.

Thus $a\mathcal{H}b$. So $\mathcal{H} = \mathcal{J}$. Now $\mathcal{J} = \mathcal{H} = \mathcal{L} \cap \mathcal{R}$ gives $\mathcal{J} \subseteq \mathcal{L}$ and $\mathcal{J} \subseteq \mathcal{R}$. Therefore $\mathcal{L} = \mathcal{J} = \mathcal{R}$.

(6) $\Rightarrow$ (1): Let $a \in S$. Since S is regular so there exists $a' \in V_{\leqslant}(a)$. Clearly $a\mathcal{L}a'a$ and $a\mathcal{R}aa'$. So by the given condition $a\mathcal{R}a'a$ and $a\mathcal{L}aa'$. Now $a \leqslant aa'a \leqslant aa'aa'a \leqslant aa'aa'aa'a \leqslant aas_1a's_2aa$ for some $s_1, s_2 \in S$. So $a \leqslant a^2pa^2$ where $p = s_1a's_2$. So S is completely regular.

Also let $a', a'' \in V_{\leqslant}(a)$. Now $a\mathcal{L}a'a\mathcal{L}a''a$ implies that $a\mathcal{R}a'a\mathcal{R}a''a$. Also by the given condition we can show that $a\mathcal{L}aa'\mathcal{L}a''a$. So it is to check that $a'\mathcal{R}a''$ and $a'\mathcal{L}a''$. So $a'\mathcal{H}a''$. Hence S is inverse ordered semigroup. □

4. Bi-ideals in inverse ordered semigroups

Following Hansda [2] an ordered semigroup S is *completely regular* if and only if for every $a \in S$ there is some $e \in E_{\leqslant}(S)$ such that $a \leqslant ae$, $a \leqslant ea$ and $B(a) = B(e)$. Here our approach allows one to see the role of principal bi-ideal generated by an inverse of an element in an inverse ordered semigroup.

Lemma 4.1. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) *S is a completely regular ordered semigroup;*

(2) *for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(a) = B(a')$;*

(3) *for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a) \cap B(a') = B(a') \cap B(a) = B(a'a)$;*

(4) *$B(a) = B(a^2)$ for any $a \in S$.*

Proof. (1) $\Rightarrow$ (2): First suppose that S is completely regular ordered semigroup. Let $a \in S$. Then by Theorem 2.2 there is $a' \in V_{\leqslant}(a)$ such that $aa' \leqslant a'ua$ and $a'a \leqslant ava'$ for some $u, v \in S$. Let $x \in B(a)$. Therefore $x \leqslant a$ or $x \leqslant as_1a$ for some $s_1 \in S$. If $x \leqslant a$ then $x \leqslant aa'a \leqslant aa'aa'a \leqslant a'uaaava' = a'za'$ where $z = uaaav$. Again if $x \leqslant as_1a$ then there is $t \in S$ such that $x \leqslant a'ta'$. Therefore in either case $x \in B(a')$. Also $a \in B(a')$. So $B(a) \subseteq B(a')$. Similarly $B(a') \subseteq B(a)$. Hence $B(a) = B(a')$.

(2) $\Rightarrow$ (3): Suppose that condition (2) holds. Let $a \in S$. Then there is $a' \in V_{\leqslant}(a)$ such that $a \leqslant aa'a$. Let $x \in B(aa')$. Then $x \leqslant aa'$ or $x \leqslant aa'saa'$ for some $s \in S$. By given condition $a' \in B(a)$. So $a' \leqslant a$ or there is $y \in S$ such that $a' \leqslant aya$. If $x \leqslant aa'saa'$ and $a' \leqslant aya$ then $x \leqslant aa'saaya$. If $x \leqslant aa'saa'$ and $a' \leqslant a$ then $x \leqslant aa'saa$. If $x \leqslant aa'$ and $a' \leqslant a$ then $x \leqslant aa$. Also if $x \leqslant aa'$ and $a' \leqslant aya$ then $x \leqslant aaya$. Therefore in either case $x \in B(a)$. Hence $B(aa') \subseteq B(a)$. Likewise $B(aa') \subseteq B(a')$ and hence $B(aa') \subseteq B(a) \cap B(a')$.

Let $w \in B(a) \cap B(a')$. So $w \in B(a)$ and $w \in B(a')$. Therefore $w \leqslant a$ or $w \leqslant as_2a$ and $w \leqslant a'$ or $w \leqslant a's_3a'$ for some $s_2, s_3 \in S$. Since S is regular, there is $d \in S$

such that $w \leqslant wdw$. If $w \leqslant a$ and $w \leqslant a'$ then $w \leqslant wdw \leqslant ada' \leqslant aa'ada'aa'$. If $w \leqslant as_2a$ and $w \leqslant a'$ then $w \leqslant wdw \leqslant as_2ada' \leqslant aa'as_2ada'aa'$. If $w \leqslant as_2a$ and $w \leqslant a's_3a'$ then $w \leqslant wdw \leqslant as_2ada's_3a' \leqslant aa'as_2ada's_3a'aa'$. If $w \leqslant a$ and $w \leqslant a's_3a'$ then $w \leqslant wdw \leqslant ada's_3a' \leqslant aa'ada's_3a'aa'$. Therefore in either case $w \in B(aa')$. Hence $B(a) \cap B(a') \subseteq B(aa')$. Thus $B(aa') = B(a) \cap B(a')$.

(3) $\Rightarrow$ (4): Suppose that condition (3) holds. Let $a \in S$. Then there exists $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a'a)$. Now $a \leqslant aa'a \leqslant aa'aa'a = a(a'a)a'(aa')a$. Now by condition (3) $a'a \leqslant aa'zaa'$ and $aa' \leqslant a'awa'a$ for some $z, w \in S$. Then $a \leqslant a(a'a)a'(aa')a$ implies that $a \leqslant a(aa'zaa')a'(a'awa'a)a = a^2(a'zaa'a'a'awa')a^2$. Thus $B(a) \subseteq B(a^2)$. It is evident that $B(a^2) \subseteq B(a)$ and hence $B(a) = B(a^2)$.

(4) $\Rightarrow$ (1): Suppose condition (4) holds. Therefore $a \leqslant a^2$ or $a \leqslant a^2s_2a^2$ and $a^2 \leqslant a$ or $a^2 \leqslant as_3a$ for some $s_2, s_3 \in S$. Therefore in either case $a\mathcal{H}a^2$. Since S is regular, so $a \leqslant aza$ for some $z \in S$. So $a \leqslant aza \leqslant a^2s_4zs_5a^2$ for some $s_4, s_5 \in S$. Hence S is completely regular ordered semigroup. □

Corollary 4.2. *A regular ordered semigroup S is completely regular if and only if for any $a \in S$ there is $a' \in V_{\leqslant}(a)$ such that $B(aa') = B(a) \cap B(a') = B(a'a) = B(a) = B(a')$.*

Proof. This follows from Lemma 4.1 . □

Theorem 4.3. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) *S is an inverse ordered semigroup;*

(2) *for any $a \in S$, $B(a') = B(a'')$ for every $a', a'' \in V_{\leqslant}(a)$;*

(3) *for any $e, f \in E_{\leqslant}(S)$, $B(ef) = B(e) \cap B(f)$;*

(4) *for any $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$, $B(ex) = B(xe)$.*

Proof. (1) $\Rightarrow$ (2): First suppose that S is an inverse ordered semigroup. Let $a \in S$ and $a', a'' \in V_{\leqslant}(a)$. Suppose $x \in B(a')$. Therefore $x \leqslant a'$ or $x \leqslant a'ya'$ for some $y \in S$. Since S is inverse, so $a'\mathcal{H}a''$. If $x \leqslant a'$ then $x \leqslant a'aa' \leqslant a''s_1as_2a''$ for some $s_1, s_2 \in S$. Therefore $x \leqslant a''sa''$ where $s = s_1as_2$. Again if $x \leqslant a'ya'$ then there is $s_3 \in S$ such that $x \leqslant a''s_3a''$. Therefore in either case $x \in B(a'')$. Also $a' \in B(a'')$. So $B(a') \subseteq B(a'')$. Similarly $B(a'') \subseteq B(a')$. Hence $B(a') = B(a'')$.

(2) $\Rightarrow$ (3): First suppose that condition (2) holds and let $e, f \in E_{\leqslant}(S)$. Let $x \in V_{\leqslant}(ef)$. Therefore $ef \leqslant efxef$ and $x \leqslant xefx$. So $fxe \leqslant fxefxe$. Therefore $fxe \in E_{\leqslant}(S)$. Also $ef \leqslant ef(fxe)ef$ and $fxe \leqslant fxe(ef)fxe$. Therefore $ef, fxe \in V_{\leqslant}(fxe)$. So by the condition $B(ef) = B(fxe)$. Clearly $ef\mathcal{H}fxe$.

Let $w \in B(ef)$. Therefore $w \leqslant ef$ or $w \leqslant efs_1ef$ for some $s_1 \in S$. If $w \leqslant ef$ then $w \leqslant ef \leqslant efxef \leqslant efxs_2fxe$ for some $s_2 \in S$. Again if $w \leqslant efs_1ef$ then $w \leqslant efs_1ef \leqslant efs_1s_2fxe$. So in either case $w \in B(e)$. Similarly $w \in B(f)$. Hence $w \in B(e) \cap B(f)$. Therefore $B(ef) \subseteq B(e) \cap B(f)$.

Again let $y \in B(e) \cap B(f)$. So $y \leqslant e$ or $y \leqslant es_4e$ and $y \leqslant f$ or $y \leqslant fs_5f$, for some $s_4, s_5 \in S$. Since S is inverse, there exists $z \in V_{\leqslant}(y)$ such that $z \leqslant zyz$ and $y \leqslant yzy$. If $y \leqslant es_4e$ and $y \leqslant fs_5f$ then $z \leqslant zyz \leqslant zes_4ez$. Therefore $es_4ez \leqslant es_4ezes_4ez$. So $es_4ez \in E_{\leqslant}(S)$. Similarly $zfs_5f \in E_{\leqslant}(S)$. Now $es_4ez \leqslant es_4ezes_4ez \leqslant es_4ez(yz)es_4ez$ and $yz \leqslant yzyz \leqslant yz(es_4ez)yz$. Therefore $es_4ez, yz \in V_{\leqslant}(es_4ez)$. So condition (2) $B(es_4ez) = B(yz)$. Similarly $B(zfs_5f) = B(zy)$. Clearly $es_4ez\mathcal{H}yz$ and $zfs_5f\mathcal{H}zy$. Now $y \leqslant yzy \leqslant es_4ezfs_5f \leqslant es_4ezyzfs_5f \leqslant ees_4ezyzfs_5ff \leqslant eyzs_6ys_7zyf \leqslant efs_5fzs_6ys_7zes_4ef$ for some $s_6, s_7 \in S$. If $y \leqslant e$ and $y \leqslant f$ then clearly $B(ez) = B(yz)$ and $B(zf) = B(zy)$. Now $y \leqslant yzy \leqslant ezf \leqslant eezyzff \leqslant eyzs_8ys_9zyf \leqslant efzs_8ys_9zef$ for some $s_8, s_9 \in S$. If $y \leqslant e$ and $y \leqslant fs_5f$ then $zfs_5f \in E_{\leqslant}(S)$. Now $y \leqslant yzy \leqslant ezfs_5f \leqslant eezfs_5ff \leqslant ezfs_5fs_{10}ef \leqslant ezfs_5ffs_{10}ef \leqslant efs_{11}zfs_5fs_{10}ef$ for some $s_{10}, s_{11} \in S$. Again if $y \leqslant es_4e$ and $y \leqslant f$ then $es_4ez \in E_{\leqslant}(S)$. Now $y \leqslant yzy \leqslant es_4ezf \leqslant ees_4ezff \leqslant efs_{12}es_4ezf \leqslant efs_{12}ees_4ezf \leqslant efs_{12}es_4ezs_{13}ef$ for some $s_{12}, s_{13} \in S$. Therefore in either case $y \in B(ef)$ and so $B(e) \cap B(f) \subseteq B(ef)$. Hence $B(e) \cap B(f) = B(ef)$.

(3) $\Rightarrow$ (4): First suppose that condition (3) holds in S. Let $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$ so $e, xe, ex \in E_{\leqslant}(S)$. By condition (3) $B(exe) = B(e) \cap B(xe)$,that is, $B(e) = B(e) \cap B(xe)$. Therefore $B(e) \subseteq B(xe)$. Again $B(xee) = B(e) \cap B(xe)$ that is $B(xe) = B(e) \cap B(xe)$. So $B(xe) \subseteq B(e)$. Therefore $B(e) = B(xe)$. Similarly $B(e) = B(ex)$. Therefore $B(xe) = B(ex)$.

(4) $\Rightarrow$ (1): Suppose that condition (4) holds in S. Now $ex \in B(e)$ and $ex \in B(x)$. So $ex \leqslant e$ or $ex \leqslant eb_1e$, and $ex \leqslant x$ or $ex \leqslant xb_2x$ for some $b_1, b_2 \in S$. If $ex \leqslant e$ and $ex \leqslant x$ then $ex \leqslant exex \leqslant xe \leqslant xexe = xae$ where $a = ex$. If $ex \leqslant e$ and $ex \leqslant xb_2x$ then $ex \leqslant exex \leqslant xb_2xe = xbe$ where $b = b_2x$. If $ex \leqslant eb_1e$ and $ex \leqslant x$ then $ex \leqslant exex \leqslant xeb_1e = xce$ where $c = eb_1$. Again if $ex \leqslant eb_1e$ and $ex \leqslant xb_2x$ then $ex \leqslant exex \leqslant xb_2xeb_1e = xde$ where $d = b_2xeb_1$. Therefore in either case $ex \leqslant xse$ for some $s \in S$. Similarly $xe \leqslant etx$ for some $t \in S$. Hence e, x are $\mathcal{H}$-commutative. So S is an inverse ordered semigroup, by Corollary 3.7. □

Corollary 4.4. *A regular ordered semigroup S is inverse if and only if for any $e \in E_{\leqslant}(S)$ and $x \in V_{\leqslant}(e)$, $B(ex) = B(e) \cap B(x) = B(xe) = B(e) = B(x)$.*

Corollary 4.5. *A regular ordered semigroup S is inverse if and only if for any $e, f \in E_{\leqslant}(S)$, $e\mathcal{L}f(e\mathcal{R}f)$ implies $B(e) = B(f)$.*

Proof. Let S be an inverse ordered semigroup. Since S is inverse, $e\mathcal{L}f(e\mathcal{R}f)$ implies $e\mathcal{H}f$ by Theorem 3.4. So it is easy to check that $B(e) = B(f)$.

Conversely suppose that the condition holds in S. Now $B(e) = B(f)$ gives that $e \in B(f)$ and $f \in B(e)$. Therefore $e \leqslant f$ or $e \leqslant fxf$ and $f \leqslant e$ or $f \leqslant eye$. In either case $e\mathcal{R}f$. So $e\mathcal{L}f$ implies $e\mathcal{H}f$. Hence S is inverse ordered semigroup by Theorem 3.4. □

Lemma 4.6. *Let S be an inverse ordered semigroup and $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$, where $a, b \in S$. Then following conditions hold on S:*

(1) $a\mathcal{L}b$ *if and only if* $B(a'a) = B(b'b)$.

(2) $a\mathcal{R}b$ *if and only if* $B(aa') = B(bb')$.

Proof. (1): Let S be an inverse ordered semigroup and $a' \in V_{\leqslant}(a)$, $b' \in V_{\leqslant}(b)$ where $a, b \in S$. So by Lemma 3.5 $a'a\mathcal{H}b'b$. Let $x \in B(a'a)$. Therefore $x \leqslant a'a$ or $x \leqslant a'as_1a'a$ for some $s_1 \in S$. So it is easy to verify that $x \in B(b'b)$. Also $a'a \in B(b'b)$. So $B(a'a) \subseteq B(b'b)$. Similarly $B(b'b) \subseteq B(a'a)$. So $B(a'a) = B(b'b)$.

The converse statement is obvious.

(2): Analogously as (1). □

Characterization of ordered semigroups which are both completely regular and inverse have been presented in the next theorem.

Theorem 4.7. *Let S be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) *S is completely regular and inverse ordered semigroup;*

(2) *for any $a, b \in S$, $B(ab) = B(ba) = B(a) \cap B(b)$;*

(3) *$B(ab) = B(ba)$ where $a, b \in S$ and $ab, ba \in E_{\leqslant}(S)$;*

(4) *for any $a, b \in S$, $a\mathcal{L}b$ implies $B(a) = B(b)$.*

Proof. (1) ⇒ (2): First suppose that S is completely regular and inverse ordered semigroup. Then any ordered idempotent of S is $\mathcal{H}$ commutative to any element of S by Theorem 3.8 . Let $a, b \in S$. Since S is regular, so there are $p, q, r \in S$ such that $a \leqslant apa$, $b \leqslant bqb$ and $ab \leqslant abrab$. Clearly $bq, pa \in E_{\leqslant}(S)$. Now $ab \leqslant abrab \leqslant abqbrapab \leqslant bqp_1abrabp_2pa = bs_2a$ where $s_2 = qp_1abrabp_2pa$. Let $x \in B(ab)$. Therefore $x \leqslant ab$ or $x \leqslant abs_1ab$ for some $s_1 \in S$. If $x \leqslant abs_1ab$, then $x \leqslant abs_1bs_2a$. So $x \leqslant aya$ where $y = bs_1bs_2$. Again if $x \leqslant ab$, then $x \leqslant abrab \leqslant abrbs_2a$. So in either case $x \in B(a)$. Also $ab \in B(a)$. Similarly $x \in B(b)$ and $ab \in B(b)$. Hence $B(ab) \subseteq B(a) \cap B(b)$.

Again let $y \in B(a) \cap B(b)$. So $y \leqslant a$ or $y \leqslant as_4a$ and $y \leqslant b$ or $y \leqslant bs_5b$ for some $s_4, s_5 \in S$. Since S is regular, So there is $z \in S$ such that $y \leqslant yzy$. Now if $y \leqslant as_4a$ and $y \leqslant bs_5b$ then $y \leqslant yzy \leqslant as_4azbs_5b \leqslant as_4azbqbs_5b \leqslant abqs_6s_4azbs_5b \leqslant abqs_6s_4apazbs_5b \leqslant abqs_6s_4azbs_5s_7pab$ for some $s_6, s_7 \in S$. Again if $y \leqslant a$ and $y \leqslant b$ then $y \leqslant yzy \leqslant azb \leqslant apazbqb \leqslant abqs_8pazb \leqslant abqs_8zs_9pab$ for some $s_8, s_9 \in S$. Again if $y \leqslant a$ and $y \leqslant bs_5b$ then $y \leqslant yzy \leqslant azbs_5b \leqslant apazbqbs_5b \leqslant abqs_{10}pazbs_5b \leqslant abqs_{10}zbs_5s_{11}pab$ for some $s_{10}, s_{11} \in S$. Also if $y \leqslant as_4a$ and $y \leqslant b$ then $y \leqslant yzy \leqslant as_4azb \leqslant as_4apazbqb \leqslant abqs_{12}s_4apazb \leqslant abqs_{12}s_4azs_{13}pab$ for some $s_{12}, s_{13} \in S$. Therefore in either case $y \in B(ab)$. Hence $B(a) \cap B(b) \subseteq B(ab)$. Therefore $B(ab) = B(a) \cap B(b) = B(b) \cap B(a) = B(ba)$.

(2) ⇒ (3): Suppose that the given condition (2) holds. Therefore $B(ab) = B(a) \cap B(b) = B(b) \cap B(a) = B(ba)$.

(3) $\Rightarrow$ (4): First suppose that condition (3) holds and let $a\mathcal{L}b$. So there exists $s, t \in S$ such that $a \leqslant sb$ and $b \leqslant ta$. Since S is regular, $a \leqslant aza$ and $z \leqslant zaz$ for some $z \in V_{\leqslant}(a)$. Clearly $az, za \in E_{\leqslant}(S)$. Now $z \leqslant zaz \leqslant zsbz$. So $zsb \leqslant zsbzsb$. Therefore $zsb \in E_{\leqslant}(S)$. Similarly $bzs \in E_{\leqslant}(S)$. So by the condition (3) $B(zsb) = B(bzs)$. Clearly $zsb\mathcal{H}bzs$. Similarly $za\mathcal{H}az$. Let $x \in B(a)$. Therefore $x \leqslant a$ or $x \leqslant as_1a$ for some $s_1 \in S$. If $x \leqslant a$ then $x \leqslant a \leqslant aza \leqslant azsb \leqslant zas_2sb \leqslant zsbs_2sb \leqslant bzss_3s_2sb$ for some $s_2, s_3 \in S$. Similarly if $x \in as_1a$ then $x \leqslant bs_4b$ for some $s_4 \in S$. So in either case $x \in B(b)$. Therefore $B(a) \subseteq B(b)$. Similarly $B(b) \subseteq b(a)$. Therefore $B(a) = B(b)$.

Conversely suppose that the given condition holds, that is $a\mathcal{L}b$ implies $B(a) = B(b)$ for any $a, b \in S$. Now $B(a) = B(b)$ implies that $a\mathcal{R}b$. So $a\mathcal{L}b$ implies that $a\mathcal{R}b$. Therefore $\mathcal{L} \subseteq \mathcal{H}$. Also $\mathcal{H} \subseteq \mathcal{L}$. Hence $\mathcal{L} = \mathcal{H}$. So S is completely regular and an inverse ordered semigroup by Theorem 3.8. □

Corollary 4.8. *A regular ordered semigroup S is completely regular and inverse if and only if for any $e \in E_{\leqslant}(S)$ and for any $a \in S$, $B(ea) = B(e) \cap B(a)$.*

Proof. For $b = e$ we obtain the result. □

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