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The transitivity of primary conjugacy in a class of semigroups

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Abstract. Elements a, b of a semigroup S are said to be *primarily conjugate* or just *p-conjugate*, if there exist $x, y \in S^1$ such that $a = xy$ and $b = yx$. The p-conjugacy relation generalizes conjugacy in groups, but for general semigroups, it is not transitive. Finding the classes of semigroups in which this notion is transitive is an open problem. The aim of this note is to show that for semigroups satisfying $xy \in \{yx, (xy)^n\}$ for some $n > 1$, primary conjugacy is transitive.

By a notion of conjugacy for a class of semigroups, we mean an equivalence relation defined in the language of that class of semigroups such that when restricted to groups, it coincides with the usual notion of conjugacy.

Before introducing the notion of conjugacy that will occupy us, we recall some standard definitions and notation (we generally follow [4]). For a semigroup S, we denote by S^1 the semigroup S if S is a monoid; otherwise S^1 denotes the monoid obtained from S by adjoining an identity element 1.

Any reasonable notion of semigroup conjugacy should coincide in groups with the usual notion. Elements a, b of a group G are conjugate if there exists $g \in G$ such that $a = g^{-1}bg$. Conjugacy in groups has several equivalent formulations that avoid inverses, and hence generalize syntactically to any semigroup. For many of these notions including the one we focus on here, we refer the reader to [2, 5, 8].

For example, if G is a group, then $a, b \in G$ are conjugate if and only if $a = uv$ and $b = vu$ for some $u, v \in G$. Indeed, if $a = g^{-1}bg$, then setting $u = g^{-1}b$ and $v = g$ gives $uv = a$ and $vu = b$; conversely, if $a = uv$ and $b = vu$ for some $u, v \in G$, then setting $g = v$ gives $g^{-1}bg = v^{-1}vuv = uv = a$.

This last formulation was used to define the following relation on a free semigroup S (see [9]):

$$a \sim_{\mathrm{p}} b \qquad \iff \qquad \exists_{u,v \in S^1} \quad a = uv \text{ and } b = vu.$$

If S is a free semigroup, then $\sim_{\mathrm{p}}$ is an equivalence relation on S [9, Cor.5.2], and so it can be considered as a notion of conjugacy in S. In a general semigroup S, the relation $\sim_{\mathrm{p}}$ is reflexive and symmetric, but not transitive. If $a \sim_{\mathrm{p}} b$ in a semigroup, we say that a and b are *primarily conjugate* or just p-conjugate for short (hence the subscript in $\sim_{\mathrm{p}}$); a and b were said to be "primarily related" in [8].

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Lallement [9] credited the idea of the relation $\sim_{\mathrm{p}}$ to Lyndon and Schützenberger [10].

In spite of its name, $\sim_{\mathrm{p}}$ is a valid notion of conjugacy only in the class of semigroups in which it is transitive. Otherwise, the transitive closure $\sim_{\mathrm{p}}^*$ of $\sim_{\mathrm{p}}$ has been defined as a conjugacy relation in a general semigroup [3, 7, 8]. Finding classes of semigroups in which $\sim_{\mathrm{p}}$ itself is transitive, that is, $\sim_{\mathrm{p}}=\sim_{\mathrm{p}}^*$, is an open problem. The aim of this note is to prove the following theorem.

Theorem. *Let $n > 1$ be an integer and let S be a semigroup satisfying the following: for all $x, y \in S$,*

$$xy \in \{yx, (xy)^n\} .$$

Then primary conjugacy $\sim_p$ is transitive in S.

There are various motivations for studying this particular class of semigroups. First, it naturally generalizes two classes of semigroups in which $\sim_{\mathrm{p}}$ is transitive.

Proposition. *Let S be a semigroup.*

(1) *If S is commutative, then $\sim_p$ is transitive.*

(2) *If S satisfies $xy = (xy)^2$ for all $x, y \in S$, then $\sim_p$ is transitive.*

Proof. (1). In a commutative semigroup, $\sim_{\mathrm{p}}$ is the identity relation and hence it is trivially transitive.

(2). If $a \sim_{\mathrm{p}} b$, then $a = uv$ and $b = vu$ for some $u, v \in S^1$. Thus $a^2 = (uv)^2 = uv = a$ and $b^2 = (vu)^2 = vu = b$ so that a, b are idempotents. In particular, a, b are completely regular elements of S. The restriction of $\sim_{\mathrm{p}}$ to the set of completely regular elements is a transitive relation [6]. □

The other motivation for studying this class of semigroups is that it has been of recent interest in other contexts. In particular, J. P. Araújo and M. Kinyon [1] showed that a semigroup satisfying $x^3 = x$ and $xy \in \{yx, (xy)^2\}$ for all x, y is a semilattice of rectangular bands and groups of exponent 2.

The proof of Theorem was found by first proving the special cases $n = 2, 3, 4$ using the automated theorem prover `Prover9` developed by McCune [11]. After studying these proofs, the pattern became apparent, leading to the proof of the general case. Note that `Prover9` and other automated theorem provers usually cannot handle statements like our theorem directly because there is not a way to specify that n is a fixed positive integer. Thus the approach of examining a few special cases and then extracting a human proof of the general case is the most efficient way to use an automated theorem prover in these circumstances.

Proof of Theorem. Suppose $a, b, c \in S$ satisfy $a \sim_{\mathrm{p}} b$ and $b \sim_{\mathrm{p}} c$. Since $a \sim_{\mathrm{p}} b$, there exist $a_1, a_2 \in S^1$ such that $a = a_1a_2$ and $b = a_2a_1$. Similarly, since $b \sim_{\mathrm{p}} c$, there exist $b_1, b_2 \in S^1$ such that $b = b_1b_2$ and $c = b_2b_1$. We want to prove there exist $x, y \in S^1$ such that $a = xy$ and $c = yx$. If $a = b$ or if $b = c$, then there is nothing to prove. Thus we may assume without loss of generality that $a_1a_2 \neq a_2a_1$ and $b_2b_1 \neq b_1b_2$.

Assume first that $n = 2$. Then

$$a = a_1a_2 = (a_1a_2)(a_1a_2) = a_1(a_2a_1)a_2 = a_1ba_2 = (a_1b_1)(b_2a_2)\,,$$

and

$$c = b_2b_1 = (b_2b_1)(b_2b_1) = b_2(b_1b_2)b_1 = b_2bb_1 = (b_2a_2)(a_1b_1)\,.$$

Thus setting $x = a_1b_1$ and $y = b_2a_2$, we have $a \sim_{\mathrm{p}} c$ in this case.

Now assume $n > 2$. We have

$$\begin{aligned} a = a_1a_2 = (a_1a_2)^n &= \underbrace{(a_1a_2)\cdots(a_1a_2)}_{n} \\ &= a_1\underbrace{(a_2a_1)\cdots(a_2a_1)}_{n-1}a_2 \\ &= a_1b^{n-1}a_2 \\ &= a_1bb^{n-2}a_2 \\ &= a_1(b_1b_2)b^{n-2}a_2 \\ &= (a_1b_1)(b_2b^{n-2}a_2) \end{aligned}$$

and

$$\begin{aligned} c = b_2b_1 = (b_2b_1)^n &= \underbrace{(b_2b_1)\cdots(b_2b_1)}_{n} \\ &= b_2\underbrace{(b_1b_2)\cdots(b_1b_2)}_{n-1}b_1 \\ &= b_2b^{n-1}b_1 \\ &= b_2b^{n-2}bb_1 \\ &= b_2b^{n-2}(a_2a_1)b_1 \\ &= (b_2b^{n-2}a_2)(a_1b_1)\,. \end{aligned}$$

Thus setting $x = a_1b_1$ and $y = b_2b^{n-2}a_2$, we have that $a \sim_{\mathrm{p}} c$. □

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