Semirings which are distributive lattices of weakly left $k$-Archimedean semirings

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Abstract. We introduce a binary relation $\rightarrow$ on a semiring $S$, and generalize the notion of left $k$-Archimedean semirings and introduce weakly left $k$-Archimedean semirings, via the relation $\rightarrow$. We also characterize the semirings which are distributive lattices of weakly left $k$-Archimedean semirings.

1. Introduction

The notion of the semirings was introduced by Vanniver [12] in 1934. The underlying algebra in idempotent analysis [6] is a semiring. Recently idempotent analysis have been used in theoretical physics, optimization etc., various applications in theoretical computer science and algorithm theory [5, 7]. Though the idempotent semirings have been studied by many authors like Monico [8], Sen and Bhuniya [11] and others as a $(2, 2)$ algebraic structure, idempotent semirings are far different from the semirings whose multiplicative reduct is just a semigroup and additive reduct is a semilattice. So for better understanding about the abstract features of the particular semirings $\mathbb{R}_{max}$ (Maslov’s dequantization semiring), Max-Plus algebra, syntactic semirings we need a separate attention to the semirings whose additive reduct is a semilattice. From the algebraic point of view while studying the structure of semigroups, semilattice decomposition of semigroups, an elegant technique, was first defined and studied by Cliford [4]. This motivated Bhuniya and Mondal to study on the structure of semirings whose additive reduct is a semilattice [1, 2, 9, 10]. In [1], Bhuniya and Mondal studied the structure of semirings with a semilattice additive reduct. There, the description of the least distributive lattice congruence on such semirings was given. In [10], Mondal and Bhuniya gave the distributive lattice decompositions of the semirings into left $k$-Archimedean semirings. In this paper we generalize the notion of left $k$-Archimedean semirings introducing weakly left $k$-Archimedean semirings, analogous to the notion of weakly left $k$-Archimedean semigroups [3] and characterize the semirings which

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are distributive lattices of weakly left \(k\)-Archimedean semirings.

The preliminaries and prerequisites for this article have been discussed in section
2. In section 3 we introduce the notion of weakly left \(k\)-Archimedean semirings.
We give a sufficient condition for a semiring \(S\) to be weakly left \(k\)-Archimedean
in terms of a binary relation \(\xrightarrow{l}\) on \(S\). We also give a condition under which a
weakly left \(k\)-Archimedean semiring becomes a left \(k\)-Archimedean semiring.
In section 4 we characterize the semirings which are distributive lattices of weakly
left \(k\)-Archimedean semirings.

2. Preliminaries and prerequisites

A semiring \((S, +, \cdot)\) is an algebra with two binary operations + and \(\cdot\) such that
both the additive reduct \((S, +)\) and the multiplicative reduct \((S, \cdot)\) are semigroups
and such that the following distributive laws hold:

\[ x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz. \]

Thus the semirings can be viewed as a common generalization of both rings and
distributive lattices. A band is a semigroup \(F\) in which every element is an idempotent.
Moreover if it is commutative, then \(F\) is called a semilattice. Throughout
the paper, unless otherwise stated, \(S\) is always a semiring with semilattice additive
reduct.

Every distributive lattice \(D\) can be regarded as a semiring \((D, +, \cdot)\) such that
both the additive reduct \((D, +)\) and the multiplicative reduct \((D, \cdot)\) are semilattices
together with the absorptive law:

\[ x + xy = x \quad \text{for all} \quad x, y \in S. \]

An equivalence relation \(\rho\) on \(S\) is called a congruence relation if it is compatible
with both the addition and multiplication, i.e., for \(a, b, c \in S\), \(apb\) implies \((a + c)\rho(b + c), ac\rho b\)
and \(a\rho b\). A congruence relation \(\rho\) on \(S\) is called a distributive lattice congruence on
\(S\) if the quotient semiring \(S/\rho\) is a distributive lattice. Let \(C\) be a class of semirings which we call \(C\)-semirings. A semiring \(S\) is called a
distribution lattice of \(C\)-semirings if there exists a congruence \(\rho\) on \(S\) such that
\(S/\rho\) is a distributive lattice and each \(\rho\)-class is a semiring in \(C\).

Let \(S\) be a semiring and \(\phi \neq A \subseteq S\). Then the \(k\)-closure of \(A\) is defined by
\(\overline{A} = \{ x \in S \mid x + a_1 = a_2 \text{ for some } a_i \in A \} = \{ x \in S \mid x + a = a \text{ for some } a \in A \},\)
and the \(k\)-radical of \(A\) by \(\sqrt{A} = \{ x \in S \mid (\exists n \in \mathbb{N}) x^n \in A \}.\) Then \(\overline{A} \subseteq \sqrt{A}\)
by definition, and \(A \subseteq \overline{A}\) since \((S, +)\) is a semilattice. A non empty subset \(L\) of \(S\) is
called a left (resp. right) ideal of \(S\) if \(L + L \subseteq L\), and \(SL \subseteq L\) (resp. \(LS \subseteq L\)). A
collection of left ideal \(L\) of \(S\) is called a left ideal of \(S\) if it is both left and a right ideal
of \(S\). An ideal (resp. left ideal) \(A\) of \(S\) is called a \(k\)-ideal (left \(k\)-ideal) of \(S\) if and
only if \(\overline{A} = A\).
Lemma 2.1. (cf. [1]) Let $S$ be a semiring.

(a) For $a, b \in S$ the following statements are equivalent

(i) There are $s_1, t_1 \in S$ such that $b + s_1a = s_2a$.

(ii) There are $s, t \in S$ such that $b + sat = sat$.

(iii) There is $x \in S$ such that $b + xax = xax$.

(b) If $a, b, c \in S$ such that $b + xax = xax$ and $c + yay = yay$ for some $x, y \in S$, then there is $z \in S$ such that $b + zaz = zaz = c + zaz$.

(c) If $a, b, c \in S$ such that $c + xax = xax$ and $c + yby = yby$ for some $x, y \in S$, then there is $z \in S$ such that $c + zaz = zaz$ and $c + zbz = zbz$.

Lemma 2.2. (cf. [1]) For a semiring $S$ and $a, b \in S$ the following statements hold.

1. $SaS$ is a $k$-ideal of $S$.

2. $\sqrt{SaS} = \sqrt{SaS}$.

3. $b^m \in \sqrt{SaS}$ for some $m \in \mathbb{N} \Leftrightarrow b^k \in \sqrt{SaS}$ for all $k \in \mathbb{N}$.

Lemma 2.3. (cf. [10]) Let $S$ be a semiring.

(a) For $a, b \in S$ the following statements are equivalent:

(i) there are $s \in S$ such that $b + sa = sa$.

(ii) there are $s \in S$ such that $b + sa = sa$.

(b) If $a, b, c \in S$ such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.

Theorem 2.4. (cf. [10]) The following conditions on a semiring $S$ are equivalent:

1. $S$ is a distributive lattice of left $k$-Archimedean semirings,

2. for all $a, b \in S$, $b \in SaS$ implies that $b \in \sqrt{Sa}$,

3. for all $a, b \in S$, $ab \in \sqrt{Sa}$,

4. $\sqrt{L}$ is a $k$-ideal of $S$, for every left $k$-ideal $L$ of $S$,

5. $\sqrt{Sa}$ is a $k$-ideal of $S$, for all $a \in S$,

6. for all $a, b \in S$, $\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$.
3. Weakly left $k$-Archimedean semirings

In [1], Bhunia and Mondal studied the structure of semirings, and during this they gave the description of the least distributive lattice congruence on a semiring $S$ stem from the divisibility relation defined by: for $a, b \in S$, $a \mid b \iff b \in \sqrt{Sa}$.

Thus it follows from the Lemma 2.1, $a \rightarrow b \iff b^n \in \sqrt{Sa}$ for some $n \in \mathbb{N}$.

In this section we introduce the relation $\rightarrow_l$ (left analogue of $\rightarrow$) on a semiring $S$, the notion of weakly left $k$-Archimedean semirings and study them.

**Proposition 3.1.** Let $S$ be a semiring. Then $Sa$ is a left $k$-ideal of $S$ for every $a \in S$.

**Proof.** For $b, c \in \sqrt{Sa}$, there is $x \in S$ such that $b + xa = xa = c + xa$, by Lemma 2.3. This implies $(b + c) + xa = xa$, i.e., $b + c \in \sqrt{Sa}$. Moreover, for any $s \in S$ we get $sb + sxa = sxu$, and so $sb \in \sqrt{Sa}$. For $u \in \sqrt{Sa}$ there is some $b \in \sqrt{Sa}$ such that $u + b = b$. Using again $b + xa = xa$ for some $x \in S$, we get $u + xa = u + b + xa = b + xa = xa$, i.e., $u \in \sqrt{Sa}$. So $\sqrt{Sa}$ is a left $k$-ideal of $S$.

Now we introduce the relation $\rightarrow_l$ on a semiring $S$ as a generalization of the division relation $\mid$, and they are given by: for $a, b \in S$, $a \mid_l b \iff b \in \sqrt{Sa}$.

Thus $a \rightarrow_l b$ if there exist some $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$, by Lemma 2.3.

In [10], Mondal and Bhunia defined *left $k$-Archimedean semirings* as: A semiring $S$ is called left $k$-Archimedean if for all $a \in S$, $S = \sqrt{Sa}$. For example, let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, define $+$ and $\cdot$ on $S = A \times A$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\max\{a, c\}, \max\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, b).$$

Then $(S, +, \cdot)$ is a left $k$-Archimedean semiring.

We now introduce a more general notion:

A semiring $S$ will be called *weakly left $k$-Archimedean* if $ab \rightarrow_l b$, for all $a, b \in S$.

**Example 3.2.** Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, define $+$ and $\cdot$ on $S = A \times A$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\max\{a, c\}, \max\{b, d\}), \quad (a, b) \cdot (c, d) = (ac, d).$$
Then \((S, +, \cdot)\) is a weakly left \(k\)-Archimedean semiring. Now let \((a, \frac{1}{2}), (c, \frac{1}{3}) \in S\). If possible, let there exist \(n \in \mathbb{N}\) and \((x, y) \in S\) satisfying \((a, \frac{1}{2})^n + (x, y) \cdot (c, \frac{1}{3}) = (x, y) \cdot (c, \frac{1}{3})\). This implies \((a^n, \frac{1}{2}) + (xc, \frac{1}{3}) = (xc, \frac{1}{3})\) so that \(\max\{a^n, xc\} = xc, \max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{3}\), which is not possible. Consequently, \((S, +, \cdot)\) is not a left \(k\)-Archimedean semiring.

Here we see that the relation \(\rightarrow\) is not symmetric on a semiring \(S\) in general. For, consider the Example 3.2, there \((a, \frac{1}{2}) \rightarrow (c, \frac{1}{3})\) but not \((c, \frac{1}{3}) \rightarrow (a, \frac{1}{2})\). Although, the semiring \(S\) is weakly left \(k\)-Archimedean. Now, in the following proposition we show that if the relation \(\rightarrow\) is symmetric on a semiring \(S\), then \(S\) is weakly left \(k\)-Archimedean.

**Proposition 3.3.** A semiring \(S\) is weakly left \(k\)-Archimedean if the relation \(\rightarrow\) is symmetric on \(S\).

**Proof.** Let \(\rightarrow\) is a symmetric relation on \(S\) and \(a, b \in S\). Now \(ab \in \sqrt{Sb}\) implies that \(b \rightarrow ab\) and so \(ab \rightarrow b\), by symmetry of \(\rightarrow\) on \(S\). Thus \(S\) is weakly left \(k\)-Archimedean. \(\Box\)

Thus the condition of symmetry of \(\rightarrow\) is only sufficient for a semiring \(S\) to be weakly left \(k\)-Archimedean, not necessary. Let \(S\) be a left \(k\)-Archimedean semiring, and \(a, b \in S\). Then \(b \in \sqrt{Sb}\) implies that \(b^n + sa = sa\) for some \(n \in \mathbb{N}\) and \(s \in S\). Multiplying \(b\) on both sides on the right we get \(b^{n+1} + sab = sab\). This yields \(ab \rightarrow b\) so that \(S\) is a weakly left \(k\)-Archimedean semiring. Thus we have the following proposition:

**Proposition 3.4.** Every left \(k\)-Archimedean semiring \(S\) is a weakly left \(k\)-Archimedean semiring.

Here in the following proposition we find a condition for which the converse holds:

**Proposition 3.5.** Let \(S\) be a semiring, and \(ab \in \sqrt{Sb}\), for all \(a, b \in S\) hold. Then \(S\) is left \(k\)-Archimedean semiring if it is weakly left \(k\)-Archimedean.

**Proof.** Let \(a, b \in S\). Then \(ba \rightarrow a\), whence by Lemma 2.3, there are \(n \in \mathbb{N}\) and \(s \in S\) such that \(a^n + sba = sba\). Again by hypothesis, there are \(m \in \mathbb{N}\) and \(t \in S\) such that \((sba)^m + tsb = tsb\). Now \(a^n + sba = sba\) implies that \(a^{nm} + (sba)^m = (sba)^m\). Adding \(tsb\) on both sides we get \(a^{nm} + [(sba)^m + tsb] = [(sba)^m + tsb]\), i.e. \(a^{nm} + tsb = tsb \in \sqrt{Sb}\). So \(a \in \sqrt{Sb}\). Thus \(S\) is a left \(k\)-Archimedean semiring. \(\Box\)

Now, by Theorem 2.4, we see that a weakly left \(k\)-Archimedean semiring will be a left \(k\)-Archimedean semiring if it is a distributive lattice of left \(k\)-Archimedean semirings.
4. Lattices of weakly left $k$-Archimedean semirings

In this section we characterize the semirings which are distributive lattices of weakly left $k$-Archimedean semirings. A semiring $S$ is called a distributive lattice of weakly left $k$-Archimedean semirings if there exists a congruence $\rho$ on $S$ such that $S/\rho$ is a distributive lattice and each $\rho$-class is a weakly left $k$-Archimedean semiring.

Lemma 4.1. Suppose $S$ is a distributive lattice $D$ of subsemirings $S_{\alpha}, \alpha \in D$. Then $a, b \in S_{\alpha}, \alpha \in D$, then $a \xrightarrow{l} b$ in $S$ implies that $a \xrightarrow{l} b$ in $S_{\alpha}$.

Proof. Let $\rho$ be a distributive lattice congruence on $S$ so that $S$ is a distributive lattice $D$ of subsemirings $S_{\alpha}, \alpha \in D$. Let $a \xrightarrow{l} b$. Then $b^n + xa = xa$ for some $n \in \mathbb{N}, x \in S$. Let $x \in S_{\beta}, \beta \in D$. Now $b^{n+1} + bxa = bxa$, and so $b\eta(b + bxa)\rho(b^{n+1} + bxa) = b\eta baxb$, i.e., $b\eta ab$. This implies $a = a\beta = a\beta$, since $D$ is a distributive lattice. Now $b^{n+1} + bxa = bxa \in S_{\alpha}b = S_{\alpha}a$ so that $b^{n+1} \in S_{\alpha}a$. Consequently, $a \xrightarrow{l} b$ in $S_{\alpha}$.

Now we are in a position to present the main result of this paper. Here we characterize the semirings which are distributive lattices of weakly left $k$-Archimedean semirings.

Theorem 4.2. The following conditions are equivalent on a semiring $S$:

(1) $S$ is a distributive lattice of weakly left $k$-Archimedean semirings,

(2) for all $a, b \in S$, $a \rightarrow b \Rightarrow ab \xrightarrow{l} b$.

Proof. (1) $\Rightarrow$ (2). Let $S$ be a distributive lattice $D = S/\rho$ of weakly left $k$-Archimedean semirings $S_{\alpha}, \alpha \in D$. $\rho$ being the corresponding distributive lattice congruence. Let $a, b \in S$ such that $a \rightarrow b$ so that there are $n \in \mathbb{N}$ and $s \in S$ such that $b^n + sas = sas$, by Lemma 2.1. Also there are $\alpha, \beta \in D$ such that $a \in S_{\alpha}, b \in S_{\beta}$. Now $(b + sas)\rho(b^n + sas) = sas\rho sas^2$. So $b\eta(b^2 + bas)\rhobas^2$, which implies $b\eta(b + ba)\rho(bas^2 + ba)\rhoba$ and thus $ba \in S_{\beta}$. Since $S_{\beta}$ is a weakly left $k$-Archimedean semiring, $b^n \in S_{\beta}bab \subseteq S_{\beta}ab$ for some $n \in \mathbb{N}$, yielding $ab \xrightarrow{l} b$.

(2) $\Rightarrow$ (1). By Lemma 2.2, for $a, b \in S, (ab)^2 \in S\alpha S$ implies that $a \rightarrow ab$. So by hypothesis, $a^2b = a(ab) \xrightarrow{l} (ab)$. This shows that $(ab)^n \in S\alpha^2b \subseteq S\alpha^2S$, for some $n \in \mathbb{N}$. Then by Theorem 4.3[1], $S$ is a distributive lattice(D = $S/\eta$) of $k$-Archimedean semirings $S_{\alpha}, \alpha \in D$, where $\eta$ is the least distributive lattice congruence on $S$. Let $a, b \in S_{\alpha}$. Then $a \rightarrow b$ and so $ab \xrightarrow{l} b$ in $S$. Then by Lemma 4.1, one gets $ab \xrightarrow{l} b$ in $S_{\alpha}$. Thus $S_{\alpha}$ is weakly left $k$-Archimedean.

Now we give an example of a semiring which is a distributive lattice of left $k$-Archimedean semirings, whence a distributive lattice of weakly left $k$-Archimedean semirings.
Example 4.3. Consider the set $\mathbb{N}$ of all natural numbers, and define $+$ and $\cdot$ on $S = \mathbb{N} \times \mathbb{N}$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\min \{a, c\}, \min \{b, d\}), \quad (a, b) \cdot (c, d) = (ac, b).$$

Then $S$ is a distributive lattice of left $k$-Archimedean semirings.

Example 4.4. Consider the set $\mathbb{N}$ of all natural numbers, and define $+$ and $\cdot$ on $S = \mathbb{N} \times \mathbb{N}$ by: for all $(a, b), (c, d) \in S$

$$(a, b) + (c, d) = (\min \{a, c\}, \min \{b, d\}), \quad (a, b) \cdot (c, d) = (ac, d).$$

Then $S$ is a distributive lattice of weakly left $k$-Archimedean semirings. But $S$ is not a distributive lattice of left $k$-Archimedean semirings. Indeed, for $(1, 2), (2, 2) \in S$ suppose there exist $n \in \mathbb{N}$ and $(x, y) \in S$ satisfying $[(1, 2) \cdot (2, 1)]^n + (x, y) \cdot (1, 2) = (x, y) \cdot (1, 2)$. This implies $(2^n, 1) + (x, 2) = (x, 2)$, i.e. $\min \{2^n, x\} = x$, $\min \{1, 2\} = 2$. The last equality is absurd.

References


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