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Complete graph decompositions and P-groupoids

John Carr and *Mark Greer*

Abstract. We study P-groupoids that arise from certain decompositions of complete graphs. We show that left distributive P-groupoids are distributive, quasigroups. We characterize some P-groupoids when the corresponding decomposition is a Hamiltonian decomposition for complete graphs of odd, prime order. We also study a specific example of a P-quasigroup constructed from cyclic groups of odd order. We show that the right multiplication group of such P-quasigroups is isomorphic to the dihedral group.

1. Introduction

The concept of graph amalgamation was introduced in 1984 by Anthony Hilton [5]. Recently, the subject has gained more attention and is becoming more widely studied. We aim to provide insight into graph amalgamation by considering the results of amalgamation in Latin squares. First, we cover some preliminaries.

Recall that a graph is an ordered pair $G = (V, E)$ comprising a set V of vertices with a set E of edges. A *complete graph*, denoted by K_n where n is the number of vertices in the graph, is a graph where every pair of vertices is connected by an edge. An *edge coloring* of a graph G is a function $\gamma : C \to E(G)$, where C is a set of colors. A *Hamiltonian decomposition* of K_{2n+1} is an edge-coloring of K_{2n+1} with n colors in which each color class is a C_{2n+1} cycle, called *Hamiltonian cycles.*

We define *graph amalgamation* in the following way.

Definition 1.1. Let G and H be two graphs with the same number of edges where G has more vertices than H. We say that H is an *amalgamation* of G if there exists a bijection $\phi : E(G) \to E(H)$ and a surjection $\psi : V(G) \to V(H)$ where the following hold

1. If x, y are two vertices in G where $\psi(x) \neq \psi(y)$, and both x and y are adjacent by edge e in G, then $\psi(x)$ and $\psi(y)$ are adjacent by edge $\phi(e)$ in H.
2. If e is a loop on a vertex $x \in V(G)$, then $\phi(e)$ is a loop on $\psi(x) \in H$.
3. If e joins $x, y \in V(G)$ where $x \neq y$, but $\psi(x) = \psi(y)$, then $\psi(e)$ is a loop on $\psi(x)$.

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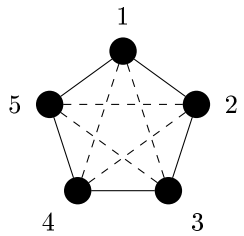
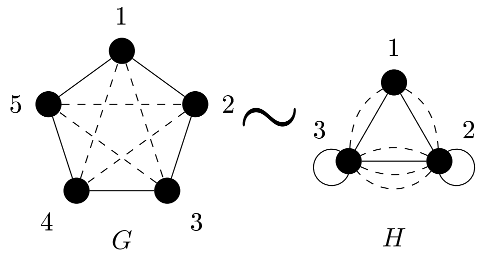
Example 1.2. K_5 and a Hamiltonian decomposition.



Example 1.3. The following is an example of a graph amalgamation of the complete graph on 5 vertices with the amalgamation $\psi(1) = 1$, $\psi(2) = 2$, $\psi(3) = 2$, $\psi(4) = 3$, $\psi(5) = 3$.

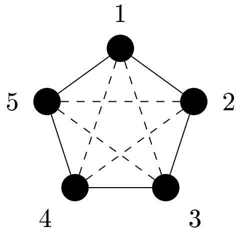


Note that since the edges between two amalgamated graphs are in bijection with each other, edge colorings are invariant to amalgamation; that is, edge colors are unchanged by amalgamation. However, more interesting is the fact that if G is a complete graph of the form K_{2n+1} and the edges are colored in such a way as to specify a Hamiltonian decomposition, then the edges also form a Hamiltonian decomposition in H.

The concept of amalgamating a larger graph down into a smaller graph is a well understood concept in graph theory. Likewise, one can *disentangle* vertices of a graph to create a larger graph. To disentangle a vertex is to split the vertex into multiple vertices. Using example 3, we could disentangle vertex 2 of graph H into vertices 2 and 3, while disentangling vertex 3 into vertices 4 and 5 to create graph G. Some graph theorists are currently studying how to take a graph with a Hamiltonian decomposition such as graph G, and to disentangle G to create a new graph, say G', where G' also has a Hamiltonian decomposition. Since the concept of amalgamation also exists in the Latin square setting, we approach the problem from an algebraic perspective.

Let K_n be a complete graph. It is well known that the edges in K_n can be decomposed into distinct cycles if and only if n is odd [9]. In this setting, Kotzig gave a complete characterization of a groupoid (termed P-groupoid) that would describe the decomposition. Indeed, let Q be a set with n elements (corresponding to the vertices in K_n) and define $xy = z$ if and only if edges (x, y) and (y, z) are in the same cycle where $x \neq y$. If $x = y$, then set $x^2 = x$.

Example 1.4. Consider the previous example of K_5, along with its associated P-groupoid.



(Q,·)	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Kotzig then showed that all decompositions of complete graphs are given by *P-groupoids*, defining them as follows.

Definition 1.5 ([9]). Let $(Q, \cdot)$ be a groupoid. Then $(Q, \cdot)$ is a P-groupoid if for all $x, y, z \in Q$,

(1.5.1) $x^2 = x$ (Idempotent).

(1.5.2) $x \neq y \Rightarrow xy \neq x$ and $xy \neq y$.

(1.5.3) $xy = z \Leftrightarrow zy = x$.

For the rest of the paper we only consider finite groupoids and P-groupoids. One can quickly show that the order of every P-groupoid is odd [9] and that the equation $xa = b$ is always uniquely solvable for x. Indeed, $xa = b \Leftrightarrow ba = x$. Hence, P-groupoids are idempotent, right quasigroups. We show that if the P-groupoid is left distributive, then it is right distributive and a quasigroup (Theorem 2.2).

Dénes and Keedwell gave the first specific example of a P-quasigroup relating to the decomposition [2]. We also note that this P-quasigroup is a quandle and use results from [10] to describe the right multiplication group and automorphism group of Dénes and Keedwell's example. We then show that if $H \leq Q$ is a subquasigroup, then $|H|$ must divide $|Q|$ (Theorem 2.6). If the graph has prime order, then Dénes and Keedwell's example is an example of a P-quasigroup relating a Hamiltonian decomposition.

2. P-groudpoids and quasigroups

A *groupoid* $(Q, \cdot)$ is a set Q with a binary operation $\cdot : Q \times Q \to Q$. A *quasigroup* $(Q, \cdot)$ is a groupoid such that for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$. We denote these unique solutions by $x = a\backslash b$ and $y = b/a$, respectively. Standard references in quasigroup theory are [1, 13]. All groupoids (quasigroups) considered here are finite.

To avoid excessive parentheses, we use the following convention:

- multiplication $\cdot$ will be less binding than divisions $/, \backslash$.

- divisions are less binding than juxtaposition.

For example $xy/z \cdot y\backslash xy$ reads as $((xy)/z)(y\backslash(xy))$.

For $x \in Q$, where Q is a quasigroup, we define the *right* and *left translations* by x by, respectively, $yR_x = yx$ and $yL_x = xy$ for all $y \in Q$. The fact that these mappings are permutations of Q follows easily from the definition of a quasigroup. It is easy to see that $yL_x^{-1} = x\backslash y$ and $yR_x^{-1} = y/x$. We define the *left multiplication group of* Q, $\mathrm{Mlt}_\lambda(Q) = \langle L_x \mid \forall x \in Q\rangle$, the *right multiplication group of* Q, $\mathrm{Mlt}_\rho(Q) = \langle R_x \mid \forall x \in Q\rangle$ and the *multiplication group of* Q, $\mathrm{Mlt}(Q) = \langle Mlt_\lambda(Q), \mathrm{Mlt}_\rho(Q)\rangle$.

Lemma 2.1. *Let Q be a P-groupoid. Then $|R_x| = 2$ for all $x \in Q$* (i.e. $R_x^2 = id_Q$).

Proof. Let $|Q| = 2n+1$ for some $n \in \mathbb{Z}$ and suppose $q_1 x = q_2$ for some $x, q_1, q_2 \in Q$. Then $q_1 R_x^2 = q_2 R_x = q_1$. Moreover, $xR_x = x$. Hence,

$$R_x = (x)(q_1q_2)(q_3q_4)\dots(q_{2n})(q_{2n+1}).$$

The desired result follows. □

A groupoid Q is *left distributive* if it satisfies $x(yz) = (xy)(xz)$ for all $x, y, z \in Q$. Similarly, it is *right distributive* if it satisfies $(yz)x = (yx)(zx)$. A *distributive* groupoid is a groupoid that is both left and right distributive.

Theorem 2.2. *Let Q P-groupoid. If Q is left distributive, then Q is a distributive quasigroup.*

Proof. Let Q be a left distributive, P-groupoid. Note that by left distributivity, we have $x \cdot yx = xy \cdot x$. Suppose that $xa = xb$ for some $x, a, b \in Q$. Then we compute

$$\begin{aligned}(ax)(ab \cdot x) &= [(ax)(ab)](ax \cdot x) && \text{by left distributivity,}\\ &= [(ax)(ab)]a && \text{by Lemma 2.1,}\\ &= [(a \cdot xb)]a && \text{by left distributivity,}\\ &= a(xb \cdot a) = a(xa \cdot a) && \text{by assumption,}\\ &= ax && \text{by Lemma 2.1}\end{aligned}$$

Hence, we have $ab \cdot x = ax$ by (1.5.2). Thus, $ab = a$ and hence, $b = a$ by (1.5.2) again. Thus, Q is a quasigroup.

For right distributive, we first note that by left distributivity $x(xy \cdot z) = (xy \cdot y)(xy \cdot z) = (xy)(yz)$. Using (1.5.3), we have

$$[x(xy \cdot z)](yz) = xy. \tag{1}$$

Similarly, $(xy \cdot z)x = (xy \cdot z)(xy \cdot y) = (xy)(zy)$ and $x(xy \cdot z) = (xy \cdot y)(xy \cdot z) = (xy)(yz)$ both by left distributivity again, thus

$$(xy \cdot z)x = (xy)(zy), \tag{2}$$

$$x(xy \cdot z) = (xy)(yz). \tag{3}$$

Hence we have

$$\begin{aligned}(x \cdot yz)(xz \cdot u) &= [(xy)(xz)](xz \cdot u) && \text{by left distributivity,}\\ &= (xy)[(xy)(xz) \cdot u] && \text{by (3) with } x \to xy, y \to yz, z \to u,\\ &= (xy)[(x \cdot yz) \cdot u] && \text{by left distributivity,}\end{aligned}$$

thus

$$(x \cdot yz)(xz \cdot u) = (xy)[(x \cdot yz) \cdot u]. \tag{4}$$

Substituting $y \to yz$ in (1) give $x(yz) = [x(x(yz) \cdot z)][yz \cdot z] = x(x(yz) \cdot z) \cdot y$. So

$$x(yz) = x(x(yz) \cdot z) \cdot y. \tag{5}$$

Hence we compute

$$\begin{aligned}x &= [x \cdot x(yz)][\underline{x(yz)}]\\ &= [x \cdot x(yz)][x(x(yz) \cdot z) \cdot y] && \text{by (5),}\\ &= [x(x(yz) \cdot z)][xz \cdot y] && \text{by (4) with } y \to x(yz), u \to y.\end{aligned}$$

Thus

$$x = [x(x(yz) \cdot z)][xz \cdot y]. \tag{6}$$

Replacing $x \to xy$ and $y \to x$ in (6) gives

$$\begin{aligned}xy &= [(xy) \cdot (xy \cdot xz)z][\underline{(xy \cdot z)x}] && \text{by (6) with } x \to xy,\\ &= [(xy) \cdot (xy \cdot xz)z](xy \cdot zy) && \text{by (2),}\\ &= [(xy) \cdot (x \cdot yz)z](xy \cdot zy) && \text{by left distributivity,}\end{aligned}$$

and therefore

$$xy = [(xy) \cdot (x \cdot yz)z](xy \cdot zy). \tag{7}$$

Recalling (3) and substituting $y \to yz$, we have $(x \cdot yz)y = (x \cdot yz)(yz \cdot z) = x \cdot (x \cdot yz)z$, so

$$(x \cdot yz)y = x \cdot (x \cdot yz)z. \tag{8}$$

We compute

$$\begin{aligned}x(yz \cdot x) = (x \cdot yz)x &= (xy \cdot xz)x && \text{by left distributivity,}\\ &= (xy) \cdot (xy \cdot xz)z && \text{by (8) } x \to xy, y \to x,\end{aligned}$$

and hence

$$x(yz \cdot x) = (xy)[(x \cdot yz)z]. \tag{9}$$

Hence, the right hand side of (7) can be rewritten as

$$xy = [x(yz \cdot x)](xy \cdot zy). \tag{10}$$

Using left distributivity, we have

$$\begin{aligned}(xy)\cdot(xz\cdot y)(zy) &= [(xy)(xz\cdot y)](xy\cdot zy) && \text{by left distributivity,}\\ &= [(xy\cdot xz)(xy\cdot y)](xy\cdot zy) && \text{by left distributivity,}\\ &= [(x\cdot yz)(xy\cdot y)](xy\cdot zy) && \text{by left distributivity,}\\ &= [(x\cdot yz)x](xy\cdot zy)\\ &= [x(yz\cdot x)](xy\cdot zy),\end{aligned}$$

and thus

$$[x(yz\cdot x)][(xy\cdot zy)] = (xy)[(xz\cdot y)(zy)]. \tag{11}$$

Therefore, the right hand side of (10) can be rewritten as $xy = (xy)[(xz\cdot y)(zy)]$

Finally, since $(xy)[(xz\cdot y)(zy)] = xy$, we have $(xz\cdot y)(zy) = xy$ by (1.5.2) and thus $(xy)(zy) = xz\cdot y$ by (1.5.3). □

We now focus on the first specific constructions of a P-quasigroup dealing with Hamiltonian decompositions given by Deńes and Keedwell [2].

Theorem 2.3 ([2])**.** *Consider $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ where $n = 2k+1$ for some $k \in \mathbb{Z}$. Define $r \circ s = 2s - r \mod n$. Then $(\mathbb{Z}_n, \circ)$ is a P-quasigroup of order n.*

Proposition 2.4. *For $(\mathbb{Z}_n, \circ)$, the following hold:*

(i) $yL_x^n = 2^n(y-x)+x$ *for all* $x, y \in Q$.

(ii) $|L_x| = k$ *where k is the smallest integer such that* $2^k \equiv 1 \mod n$.

(iii) $L_x^n R_x = R_x L_x^n$.

Proof. Let $x, y \in Q$. For (i), $yL_x = 2y - x = 2(y-x)+x$. By induction,

$$yL_x^{n+1} = (2y-x)L_x^n = 2^n((2y-x)-x)+x = 2^{n+1}(y-x)+x.$$

For (ii), let $k > 0$ be the smallest integer such that $yL_x^k = y$. Then, by (1),

$$2^k(y-x)+x \equiv y \Leftrightarrow 2^k y - y - 2^k x + x \equiv 0 \Leftrightarrow (y-x)(2^k-1) \equiv 0.$$

Hence, $2^k \equiv 1 \mod n$. Finally,

$$yL_xR_x = (2y-x)R_x = 3x - 2y = (2x-y)L_x = yR_xL_x.$$

Since $\mathrm{Mlt}(Q)$ is a group, (iii) follows. □

$(\mathbb{Z}_n, \circ)$ is well known. A quasigroup Q is medial (or entropic) if $(xy)(zw) = (xz)(yw)$ for all $x, y, z, w \in Q$. Idempotent medial quasigroups are distributive [15]. There is a well-known correspondence between abelian groups and medial quasigroups, the Toyoda-Bruck theorem. That is, $(Q, \cdot)$ is a medial quasigroup if and only if there is an abelian group $(Q,+)$ such that $x \cdot y = f(x) + g(y) + c$ for all $x, y \in Q$ for some commuting $f, g \in \text{Aut}(Q)$ and $c \in Q$ [14]. If $(G,+)$ is an abelian group of odd order, then both $f(x) = -x$ and $g(y) = 2y$ are automorphisms of G. Hence, Deńes and Keedwell's P-quasigroup is precisely the medial quasigroup of the form $x \circ y = f(x) + g(y) + 0$.

Definition 2.5. A groupoid $(Q, \cdot)$ is a *quandle* if

1. $a^2 = a$ for all $a \in Q$,
2. For all $a, b \in Q$, the equations $xa = b$ have a unique solution,
3. $(ab)c = (ac)(bc)$ for all $a, b, c \in Q$.

Note that quandles are idempotent, right distributive, and right quasigroups.

$(\mathbb{Z}_n, \circ)$ is also referred to as the *dihedral quandle* of order n with $\text{Mlt}_\rho(Q) \cong D_{2n}$ [10], the dihedral group of order $2n$. For a quandle Q, the *inner automorphism group of* Q, $\text{Inn}(Q)$ is the subgroup generated by L_x for all $x \in Q$. Thus, $\text{Inn}(\mathbb{Z}_n, \circ)$ is isomorphic to the dihedral group of order n. Moreover, both L_x and R_y are affine maps for all x, y. Indeed,

$$[(1-t)a+tb]L_x = 2[(1-t)a+tb]-x = (1-t)(2a-x)+j(2b-x) = (1-t)(aL_x)+t(bL_x),$$

$$[(1-t)a+tb]R_y = 2y-[(1-t)a+tb] = (1-t)(2y-a)+t(2y-b) = (1-t)(aR_y)+t(bR_y),$$

for all $a, b, t \in \mathbb{Z}_n$. That is $\text{Aut}(\mathbb{Z}_n, \circ)$ is isomorphic to the affine group $\text{Aff}(\mathbb{Z}_n)$ [10].

Note that P-quasigroups always have subgroups $\langle x \rangle$ for all x. It is well-known that in general, the order of a subquasigroup doesn't divide the order of the quasigroup. However, for $Q = (\mathbb{Z}_n, \circ)$, the order of the subquasigroup always divides the order of the quasigroup.

Theorem 2.6. *Let $Q = (\mathbb{Z}_n, \circ)$. If $H \leq Q$, then $|H|$ divides $|Q|$. Hence, if Q has prime order and $|H| \leq |Q|$, then $H = \langle x \rangle$ for some $x \in Q$ or $H = Q$.*

Proof. Let $H \leq Q$. If $|H| = \langle x \rangle$, then $|H| = 1$ and we are done. Let $x, y \in Q$. Then $y = x + k$, since both $x, y \in \mathbb{Z}_n$. Then $x \circ y = x + 2k \in H$. Continuing, $x \circ (x + 2k) = x + 3k$, and thus, elements of H are of the form $x + lk$. Since Q is finite, we must have $x + l_1 k = x + l_2 k$. Thus, $k(l_1 - l_2) \equiv 0 \mod n$. Thus, k is a divisor of n. Let $kl = n$. Then $H = \{x, x + k, x + 2k, \ldots x + (l - 1)k\}$, and therefore $|H| = l$, a divisor of n. □

The following is a minimal example of a P-groupoid that is not a quandle, found by Mace4 [11].

Example 2.7. A P-groupoid of order 5 that is not a quandle.

$(Q,\cdot)$	1	2	3	4	5
1	1	3	2	3	4
2	3	2	1	5	3
3	2	1	3	1	2
4	5	5	5	4	1
5	4	4	4	2	5

3. Hamiltonian decompostions and P-quasigroups

Theorem 3.1. *Let Q_1 and Q_2 be two P-groupoids. Then, $Q_1 \cong Q_2$* if and only if *the corresponding decompositions of the associated complete graph is isomorphic.*

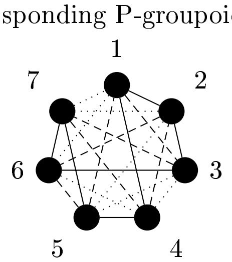
Proof. Suppose ϕ is an isomorphism between Q_1 and Q_2 where both Q_1 and Q_2 correspond to decompositions of K_n. By definition $(a,b)(b,c)$ belong to the same cycle in the decomposition of K_n if and only if $ab = c$ for all $a, b, c \in Q_1$. Then $(\phi(a), \phi(b))(\phi(b), \phi(c))$ belong to the same cycle in K_n if and only if $\phi(a)\phi(b) = \phi(c)$. Since this is precisely the correspondence between Q_2 and its Hamiltonian decompostion of K_n, the decompositions must be isomorphic.

Alternatively, suppose ϕ is an isomorphism between two decompositions of K_n. If $(\phi(a), \phi(b))(\phi(b), \phi(c))$ belong to the same cycle in K_n for some $a, b, c \in Q_1$, then $\phi(a)\phi(b) = \phi(c)$ for $\phi(a), \phi(b), \phi(c) \in Q_2$. Again, since this is precisely how we establish a correspondence between P-groupoids and complete undirected graphs, we conclude that $Q_1 \cong Q_2$. □

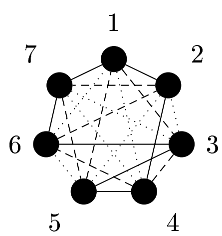
Theorem 3.2. ([2]) *Let p be an odd prime. Then $(\mathbb{Z}_n, \circ)$ corresponds to a Hamiltonian decomposition in K_p.*

Note that Theorem 3.1 does not imply all P-groupoids of prime order corresponding to a Hamiltonian decomposition of K_p are quasigroups. Below are two Hamiltonian decompositions of K_7 with non isomorphic corresponding P-groupoids.

Example 3.3. Two non-isomorphic Hamiltonian decompositions and their corresponding P-groupoids.



Q_1	1	2	3	4	5	6	7
1	1	3	7	5	6	4	2
2	4	2	6	7	3	5	1
3	5	1	3	6	2	7	4
4	2	6	5	4	7	1	3
5	3	7	4	1	5	2	6
6	7	4	2	3	1	6	5
7	6	5	1	2	4	3	7



Q_2	1	2	3	4	5	6	7
1	1	4	4	7	7	5	6
2	7	2	7	5	6	4	5
3	5	5	3	6	4	7	4
4	6	1	1	4	3	2	3
5	3	3	6	2	5	1	2
6	4	7	5	3	2	6	1
7	2	6	2	1	1	3	7

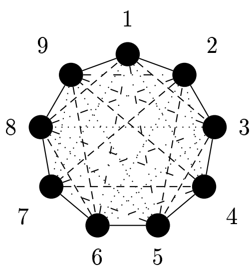
The following is motivated by Theorem 2.6.

Theorem 3.4. *Let K_n have a Hamiltonian decomposition and let Q be the corresponding P-groupoid. Then Q doesn't contain any nontrivial subgroupoids.*

Proof. Let $|Q| = n$ correspond to a complete graph K_n with a Hamiltonian decomposition. For the sake of contradiction, suppose $\exists H < Q$ where $|H| > 1$. Since H is a subgroupoid, H is closed and multiplying the elements of H will create a cycle with length less than n. However, this contradicts our assumption that K_n has a Hamiltonian decomposition. Therefore, we conclude that Q doesn't contain any subgroupoids with order greater than 1. $\square$

The following is an example of a P-groupoid corresponding to a Hamiltonian decomposition of K_9. It is currently unknown if a P-quasigroup exists that corresponds to a Hamiltonian decomposition of K_9 (note that $(\mathbb{Z}_9, \circ)$) is not a quasigroup).

Example 3.5. A P-groupoid of order 9 corresponding to a Hamiltonian decomposition.



K_9	1	2	3	4	5	6	7	8	9
1	1	3	5	2	7	4	9	6	8
2	9	2	4	1	3	3	4	4	3
3	8	1	3	5	2	2	5	5	2
4	7	6	2	4	6	1	2	2	6
5	6	7	1	3	5	7	3	3	7
6	5	4	8	8	4	6	8	1	4
7	4	5	9	9	1	5	7	9	5
8	3	9	6	6	9	9	6	8	1
9	2	8	7	7	8	8	1	7	9

Further work would consist of finding all necessary and sufficient conditions such that a P-groupoid of odd nonprime order corresponds to a Hamiltonian decomposition of a complete graph. Hilton gave necessary and sufficient conditions for a Hamiltonian decomposition of K_{2n+1} corresponding to a Hamiltonian circuit [5]. The proof relies heavily on Hall's work with completing partial Latin squares [4]. Thus, using P-groups to classify Hamiltonian decompositions is a natural choice. Moreover, due to the connection to quandles in the prime order case, perhaps finding a relationship between P-groupoids and quandles could lead to new results in both fields.

Acknowledgment. Some investigations in this paper were assisted by the automated deduction tool, PROVER9, and the finite model builder, MACE4, both developed by McCune [11]. Similarly, all presented examples were found and verified using the GAP system [3] together with the LOOPS package [12]. We also thank the anonymous referee for several comments that improved the manuscript.

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