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A characterization of elementary abelian 3-groups

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The author dedicates this paper to Professor Sarah Hart with admiration and respect.

Abstract. We give a characterization of elementary abelian 3-groups in terms of their maximal sum-free sets. A corollary to our result is that the number of maximal sum-free sets in an elementary abelian 3-group of finite rank n is $3^n - 1$.

1. Preliminaries

The well-known result of Schur which says that whenever we partition the set of positive integers into a finite number of parts, at least one of the parts contains three integers x, y and z such that $x + y = z$ introduced the study of sum-free sets. Schur [13] gave the result while showing that the Fermat's last theorem does not hold in F_p for sufficiently large p. The concept was later extended to groups as follows: A non-empty subset S of a group G is sum-free if for all $s_1, s_2 \in S$, $s_1 s_2 \notin S$. (Note that the case $s_1 = s_2$ is included in this restriction.) An example of a sum-free set in a finite group G is any non-trivial coset of a subgroup of G. Sum-free sets have applications in Ramsey theory and are also closely related to the widely studied concept of caps in finite geometry.

Some questions that appear interesting in the study of sum-free sets are:

(i) How large can a sum-free set in a finite group be?

(ii) Which finite groups contain maximal by inclusion sum-free sets of small sizes?

(iii) How many maximal by cardinality sum-free sets are there in a given finite group?

Each of these questions has been attempted by several researchers; though none is fully answered. For question (i), Diananda and Yap [7], in 1969, following an earlier work of Yap [18], determined the sizes of maximal by cardinality sum-free sets in finite abelian groups G, where $|G|$ is divisible by a prime $p \equiv 2(\mathrm{mod}\, 3)$, and where $|G|$ has no prime factor $p \equiv 2(\mathrm{mod} 3)$ but 3 is a factor of $|G|$. They gave a good bound in the case where every prime factor of $|G|$ is congruent to

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1(mod 3). Green and Ruzsa [10] in 2005 completely answered question (i) in the finite abelian case. The question is still open for the non-abelian case, even though there has been some progress by Kedlaya [11, 12], Gowers [9], amongst others.

For question (ii), Street and Whitehead [14] began research in that area in 1974. They called a maximal by inclusion sum-free set, a locally maximal sum-free set (LMSFS for short), and calculated all LMSFS in groups of small orders, up to 16 in [14, 15] as well as a few higher sizes. In 2009, Giudici and Hart [8] started the classification of finite groups containing LMSFS of small sizes. Among other results, they classified all finite groups containing LMSFS of sizes 1 and 2, as well as some of size 3. The size 3 problem was resolved in [5]. Question (ii) is still open for sizes $k \geqslant 4$; though some progress has been made in [1]. For other works on LMSFS, the reader may see [2, 3, 4, 6].

To be consistent with our notations, we will use the term 'maximal' to mean 'maximal by cardinality' and 'locally maximal' to mean 'maximal by inclusion'. Tărnăuceanu [16] in 2014 gave a characterization of elementary abelian 2-groups in terms of their maximal sum-free sets. His theorem (see Theorem 1.1 of [16]) states that "a finite group G is an elementary abelian 2-group if and only if the set of maximal sum-free sets coincides with the set of complements of the maximal subgroups". The author of [16] didn't define the term maximal sum-free sets. Unfortunately, the theorem is false whichever definition is used. If we take "maximal" in the theorem to mean 'maximal by cardinality', then a counterexample is the cyclic group C_4 of order 4, given by $C_4 = \langle x \mid x^4 = 1\rangle$. Here, there is a unique maximal (by cardinality) sum-free set namely $\{x, x^3\}$, and it is the complement of the unique maximal subgroup. But C_4 is not elementary abelian. On the other hand, if we take "maximal" to mean 'maximal by inclusion', then the theorem will still be wrong since $S = \{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\}$ is a maximal by inclusion sum-free set in $C_2^4 = \langle x_1, x_2, x_3, x_4 \mid x_i^2 = 1, x_ix_j = x_jx_i \text{ for } 1 \leqslant i, j \leqslant 4\rangle$, but does not coincide with any complement of a maximal subgroup of C_2^4. These counterexamples were first pointed out in the arXiv manuscript at https://arxiv.org/abs/1611.06546, which prompted an erratum to be published by the author (see [17]).

For a prime p and $n \in \mathbb{N}$, we write $\mathbb{Z}_p^n$ for the elementary abelian p-group of finite rank n. We recall here that the number of maximal subgroups of $\mathbb{Z}_p^n$ is $\sum\limits_{k=0}^{n-1} p^k$. Corollary 1.2 of [16] is that the number of maximal sum-free sets in $\mathbb{Z}_2^n$ is $2^n - 1$. This result is correct in its own right and can be proved by showing that each maximal sum-free set in $\mathbb{Z}_2^n$ is the non-trivial coset of a maximal subgroup of $\mathbb{Z}_2^n$, and every maximal subgroup of $\mathbb{Z}_2^n$ is the complement of a maximal sum-free set in $\mathbb{Z}_2^n$. In this paper, we give a characterization of elementary abelian 3-groups in terms of their maximal sum-free sets. Moreover, for prime $p > 3$, we show that there is no direct analogue of our result for elementary abelian p-groups of finite ranks. For the rest of this section, we state the main result of this paper and its immediate corollary. We remind the reader that $\Phi(G)$ denotes the Frattini subgroup of G.

Theorem 1.1. *A finite group G is an elementary abelian 3-group if and only if the set of non-trivial cosets of each maximal subgroup of G coincides with two maximal sum-free sets in G, every maximal sum-free set is a non-trivial coset of a maximal subgroup, and $\Phi(G) = 1$.*

Corollary 1.2. *The number of maximal sum-free sets in $\mathbb{Z}_3^n$ is $3^n - 1$.*

2. Proof of Theorem

Let S be a sum-free set in a finite group G. We define $SS = \{xy \mid x, y \in S\}$, $S^{-1} = \{x^{-1} \mid x \in S\}$ and $SS^{-1} = \{xy^{-1} \mid x, y \in S\}$. Clearly, $S \cap SS = \varnothing$. Moreover, $S \cap SS^{-1} = \varnothing$ as well; for if $x, y, z \in S$ with $x = yz^{-1}$, then $xz = y$, contradicting the fact that S is sum-free.

Lemma 2.1. *Let S be sum-free in $G = \mathbb{Z}_3^n$ $(n \in \mathbb{N})$, and let $x \in S$. Then the following hold:*

(*i*) *any two sets in $\{S, x^{-1}S, xS\}$ are disjoint;*

(*ii*) *any two sets in $\{S, SS^{-1}, S^{-1}\}$ are disjoint.*

Moreover, if S is maximal, then the following also hold:

(*iii*) *$S \cup x^{-1}S \cup xS = G$ and $|S| = \frac{|G|}{3}$;*

(*iv*) *$S \cup SS^{-1} \cup S^{-1} = G$.*

Proof. (*i*). As S is sum-free, $S \cap xS = \varnothing = S \cap x^{-1}S$. So we only need to show that $xS \cap x^{-1}S = \varnothing$. Suppose for contradiction that $xS \cap x^{-1}S \neq \varnothing$. Then there exist $y, z \in S$ such that $xy = x^{-1}z$. This means that $y = xz$; a contradiction. Therefore $xS \cap x^{-1}S = \varnothing$.

The proof of (*ii*) is similar to (*i*).

For (*iii*), as $S \cup x^{-1}S \cup xS \subseteq G$, we have that $3|S| \leq |G|$; whence $|S| \leq \frac{|G|}{3}$. Each maximal subgroup of G has size $\frac{|G|}{3}$. As any non-trivial coset of such a subgroup is sum-free and has size $\frac{|G|}{3}$; such a coset of the maximal subgroup must be maximal sum-free. Thus, $|S| = \frac{|G|}{3}$, and $S \cup x^{-1}S \cup xS = G$.

The proof of (*iv*) is similar. □

Proposition 2.2. *Suppose S is a maximal sum-free set in an elementary abelian 3-group G, and let $x \in S$. Then $xS = S^{-1} = SS$.*

Proof. Let S be a maximal sum-free set in an elementary abelian 3-group G, and $x \in S$. In the light of Lemma 2.1(*iv*), we deduce that $x^{-1}S = S^{-1}S$. Let $y \in xS$. By Lemma 2.1(*i*) therefore $y \notin S \dot{\cup} SS^{-1}$. So Lemma 2.1(iv) tells us that $y \in S^{-1}$, and we conclude that $xS \subseteq S^{-1}$. On the other hand, if $y \in S^{-1}$, then Lemma 2.1(ii) and Lemma 2.1(iii) yield $y \in xS$; so $S^{-1} \subseteq xS$. Therefore $xS = S^{-1}$. Now,

$$SS = \bigcup_{x \in S} xS = \bigcup_{x \in S} S^{-1} = S^{-1}. \tag{1}$$

Thus, $xS = S^{-1} = SS$ as required. □

Suppose p is the smallest prime divisor of the order of a finite group G, and H is a subgroup of index p in G. Then H is normal in G. This fact is well-known but we include a short proof for the reader's convenience. Suppose for a contradiction that H is not normal. Then for some $g \in G$, we have $H^g \neq H$. But $|H^g H| = \frac{|H^g||H|}{|H^g \cap H|} = \frac{|H|^2}{|H^g \cap H|} = |H|\frac{|H|}{|H^g \cap H|} \geqslant |H|p = |G|$; thus $H^g H = G$. Therefore, $g = (gh_1g^{-1})h_2$ for some $h_1, h_2 \in H$. So $g = h_2h_1 \in H$, and we conclude that $H^g = H$; a contradiction. Therefore H is normal in G.

We now prove Theorem 1.1

Proof. Let G be an elementary abelian 3-group of finite rank n. Clearly, every maximal subgroup of G has size 3^{n-1}, and the non-trivial cosets of any maximal subgroup of G yield two maximal sum-free sets in G. Next, we show that every maximal sum-free set in G is a non-trivial coset of a maximal subgroup of G. Suppose S is a maximal sum-free set in G. Let $x \in S$ be arbitrary, and define $H := x^{-1}S$. We show that H is a subgroup of G. Let a and b be elements of H. Then $a = x^{-1}y$ and $b = x^{-1}z$ for some $y, z \in S$. Since $ab = x^{-1}(x^{-1}yz)$, it is sufficient to show that $x^{-1}yz \in S$. Recall from Lemma 2.1(iii) that $G = S \cup x^{-1}S \cup xS$. From Proposition 2.2 therefore, $G = S \cup x^{-1}S \cup S^{-1}$. Now, suppose $x^{-1}yz \in x^{-1}S$. Then there exists $q \in S$ such that $x^{-1}yz = x^{-1}q$. This implies that $yz = q$; a contradiction. Next suppose $x^{-1}yz \in S^{-1}$. Then there exists $q \in S$ such that $x^{-1}yz = q^{-1}$. So $yz = xq^{-1}$, and we obtain that $x^{-1}q = y^{-1}z^{-1} = (yz)^{-1}$; a contradiction as $x^{-1}q \in x^{-1}S$, $(yz)^{-1} \in (SS)^{-1} = S$ by Equation 1, and Lemma 2.1(i) tells us that $x^{-1}S \cap S = \varnothing$. We have shown that $x^{-1}yz \notin x^{-1}S \cup S^{-1}$. In the light of $G = S \cup x^{-1}S \cup S^{-1}$ therefore, $x^{-1}yz \in S$; whence, H is closed. So H is a subgroup of G. As $|H| = |x^{-1}S| = |S| = \frac{|G|}{3}$, we conclude that H is a maximal subgroup of G, and $S = xH$ is a non-trivial coset of H in G. So we have shown now that every maximal sum-free set in G is a non-trivial coset of a maximal subgroup of G. The third part that $\Phi(G) = 1$ follows from the fact that the intersection of maximal subgroups of G is trivial.

Conversely, suppose G is a finite group such that the set of non-trivial cosets of each maximal subgroup of G coincides with two maximal sum-free sets in G, every maximal sum-free set of G is a coset of a maximal subgroup of G, and $\Phi(G) = 1$. First and foremost, G has no subgroup of index 2; otherwise it will have a maximal sum-free set which is not a coset of a subgroup of index 3. As the smallest index of a maximal subgroup of G is 3, any such subgroup must be normal in G. Let H be a Sylow 3-subgroup of G. Then either $H = G$ or H is contained in a maximal subgroup (say M) of G. Suppose H is contained in such maximal subgroup M. As $|G/M| = 3$, we deduce immediately that $|G : H|$ is divisible by 3; a contradiction! Therefore, $H = G$, and we conclude that G is a 3-group. Now, G is an elementary abelian 3-group follows from the fact that $\Phi(G) = 1$ and $P/\Phi(P)$ is elementary abelian for every p-group P. □

Let $p > 3$ and prime, and suppose $n \in \mathbb{N}$. If $G = \mathbb{Z}_p^n$, then there exists a normal subgroup N of G such that $G/N \cong \mathbb{Z}_p$, and $\mathbb{Z}_p$ has a maximal sum-free set of size at least 2 (the latter fact follows from the classification of groups containing maximal by inclusion sum-free sets of size 1 in [8, Theorem 4.1]). The union of non-trivial cosets of N corresponding to this maximal sum-free set of $\mathbb{Z}_p$ is itself sum-free in G. So G has a maximal sum-free set of size at least $2|N|$. This argument shows that for $p > 3$, no direct analogue of Theorem 1.1 holds for elementary abelian p-groups of finite ranks.

References

[1] **C.S. Anabanti**, *On finite groups containing locally maximal product-free sets of size* 4, Algebra Discrete Math., to appear.

[2] **C.S. Anabanti**, *Three questions of Bertram on locally maximal sum-free sets*, Appl. Algebra Engineering, Comm. Comput., **30** (2019), 127–134.

[3] **C.S. Anabanti**, *On filled soluble groups*, Commun. Algebra, **46** (2018), 4914–4917.

[4] **C.S. Anabanti, G Erskine and S.B. Hart**, *Groups whose locally maximal product-free sets are complete*, Australas. J. Combin. **71** (2018), 544–563.

[5] **C.S. Anabanti and S.B. Hart**, *Groups containing small locally maximal product-free sets*, Int. J. Comb. 2016, Art. ID 8939182, 5 pp.

[6] **C.S. Anabanti and S. B. Hart**, *On a conjecture of Street and Whitehead on locally maximal product-free sets*, Australas. J. Combin. **63** (2015), 385–398.

[7] **P.H. Diananda and H. P. Yap**, *Maximal sum-free sets of elements of finite groups*, Proc. Japan Acad., **45** (1969), 1–5.

[8] **M. Giudici and S. Hart**, *Small maximal sum-free sets*, Electronic J. Combin., **16** (2009), 17pp.

[9] **W.T. Gowers**, *Quasirandom groups*, Combin., Probability Computing, **17** (2008), 363–387.

[10] **B. Green and I. Z. Ruzsa**, *Sum-free sets in abelian groups*, Israel J. Math., **147** (2005), 157–188.

[11] **K.S. Kedlaya**, *Large product-free subsets of finite groups*, J. Combin. Theory Series A, **77** (1997), 339–343.

[12] **K.S. Kedlaya**, *Product-free subsets of groups*, Amer. Math. Monthly, **105** (1998), 900–906.

[13] **I. Schur**, *Über die Kongruenz* $x^m + y^m \equiv z^m (\mathrm{mod}\, p)$, Jahresbericht der Deutschen Mathematiker-Vereinigung, **25** (1917), 114–116.

[14] **A.P. Street and E.G. Whitehead Jr.**, *Group Ramsey Theory*, J. Combin. Theory Series A, **17** (1974), 219–226.

[15] **A.P. Street and E.G. Whitehead, Jr.**, *Sum-free sets, difference sets and cyclotomy*, Comb. Math., Lect. Notes Math., **403** (1974), 109–124.

[16] **M. Tărnăuceanu**, *A characterization of elementary abelian 2-groups*, Archiv Math., **102** (2014), 11–14.

[17] **M. Tărnăuceanu**, *Erratum: A characterization of elementary abelian 2-groups*, Archiv Math., **108** (2017), 223–224.

[18] **H.P. Yap**, *Maximal sum-free sets of group elements*, J. London Mat. Soc., **44** (1969), 131–136.

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