Quasigroups and Related Systems **27** (2019), 157 – 160

On T^*-pure ordered semigroups

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Abstract. The concepts of T^**-pure* ordered semigroups is introduced. We characterize T^*-pure archimedean ordered semigroups and prove that any T^*-pure ordered semigroup is a semilattice of archimedean semigroups.

A bi-ideal A of a semigroup S is said to be *two-sided pure* if $A \cap xSy = xAy$ for all $x, y \in S$. A semigroup S is said to be T^**-pure* if every bi-ideal of S is two-sided pure. T^**-pure* semigroups has been studied by N. Kuroki [9].

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, i.e., for any $x, y, z \in S$,

$$x \leqslant y \text{ implies } zx \leqslant zy \text{ and } xz \leqslant yz,$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*).

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For any nonempty subsets A of S we define

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

It was shown in [8] that for any nonempty subsets A, B of S the following holds: (1) $A \subseteq (A]$; (2) $A \subseteq B$ implies $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $(A \cup B] = (A] \cup (B]$; (5) $((A]] = (A]$.

A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S (cf. [4]), if $SA \subseteq A$ (resp., $AS \subseteq A$) and $A = (A]$, that is, for any $x \in A$, $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union and intersection of two ideals of S are an ideal of S.

A left (right) ideal A of S is said to be *proper* if $A \subset S$. S is said to be *left* (resp., *right*) *simple* if S does not contain proper left (resp., right) ideals. If S does not contain proper ideals then we call S *simple*. A proper ideal A of S is said to be *maximal* if for any ideal B of S, if $A \subset B \subseteq S$, then $B = S$. In an ordered semigroup $(S, \cdot, \leqslant)$, the *principal ideal generated* by a is of the form $I(a) = (a \cup Sa \cup aS \cup SaS]$.

A subsemigroup B is called a *bi-ideal* of S if (i) $BSB \subseteq B$; (ii) for any $x \in B$ and $y \in S$, $y \leqslant x$ implies $y \in B$ ([5]).

A *bi-ideal generated* by a has the form $B(a) = (a \cup a^2 \cup aSa]$.

A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. A semilattice congruence σ on S is *complete* if $a \leqslant b$ implies $(a, ab) \in \sigma$. An ordered semigroup S is a *semilattice of archimedean semigroups* (resp., *complete semilattice of archimedean*) if there exists a semilattice congruence (resp., complete semilattice congruence) σ on S such that for each $x \in S$ the σ-class $(x)_\sigma$ is an archimedean subsemigroup of S.

A subsemigroup F is called a *filter* of S if (i) $a, b \in S$, $ab \in S$ implies $a \in F$ and $b \in F$; (ii) if $a \in F$ and b in S, $a \leqslant b$, then $b \in F$ ([3]).

For an element x of S, we denote by $N(x)$ the filter of S generated by x and consider the equivalence relation $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$. The relation $\mathcal{N}$ is the lest complete semilattice congruence on S.

2010 Mathematics Subject Classification: 06F05.
Keywords: Ordered semigroup, weakly commutative semigroup, semilattice of archimedean semigroups, bi-ideal.

An element e of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *ordered idempotent* if $e \leqslant e^2$. The set of all ordered idempotent of an ordered semigroup S denoted by $E(S)$. An ordered semigroup S is *idempotent ordered* if $S = E(S)$.

An ordered semigroup $(S, \cdot, \leqslant)$ is called *archimedean* [2] if for any $a, b \in S$ there exits a positive integer n such that $a^n \in (SbS]$. If for any $a, b \in S$ there exists positive integer n such that $(ab)^n \in (bSa]$, the S is called *weakly commutative* [7].

An element $a \in S$ is *regular* (resp., *completely regular*) if $a \in (aSa]$ (resp., $a \in (a^2Sa^2]$). A semigroup S is *regular* (resp., *completely regular*) if each its element is *regular* (resp., *completely regular*).

Definition 1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A bi-ideal A of S is said to be *two-sided pure* if $A \cap (xSy] = (xAy]$ for all $x, y \in S$. An ordered semigroup S is said to be T^**-pure* if every bi-ideal of S is two-sided pure.

Example 1. Let $S = \{a, b, c, d\}$ and $\leqslant = \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d)\}$. Then $(S, \cdot, \leqslant)$ with the multiplication $cc = dc = dd = b$ and $xy = a$ in all other cases, is an ordered semigroup and all its bi-ideals, namely $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, S, are pure. So, it is the T^*-pure ordered semigroup.

First, we have the following proposition.

Proposition 1. *Any T^*-pure ordered semigroup is weakly commutative.*

Proof. Let S be T^*-pure ordered semigroup and $a, b \in S$. Then $(bSa]$ is two-sided pure and

$$(ab)^3 = ababab \in (a(bSa]b] = (aSb] \cap (bSa] \subseteq (bSa].$$

Hence S is weakly commutative. □

Proposition 2. *Let S be T^*-pure ordered semigroup. Then S has the following properties:*

(1) $(aSb] = (a^2Sb^2]$ *for all* $a, b \in S$.

(2) *For any* $a \in S$, a^n *is completely regular for all positive integer* $n \geqslant 3$.

(3) *For each* $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy]$ *for some* $n \in N\}$.

(4) $(eS] = (Se]$ *for all* $e \in E(S)$.

Proof. (1). Since S is T^*-pure, $(aSb]$ is a two-sided pure bi-ideal. Thus

$$(aSb] = (aSb] \cap (aSb] = (a(aSb]b] \subseteq (a^2Sb^2].$$

The converse is obvious. Hence $(aSb] = (a^2Sb^2]$.

(2). By (1), $a^n = aa^{n-2}a \in (aSa] = ((a^n)^2S(a^n)^2]$ for any $a \in S$ and $n \geqslant 3$. Hence a^n is completely regular.

(3). This follows from Proposition 1 and Lemma in [7].

(4). Let $e \in E(S)$ and $x \in (Se]$. Then $x \leqslant ae$ for some $a \in S$. Since S is T^*-pure, $(eSe]$ is two-sided pure. Thus

$$x \leqslant ae \leqslant aeeee \in (a(eSe]e] = (aSe] \cap (eSe] \subseteq (eSe] \subseteq (eS].$$

Similarly, $(eS] \subseteq (Se]$. Hence $(eS] = (Se]$. □

Theorem 1. *Let $(S, \cdot, \leqslant)$ be a regular ordered semigroup. The following statements are equivalent:*

(1) *S is T^*-pure.*

(2) *S is weakly commutative.*

(3) *For each* $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy]$ *for some* $n \in N\}$.

(4) $(Se] = (eS]$ *for all* $e \in E(S)$.

Proof. (1) $\Rightarrow$ (2) by Proposition 1.

(2) $\Leftrightarrow$ (3) by Lemma in [7].

(2) $\Rightarrow$ (4). Let $e \in E(S)$ and $x \in (eS]$. Then $x \leqslant ea$ for some $a \in S$. Since S is regular, $ea \leqslant eabea$ for some $b \in S$. Then $bea \leqslant beabea = (bea)^2$. Since S is weakly commutative, then there exists positive integer n such that $(bea)^n \in (aSbe]$. Thus,

$$x \leqslant ea \leqslant eabea = ea(bea) \leqslant ea(bea)^n \in ea(aSbe] \subseteq (ea(aSbe]] \subseteq (eaaSbe] \subseteq (Se].$$

Similarly, $(eS] \subseteq (Se]$. Hence $(Se] = (eS]$.

(4) $\Rightarrow$ (1). Let A be bi-ideal of S, and $x, y \in S$. It is obvious that $(xAy] \subseteq (xSy]$. Let $z \in (xAy]$. Then $z \leqslant xay$ for some $a \in A$. Since S is regular, $a \leqslant aba$ for some $b \in S$. This implies that $ba, ab \in E(S)$. We have

$$\begin{aligned} z &\leqslant xay \leqslant xabay \leqslant xababab ay = x(bs)aba(ba)y \in (SabSbaS] \subseteq ((Sab]S(baS]] \\ &= ((abS]S(Sba]] \subseteq (ASA] \subseteq A. \end{aligned}$$

Hence $(xAy] \subseteq A \cap (xSy]$.

Let $a \in A \cap (xSy]$. Then $a \leqslant xzy$ for some $z \in S$. Since S is regular, $a \leqslant aba$ for some $b \in S$. This implies that $ba, ab \in E(S)$. We have

$$\begin{aligned} a &\leqslant aba \leqslant abababa \leqslant ababababa \leqslant xzybababababxzy \\ &= xzyb(ab)aba(ba)bxzy \in (xSabSbaSy] \subseteq (x(Sab]S(baS]y] \\ &= (x(abS]S(Sba]y] \subseteq (xASAy] \subseteq (xAy]. \end{aligned}$$

Thus $A \cap (xSy] \subseteq (xAy]$. Hence $A \cap (xSy] = (xAy]$. This complete the proof. $\square$

The following theorem can be obtained from Proposition 1 and Theorem in [7].

Theorem 2. *Any T^*-pure ordered semigroup is a semilattice of archimedean semigroups.*

Now we give a characterization of T^*-pure archimedean ordered semigroups.

Theorem 3. *For a T^*-pure ordered semigroup S the following statements are equivalent:*

(1) *S is archimedean.*

(2) *Every bi-ideal of S is archimedean.*

(3) *For any $e, f \in E(S)$, $(e, f) \in \mathcal{N}$.*

Proof. It is clear that (2) implies (1).

(3) $\Rightarrow$ (2). Let A be a bi-deal of S and $a, b \in A$. Since S is T^*-pure, a^3 and b^3 are regular by Proposition 2. Then $a^3 \leqslant a^3xa^3$ and $b^3 \leqslant b^3yb^3$ for some $x, y \in S$. This implies that $a^3x, b^3y \in E(S)$. We have $b^3y \in N(a^3x)$. Then $(a^3x)^n \in (b^3ySb^3y]$ for some positive integer n. Thus $(a^3x)^n \leqslant b^3yzb^3y$ for some $z \in S$. We have

$$\begin{aligned} a^3 &\leqslant a^3xa^3 \leqslant a^3xa^3xa^3 = (a^3x)a^3xa^3 \leqslant (a^3x)^n a^3xa^3 \leqslant (b^3yzb^3y)a^3xa^3 \\ &= bb(b(yzb^3ya^3xa^2)a) \in (Ab(ASA)] \subseteq (AbA]. \end{aligned}$$

Hence A is archimedean.

(1) $\Rightarrow$ (3). Let $e, f \in E(S)$. Since S is archimedean, there exists positive integer n such that $e^n \in (SfS]$. Since S is T^*-pure, $(fSf]$ is two-sided pure ideal. Then we have

$$e^n \in (SfS] \subseteq (SfffS] \subseteq (SfSfS] \subseteq (S(fSf]S] = (SSS] \cap (fSf] \subseteq (fSf].$$

Thus $f \in N(e)$. Hence $N(f) \subseteq N(e)$ Similarly, we have $N(e) \subseteq N(f)$. Hence $(e, f) \in \mathcal{N}$. $\square$

Theorem 4. *Any T^*-pure archimedean regular ordered semigroup S does not contain proper bi-ideals.*

Proof. Let A be any bi-ideal of S. Let $a \in A$ and $b \in S$. Since S is archimedean, then there exists positive integer n such that $b^n \in (SaS]$. Since S is T^*-pure, $(aSa]$ is two-sided pure. Then by regularity of S and Theorem 2, we have

$$b \in (bSb] = (b^n S b^n] \subseteq ((SaS]S(SaS]] \subseteq (SaSSSaS] \subseteq (S(aSa)S] \subseteq (S(aSa]S]$$
$$= (SSS] \cap (aSa] \subseteq (ASA] \subseteq A.$$

Thus $S \subseteq A$. Hence $S = A$. □

The following theorem can be obtained from Theorem 4.

Theorem 5. *Any T^*-pure archimedean regular ordered semigroup is left and right simple.*

Theorem 6. *For a T^*-pure archimedean ordered semigroup S the following statements are equivalent:*

(1) *S is regular.*

(2) *S does not contain proper bi-ideals.*

(3) *S are left and right simple.*

Proof. By Theorem 4, (1) implies (2). It is clear that (2) implies (3).

(3) $\Rightarrow$ (1). Let $a \in S$. Since S are left and right simple, $S = (Sa]$ and $S = (aS]$ by Corollary 2 in [6]. We have $a \in (aS] = (a(Sa]] \subseteq (aSa]$. This completes the proof. □

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Received February 27, 2018

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