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Characterizations of π-t-simple ordered semigroups by their ordered idempotents

Shauli Sadhya and Kalyan Hansda

Abstract. Here we extend the notion of π-groups in semigroups without order to ordered semigroups. We call them π-t-simple ordered semigroups. Our approach allows one to see the relations between Archimedean (t-Archimedean) ordered semigroups and π-t-simple ordered semigroups. Furthermore we show that a completely π-regular ordered semigroup S such that for any $a, b \in S$ there exists an ordered idempotent $e \in S$ with the property that $ab,\ b^r a^r \in \sqrt{H(e)}$ for any $r \in \mathbb{N}$, is a complete semilattice of π-t-simple ordered semigroups and conversely.

1. Introduction

Due to Cao and Xu [3], t-simple ordered semigroups play the same role in the theory of ordered semigroups as groups in the theory of semigroups without order. Bhuniya and Hansda [2] studied these ordered semigroups under the name of group like ordered semigroups. Here we extend t-simple ordered semigroups to π-t-simple ordered semigroups. Though these ordered semigroups were studied by Cao and Xu [3], but not under the name of π-t-simple ordered semigroups. The successful part of this paper is that our observation on π-t-simple ordered semigroups coincides with [3]. This paper is inspired by the work done by Cao and Xu [3].

Our paper is organized as follows. The basic definitions and properties of ordered semigroups are presented in Section 2. Section 3 is devoted to π-t-simple ordered semigroups and their relations with Archimedean ordered semigroups by their ordered idempotents.

2. Preliminaries

By an ordered semigroup we mean a partially ordered set $(S, \leqslant)$ which is at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$, $a \leqslant b$ implies $xa \leqslant xb$ and $ax \leqslant bx$. It is denoted by $(S, \cdot, \leqslant)$. For an ordered semigroup S, we denote $S^1 = S \cup \{1\}$, where 1 is a symbol, such that $1a = a,\ a1 = a$ for each $a \in S$ and $1 \cdot 1 = 1$. For every subset $H \subseteq S$, denote $(H] = \{t \in S : t \leqslant h, \text{ for some } h \in H\}$. An

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element 0 in S is called a zero of S if $0 \leqslant x$ and $0x = x0 = 0$ for every $x \in S$. An ordered semigroup S with 0 is called nil if for every $a \in S$ there is $n \in \mathbb{N}$ such that $a^n = 0$.

Let S be an ordered semigroup. A empty subset I of S is said to be a left (right) ideal of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] \subseteq I$. If I is both a left and right ideal, then it is called an ideal of S. We call S a (left, right) simple ordered semigroup if it does not contain any proper (left, right) ideal.

Due to Kehayopulu [7], Green's relation $\mathcal{H}$ on an ordered semigroup S is defined as follows: For $a, b \in S$, $a\mathcal{H}b$ if and only if $a \leqslant xb,\ b \leqslant ya,\ a \leqslant bu,\ b \leqslant av$ for some $x, y, u, v \in S^1$. For $a \in S$, the $\mathcal{H}$-class of a is denoted by $H(a)$.

By radical of a subset A of an ordered semigroup S we shall mean the set $\sqrt{A}$ defined by $\sqrt{A} = \{x \in S : (\exists\ m \in \mathbb{N})\ x^m \in A\}$. From [1], by the radical of a relation ρ on an ordered semigroup S we mean the relation denoted by $\sqrt{\rho}$ and defined by $a\sqrt{\rho}\,b$ if and only if there exist $m,\ n \in \mathbb{N}$ such that $a^m \rho b^n$. Let ρ be an equivalence relation on an ordered semigroup S. In a broad sense by ρ-unique we shall mean the uniqueness with respect to the relation ρ. Thus if for $a, b \in S$ we have $a\rho b$, then we say that a and b are the same with respect to ρ. An equivalence relation ρ on S is called a congruence if for all $a, b, c \in S$, $a\rho b$ implies $ca\rho cb$ and $ac\rho bc$. A congruence ρ on S is called a semilattice congruence if for every $a, b \in S$, $a\rho a^2$ and $ab\rho ba$. By a complete semilattice congruence we mean a semilattice congruence σ on S such that for $a, b \in S$, $a \leqslant b$ implies that $a\sigma ab$. An ordered semigroup S is called a complete semilattice of subsemigroups of type τ if there exists a complete semilattice congruence ρ such that each $\rho-$congruence class $(x)_\rho$ is a type τ subsemigroup of S. Equivalently [8], there exist a semilattice Y and a family of subsemigroups $\{S_\alpha\}_{\alpha \in Y}$ of type τ of S such that:

1. $S_\alpha \cap S_\beta = \ \phi$ for any $\alpha,\ \beta \in\ Y\ with\ \alpha \neq \beta$,
2. $S = \bigcup_{\alpha\, \in\, Y}\ S_\alpha$,
3. $S_\alpha S_\beta\ \subseteq\ S_{\alpha\,\beta}$ for any $\alpha,\ \beta \in\ Y$,
4. $S_\beta \cap (S_\alpha] \neq \phi$ implies $\beta\ \ \preceq\ \ \alpha$, where $\preceq$ is the order of the semilattice Y defined by $\ \preceq := \{(\alpha,\ \beta)\ \mid\ \alpha = \alpha\,\beta\ (\beta\,\alpha)\}$.

An ordered semigroup S is said to be *regular* (resp. *completely regular*) ordered semigroup if for every $a \in S$, $a \in (aSa]$ (resp. $a \in (a^2Sa^2]$). If $a \in (Sa^2S]$ for every $a \in S$, then S is called intra-regular. An ordered semigroup S is called *π-regular* (resp. *completely π-regular*) if for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \in (a^mSa^m]$ (resp. $a^m \in (a^{2m}Sa^{2m}]$). The set of regular, completely regular, intra-regular and π-regular elements in an ordered semigroup S is denoted by $Reg_\leqslant(S)$, $Gr_\leqslant(S)$, $Intra(S)$ and $\pi Reg_\leqslant(S)$ respectively. An element $e \in S$ is called an *ordered idempotent* (cf. [2]) if $e \leqslant e^2$. We denote the set of all ordered idempotents of an ordered semigroup S by $E_\leqslant(S)$. An ordered semigroup S is said to be *weakly commutative* if for all $a, b \in S$, $(ab)^n \in (bSa]$ for some $n \in \mathbb{N}$.

An ordered semigroup S is called a *t-simple ordered semigroup* (cf. [3]) if for all $a, b \in S$ there are $x, y \in S$ such that $a \leqslant xb$ and $a \leqslant by$. For $e \in S$, denote $G_e = \{a \in S : \ a \leqslant ea, \ a \leqslant ae \text{ and } e \leqslant za, \ e \leqslant az \text{ for some } z \in S\}$ (cf. [2]). Now if S is completely regular then we can find $z \in G_e$ and G_e forms a t-simple ordered subsemigroup of S (see [2]).

An ordered semigroup S is said to be *Archimedean* if for every $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in (SbS]$. An ordered semigroup S is said to be *left* (*right*) *Archimedean* if for every $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in (Sb]$ ($a^n \in (bS]$). An ordered semigroup S is said to be *t-Archimedean* if for every $x, y \in S$ there exists $m \in \mathbb{N}$ such that $y^m \in (xSx]$.

Theorem 2.1. (cf. [2]) *Every t-simple ordered semigroup is completely regular.*

Theorem 2.2. (cf. [2]) *A regular ordered semigroup S is a t-simple ordered semigroup if and only if for all $e, f \in E_{\leqslant}(S)$, $e\mathcal{H}f$.*

Cao and Xu [3] defined a nil-extension of an ordered semigroup as follows:

Let I be an ideal of an ordered semigroup S. Then $(S/I, \cdot, \preceq)$ is called the Rees factor ordered semigroup of S modulo I, and S is called an ideal extension of I by the ordered semigroup S/I. Moreover S is said to be a nil-extension of I if $(S/I, \cdot, \preceq)$ is a nil ordered semigroup.

3. Main results

Due to Cao and Xu [3, Corollary 5.2], an ordered semigroup S is a nil-extension of a t-simple ordered semigroup if and only if for every $a, b \in S$ there exists $m \in \mathbb{N}$ such that $a^m \in (b^n S b^n]$ for every $n \in \mathbb{N}$. Thus there may exist a t-simple ordered subsemigroup H of an ordered semigroup S such that $a^m \in H$ for every $a \in S$ and some $m \in \mathbb{N}$. So it is a very logical step to study the class of ordered semigroups of this type . This section is devoted to characterize these ordered semigroups.

Example 3.1. The set $S = \{a, b, c, d, e\}$ with respect to the multiplication $'\cdot'$ and the order $' \leqslant'$ below is an ordered semigroup.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	b	b	b	b	b
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	d

$\leqslant = \{(a, a), (a, b), (a, d), (a, e), (b, b), (c, b), (c, c), (c, e), (d, b), (d, d), (d, e), (e, e)\}$.
Now the subsets $H_1 = \{a, b\}$ and $H_2 = \{a, b, c\}$ are t-simple ordered subsemigroups of S, and for every $x \in S$ there exist $m, n \in \mathbb{N}$ such that $x^m \in H_1$ and $x^n \in H_2$.

Definition 3.2. Let S be an ordered semigroup. Then S is said to be a *π-t-simple ordered semigroup* if there exists a t-simple ordered subsemigroup H of S with the property that for every $a \in S$ there exists $m \in \mathbb{N}$ such that $a^m \in H$.

The ordered semigroup S, in Example 3.1 is a π-t-simple ordered semigroup. It is noted that H may not be unique.

A completely π-regular ordered semigroup may not be a π-t-simple ordered semigroup.

Example 3.3. The set $S = \{a, b, c, d, e\}$ with respect to the multiplication $'\cdot'$ and the order $'\leqslant'$ defined below is a completely π-regular ordered semigroup.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	b	b	c	b	b
d	a	b	b	d	b
e	b	b	b	b	b

$\leqslant = \{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,b),(d,d),(d,e),(e,e)\}$. The subset $H = \{a, b, c, d\}$ is a subsemigroup of S. Here H is not a t-simple ordered semigroup but still there is $m \in \mathbb{N}$ such that $x^m \in H$ for every $x \in H$.
If we take $\{a, c\}$ *or* $\{a, b\}$ *or* $\{a, d\}$ *or* $\{b, c\}$ *or* $\{b, d\}$, then also the conditions of S to be π-t-simple ordered semigroup do not hold.

Some characterizations of a π-t-simple ordered semigroup by its ordered idempotents have been given in the following theorem.

Theorem 3.4. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is a π-t-simple ordered semigroup;*

(2) *S is a nil-extension of a t-simple ordered semigroup;*

(3) *For any $a, b \in S$ there exists $m \in \mathbb{N}$ such that $a^m \in (ba^m S a^m b]$;*

(4) *S is completely π-regular and contains an $\mathcal{H}$-unique ordered idempotent;*

(5) *S is π-regular and contains an $\mathcal{H}$-unique ordered idempotent;*

(6) *S is t-Archimedean with an ordered idempotent.*

Proof. (1) $\Rightarrow$ (4): Suppose S satisfies (1) and let $a \in S$. Then there exist a t-simple ordered subsemigroup H of S and $m \in \mathbb{N}$ such that $a^m \in H$. Since also $a^{2m} \in H$, there exists $s \in H$ such that $a^m \leqslant a^{2m}s$. Furthermore, since $s, a^{2m} \in H$, we obtain $s \leqslant ta^{2m}$ for some $t \in H$. Hence $a^m \leqslant a^{2m}s \leqslant a^{2m}ta^{2m}$, and thus $a^m \in (a^{2m}Sa^{2m}]$, which shows that S is completely π-regular.

Let $e, f \in E_{\leqslant}(S)$. Then there are $m, n \in \mathbb{N}$ such that $e^m, f^n \in H$. Since H is a t-simple ordered subsemigroup, $e^m \leqslant f^n x$, $e^m \leqslant y f^n$, $f^n \leqslant e^m u$, $f^n \leqslant v e^m$ for some $x, y, u, v \in H$. Thus $e \leqslant e^m \leqslant f^n x = f(f^{n-1}x)$, which implies that $e \leqslant f s_1$ for some $s_1 \in S^1$. Similarly we obtain that $e \leqslant s_2 f$, $f \leqslant e s_3$, $f \leqslant s_4 e$ with s_2, $s_3, s_4 \in S^1$. Hence $e\mathcal{H}f$.

(4) $\Rightarrow$ (5): This implication is obvious.

(5) $\Rightarrow$ (6): Assume (5) holds and let $a, b \in S$. Since S is π-regular, $a^m \leqslant a^m s a^m$ and $b^n \leqslant b^n t b^n$ for some $s, t \in S$ and $m, n \in \mathbb{N}$. Since $a^m s, b^n t, s a^m, t b^n \in E_{\leqslant}(S)$ and ordered idempotents of S are $\mathcal{H}$-unique, there exists $x, y \in S^1$ such that $a^m s \leqslant b^n t x$ and $s a^m \leqslant y t b^n$. Hence $a^m \leqslant a^m s a^m \leqslant (a^m s) a^m (s a^m) \leqslant b^n t x a^m y t b^n$ and thus $a^m \in (bSb]$, which shows that S is t-Archimedean with an ordered idempotent.

(6) $\Rightarrow$ (3): Assume S is t-Archimedean, $e \in E_{\leqslant}(S)$ and $a, b \in S$. Since S is t-Archimedean, $a^m \leqslant ese$ for some $m \in \mathbb{N}$ and $s \in S$. Furthermore, since $e \in E_{\leqslant}(S)$ and S is t-Archimedean, for some $x, y \in S$ we have $e \leqslant b a^m x b a^m$ and $e \leqslant a^m b y a^m b$. Hence $a^m \leqslant ese \leqslant (b a^m x b a^m) s (a^m b y a^m b)$, and thus $a^m \in (b a^m S a^m b]$.

(3) $\Rightarrow$ (2): Assume (3) and let $a, b \in S$. Then $a^m \leqslant b a^m s a^m b$ for some $m \in \mathbb{N}$ and $s \in S$. Hence $a^m \leqslant b a^m s a^m b \leqslant b(b a^m s a^m b) s a^m b = b^2 a^m (s a^m b)^2 \leqslant b^2 (b a^m s a^m b)(s a^m b)^2 = b^3 a^m (s a^m b)^3 \leqslant \cdots$. Continuing in this way we obtain $a^m \leqslant b^{m+1} a^m (s a^m b)^{m+1}$ and thus $a^m \in (b^{m+1} S b]$. Hence (2) holds by [3, Corollary 5.2].

(2) $\Rightarrow$ (1): This implication is obvious. □

Corollary 3.5. *Every π-t-simple ordered semigroup is a nil-extension of a completely regular ordered semigroup.*

Proof. The result follows from Theorems 2.1 and 3.4. □

Theorem 3.6. *An ordered semigroup S is a π-t-simple ordered semigroup if and only if S is weakly commutative and Archimedean with an $\mathcal{H}$-unique ordered idempotent.*

Proof. First suppose that S is a weakly commutative and Archimedean ordered semigroup with an $\mathcal{H}$-unique ordered idempotent e. Then $e \leqslant e(e^2)e$. Therefore $e \in Intra(S)$ so that $Intra(S) \neq \phi$. Then S is a nil-extension of a simple ordered semigroup K by Theorem 3.5 of [3]. Let $a \in K$. Since K is simple, $a \leqslant x a^3 y$ for some $x, y \in K$. Then $a \leqslant x a^3 y \leqslant (xa) a (ay) \leqslant (xa)^2 a (ay)^2 \leqslant \ldots \leqslant (xa)^r a (ay)^r$ for every $r \in \mathbb{N}$. Since S is weakly commutative, there exist $m, n \in \mathbb{N}$ such that $(xa)^m \in (aSx]$ and $(ay)^n \in (ySa]$. Thus there exist $z, w \in S$ such that $(xa)^{mn} \leqslant azx$ and $(ay)^{mn} \leqslant ywa$. Hence $a \leqslant (xa)^{mn} a (ay)^{mn} \leqslant azxaywa$. Since K is an ideal of S, it follows that $zxayw \in K$ and thus $a \in (aKa]$. Hence K is a regular ordered semigroup.

Let $e, f \in E_{\leqslant}(K)$. Then by given condition $e\mathcal{H}f$ in S. Since K is an ideal of S, it is evident that $e\mathcal{H}f$ in K also. Therefore K is a t-simple ordered semigroup by

Theorem 2.2 and S is a nil-extension of the t-simple ordered semigroup K. Hence S is a π-t-simple ordered semigroup by Theorem 3.4.

Conversely suppose that S is a π-t-simple ordered semigroup and H is a t-simple ordered subsemigroup of S such that for every $x \in S$, $x^m \in H$ for some $m \in \mathbb{N}$. Let $a, b \in S$. Then $(ab)^m, (ba)^n \in H$ for some $m, n \in \mathbb{N}$. Since H is a t-simple ordered semigroup, $(ab)^m \in ((ba)^n H(ba)^n] \subseteq ((ba)^n S(ba)^n]$ $\subseteq (bSa]$. Thus S is weakly commutative.

Also $a^p, b^q \in H$ for some $p, q \in \mathbb{N}$. Since H is a t-simple ordered subsemigroup, we have $a^p \leqslant xb^q = xbb^{q-1}$ for some $x \in S$. Therefore, $a^p \in (SbS]$ for some $p \in \mathbb{N}$, which shows that S is Archimedean. Also there is an $\mathcal{H}$-unique ordered idempotent by Theorem 3.4. □

Lemma 3.7. *Let S be a completely π-regular ordered semigroup. Then for every $a \in S$ there exist $e \in E_{\leqslant}(S)$ and $m \in \mathbb{N}$ such that $a^m \leqslant a^m e$, $a^m \leqslant ea^m$, $e \leqslant za^m$, and $e \leqslant a^m z$ for some $z \in S$, that is $a^m \in G_e$.*

Proof. Let S be a completely π-regular ordered semigroup. Let $a \in S$. Then there exists $x \in S$ such that $a^m \leqslant a^{2m}xa^{2m}$ for some $m \in \mathbb{N}$. Let $e = a^{2m}xa^{2m}xa^{2m}$. Then we have $e = a^{2m}xa^{2m}xa^{2m} \leqslant a^{2m}xa^m a^m xa^{2m} \leqslant (a^{2m}xa^{2m}xa^{2m})a^m xa^{2m} \leqslant e(a^{2m}xa^{2m}xa^{2m}) = e^2$. Therefore $e \in E_{\leqslant}(S)$.

Now $a^m \leqslant a^{2m}xa^{2m} \leqslant a^m(a^{2m}xa^{2m}xa^{2m}) = a^m e$ and $a^m \leqslant a^{2m}xa^{2m} \leqslant (a^{2m}xa^{2m}xa^{2m})a^m = ea^m$. Also $e = a^{2m}xa^{2m}xa^{2m} \leqslant (a^{2m}xa^{2m}xa^{2m}xa^{2m})a^m$ and likewise $e \leqslant a^m(a^{2m}xa^{2m}xa^{2m}xa^{2m})$. Denote $z = a^{2m}xa^{2m}xa^{2m}xa^{2m}$. Then $e \leqslant za^m$ and similarly $e \leqslant a^m z$. Thus $a^m \in G_e$. This completes the proof. □

In the above lemma, it should be noted that $z = a^{2m}xa^{2m}xa^{2m}xa^{2m} \leqslant a^{2m}xa^{2m}xa^{2m}xa^{3m}xa^{2m} = za^m xa^{2m} \leqslant za^{2m}xa^{2m}xa^{2m} = ze$. Similarly $z \leqslant ez$. This shows that $z \in G_e$.

Lemma 3.8. *Let S be a completely π-regular ordered semigroup. Then the following statements hold in S:*

(1) *For every $e \in E_{\leqslant}(S)$, $G_e \subseteq H(e)$.*

(2) *For every $a \in S$, there are $e \in E_{\leqslant}(S)$ and $m \in \mathbb{N}$ such that $a^m \in H(e)$.*

Proof. (1): Let $x \in G_e$. Then $x \leqslant xe$, $x \leqslant ex$, $e \leqslant xz$, $e \leqslant zx$ for some $z \in S$. Hence $x\mathcal{H}e$ that is $x \in H(e)$. Therefore $G_e \subseteq H(e)$.

(2): This follows from Lemma 3.7. □

The following theorem is an extension of Corollary 5.3 of [3], that enables one to see the complete semilattice decomposition of π-t-simple ordered semigroups by their ordered idempotents.

Theorem 3.9. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is a complete semilattice of π-t-simple ordered semigroups;*

(2) *S is completely π-regular and for every $a, b \in S$ there is $e \in E_{\leqslant}(S)$ such that $ab, b^r a^r \in \sqrt{H(e)}$ for any $r \in \mathbb{N}$;*

(3) *for all $a, b \in S$ there exists $n \in \mathbb{N}$ such that $(ab)^n \in (b^{2n}Sa^{2n}]$.*

Proof. (1) $\Rightarrow$ (2): Let S be a complete semilattice Y of π-t-simple ordered semigroups $\{S_\alpha\}_{\alpha\in Y}$. By Theorem 3.4 all the semigroups S_α are completely π-regular and thus so is S. Let $a, b \in S$. Then $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. For any $r \in \mathbb{N}$ we have that $a^r \in S_\alpha$ and $b^r \in S_\beta$, and thus $ab, b^r a^r \in S_{\alpha\beta}$. From Theorem 3.4 it follows that $S_{\alpha\beta}$ contains an ordered idempotent e which is $\mathcal{H}$-unique in $S_{\alpha\beta}$. By Lemma 3.8 (applied to the semigroup $S_{\alpha\beta}$) there exists $m \in \mathbb{N}$ such that $(ab)^m \in H(e)$ and, by the same lemma, for any $r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $(b^r a^r)^n \in H(e)$. Thus $ab, b^r a^r \in \sqrt{H(e)}$.

(2) $\Rightarrow$ (3): Suppose that the condition (2) holds in S. Let $a, b \in S$. Since S is completely π-regular, there exists $n \in \mathbb{N}$ such that $(ab)^n \leqslant (ab)^{2n}x(ab)^{2n}$ for some $x \in S$. By given condition there is $e \in E_{\leqslant}(S)$ such that ab, $b^r a^r \in \sqrt{H(e)}$ for all $r \in \mathbb{N}$. This implies $(ab)^s$, $(b^r a^r)^t \in H(e)$ for some s, $t \in \mathbb{N}$. Taking $r = 2n$ we have, $(ab)^s$, $(b^{2n}a^{2n})^t \in H(e)$. Now $(ab)^s \leqslant y_1(b^{2n}a^{2n})^t \leqslant y_1(b^{2n}a^{2n})^{t-1}b^{2n}a^{2n}$ and $(ab)^s \leqslant (b^{2n}a^{2n})^t z_1 \leqslant b^{2n}a^{2n}(b^{2n}a^{2n})^{t-1}z_1$ for some $y_1, z_1 \in S^1$. Therefore we have $(ab)^s \leqslant yb^{2n}a^{2n}$ and $(ab)^s \leqslant b^{2n}a^{2n}z$ for $y = y_1(b^{2n}a^{2n})^{t-1}$, $z = (b^{2n}a^{2n})^{t-1}z_1 \in S^1$. Also we have $(ab)^n \leqslant (ab)^{2n}x(ab)^{2n}$. If $s \leqslant n$, then $(ab)^n \leqslant (ab)^{2n}x(ab)^{2n} \leqslant (ab)^{n+s}u(ab)^{n+s} \leqslant (ab)^s(ab)^n u(ab)^n(ab)^s \leqslant b^{2n}a^{2n}z(ab)^n u(ab)^n yb^{2n}a^{2n}$ for some $u \in S$. Therefore $(ab)^n \in (b^{2n}Sa^{2n}]$. If $s \geq n$, then $(ab)^n \leqslant (ab)^{2n}x(ab)^{2n} \leqslant (ab)^{3n}x(ab)^{2n}x(ab)^{2n}x(ab)^{3n} \leqslant \cdots$ and thus for any $k \in \mathbb{N}$ there exists $w \in S$ such that $(ab)^n \leqslant (ab)^{k+n}w(ab)^{k+n}$. By taking $k = s$ and proceeding as in the previous case we get $(ab)^n \in (b^{2n}Sa^{2n}]$.

(3) $\Rightarrow$ (1): Suppose that the condition (3) holds in S. Then for all $a, b \in S$ there exists $n \in \mathbb{N}$ such that $(ab)^n \in (b^{2n}Sa^{2n}] \subseteq (bSa]$. Thus S is weakly commutative. Taking $b = a$ we can prove that S is π-regular. Hence S is a complete semilattice of π-t-simple ordered semigroups, by [3, Corollary 5.3]. □

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References

[1] **A.K. Bhuniya and K. Hansda**, *On radicals of Green's relations in ordered semigroups*, Canadian Math. Bull., **60** (2017), 246–252.

[2] **A.K. Bhuniya and K. Hansda**, *On completely regular and Clifford ordered semigroups*, arXiv: 1707.01282v1. Submitted to Afrika Math.

[3] **Y.L. Cao and X.H. Xu**, *Nil-extensions of simple po-semigroups*, Commun. Algebra, **28**(5), (2000), 2477–2496.

[4] **J.L. Galibati and M.L. Veronesi**, *On quasicompletely regular semigroups*, Semigroup Forum, **29** (1984), 271–275.

[5] **K. Hansda**, *Regularity of subsemigroups generated by ordered idempotents*, Quasigroups Related Systems, **22** (2014), 217–222.

[6] **N. Kehayopulu**, *Remarks on ordered semigroups*, Math. Japon., **35** (1990), 1061–1063.

[7] **N. Kehayopulu**, *Ideals and Green's relations in ordered semigroups*, Int. J. Math. and Math. Sci., (2006) Article ID 61286.

[8] **N. Kehayopulu and M. Tsingelis**, *Semilattices of Archimedean ordered semigroups*, Algebra Coll., **15**(3) (2008), 527–540.

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Department of Mathematics
Visva Bharati University
Santiniketan, Bolpur - 731235
West Bengal, India
E-mails: sadhya.shauli567@gmail.com, kalyanh4@gmail.com