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b-lattice of nil-extensions of rectangular skew-rings

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Abstract. Every quasi completely regular semiring is a b-lattice of completely Archimedean semirings, i.e., a b-lattice of nil-extensions of completely simple semirings. In this paper we consider the semiring which is a b-lattice of nil-extensions of orthodox completely simple semirings.

1. Introduction

Semiring is one of the many concepts of universal algebra which is an established and recognized area of study. The structure of semirings has been studied by many authors, for example, by F. Pastijn, Y. Q. Guo, M. K. Sen, K. P. Shum and others ([7], [9]). In [4] the authors studied properties of quasi completely regular semirings. They proved that a semiring is a quasi completely regular semiring if and only if it is a b-lattice of completely Archimedean semirings. In [5], the authors proved that a semiring is a completely Archimedean semiring if and only if it is nil-extension of a completely simple semiring. In this paper we intend to study the semirings which are b-lattices of orthodox completely Archimedean semirings. The preliminaries and prerequisites we need for this article are discussed in section 2. In section 3, we discuss our main result.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a type (2, 2)-algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. An element p in a semiring S is said to be *infinite* [2] if and only if $p + x = p = x + p$ for all $x \in S$. Infinite element in a semiring is unique and is denoted by ∞. An infinite element ∞ in a semiring S having the property that $x \cdot \infty = \infty = \infty \cdot x$ for all $x(\neq 0) \in S$ is called *strongly infinite* [2]. An element a in a semiring $(S, +, \cdot)$ is said to be *additively regular* if there exists an element $x \in S$ such that $a = a + x + a$. An element a in a semiring $(S, +, \cdot)$ is called *completely regular* [9] if there exists an element $x \in S$ such that $a = a + x + a$, $a + x = x + a$ and $a(a + x) = a + x$. We call a semiring S *completely regular* if every element of S is completely regular. A semiring $(S, +, \cdot)$

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is called a *skew-ring* if its additive reduct $(S,+)$ is a group. A semiring $(S,+,\cdot)$ is called a *b-lattice* if $(S,\cdot)$ is a band and $(S,+)$ is a semilattice. If both the reducts $(S,+)$ and $(S,\cdot)$ are bands, then the semiring $(S,+,\cdot)$ is said to be an *idempotent semiring*. An element a in a semiring $(S,+,\cdot)$ is said to be *additively quasi regular* if there exists a positive integer n such that na is additively regular. An element a in a semiring $(S,+,\cdot)$ is said to be *quasi completely regular* [4] if there exists a positive integer n such that na is completely regular. Naturally, a semiring S is said to be quasi completely regular if all the elements of S are quasi completely regular.

A semigroup S is a *rectangular band* if it satisfies the identity $a = axa$ for all $a, x \in S$. An element x in a semigroup S is a *left zero* of S if $xa = x$ for all $a \in S$. A semigroup all of whose elements are left zeros is said to be a *left zero semigroup*. A *rectangular group* is isomorphic to a direct product of a rectangular band and a group. Again a *left group* is isomorphic to a direct product of a left zero semigroup and a group. A semigroup S in which the idempotents form a subsemigroup is *orthodox*. A completely simple orthodox semigroup is a rectangular group, and conversely. In this article, our aim is to extend the concepts of nil extensions of rectangular groups and left groups to semirings.

Throughout this paper, we always let $E^+(S)$ and Reg^+S respectively be the set of all additive idempotents and the set of all additively regular elements of the semiring S. Also we denote the set of all additive inverse elements of an additively regular element a in a semiring $(S,+,\cdot)$ by $V^+(a)$. As usual, we denote the Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{J}$ and $\mathscr{H}$ on the additive reduct $(S,+)$ of a semiring $(S,+,\cdot)$ by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{D}^+$, $\mathscr{J}^+$ and $\mathscr{H}^+$ respectively. It follows from the definition that every $\mathscr{K} \in \{\mathscr{L}^+, \mathscr{R}^+, \mathscr{D}^+, \mathscr{J}^+, \mathscr{H}^+\}$ is a congruence on the multiplicative reduct $(S,\cdot)$, and hence is a congruence on the semiring $(S,+,\cdot)$ if and only if it is a congruence on $(S,+)$. For other notations and terminologies not given in this paper, the reader is referred to the texts of Howie [3] & Golan [2].

Definition 2.1. (cf. [9]) A completely regular semiring $(S,+,\cdot)$ is called a *completely simple semiring* if $\mathscr{J}^+ = S \times S$.

Theorem 2.2. (cf. [10]) *Let R be a skew-ring, $(I,\cdot)$ and $(\Lambda,\cdot)$ are bands such that $I \cap \Lambda = \{o\}$. Let $P = (p_{\lambda,i})$ be a matrix over R, $i \in I$, $\lambda \in \Lambda$ and assume*

1. $p_{\lambda,o} = p_{o,i} = 0,$
2. $p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj},$
3. $p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk},$
4. $ap_{\lambda,i} = p_{\lambda,i}a = 0,$
5. $ab + p_{o\mu,io} = p_{o\mu,io} + ab,$
6. $ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab,$ *for all $i,j,k \in I$, $\lambda,\mu,\nu \in \Lambda$ and $a,b \in R$.*

On $\mathscr{M} = I \times R \times \Lambda$, we define operations '$+$' and '$\cdot$' on $\mathscr{M}$ by

$$(i,a,\lambda)+(j,b,\mu)=(i,a+p_{\lambda,j}+b,\mu)$$
and
$$(i,a,\lambda)\cdot(j,b,\mu)=(ij,-p_{\lambda\mu,ij}+ab,\lambda\mu).$$

Then $(\mathscr{M},+,\cdot)$ is a completely simple semiring. Conversely, every completely simple semiring is isomorphic to such a semiring.

The semiring constructed in the above theorem is denoted by $\mathscr{M}(I,R,\Lambda;P)$ and is called a *Rees matrix semiring.*

Corollary 2.3. (cf. [10]) *Let $\mathscr{M}(I,R,\Lambda;P)$ be a Rees matrix semiring. Then $p_{\lambda\mu,ij}=p_{\lambda o,oj}+p_{o\mu,io}$ holds for all $i,j\in I$; $\lambda,\mu\in\Lambda$. This yields $p_{\lambda,i}=p_{\lambda o,oi}+p_{o\lambda,io}$ and hence by assumption* (5) *and* (6) *stated in Theorem* 2.2*, $ab+p_{\lambda,i}=p_{\lambda,i}+ab$ for all $i\in I$; $\lambda\in\Lambda$ and $a,b\in R$.*

Definition 2.4. Let $(S,+,\cdot)$ be an additively quasi regular semiring. We consider the relations $\mathscr{L}^{*^+}$, $\mathscr{R}^{*^+}$, $\mathscr{J}^{*^+}$, $\mathscr{H}^{*^+}$ and $\mathscr{D}^{*^+}$ on S defined by: for $a,b\in S$;

$$\begin{aligned} a\,\mathscr{L}^{*^+}\,b &\text{ if and only if } ma\,\mathscr{L}^+\,nb,\\ a\,\mathscr{R}^{*^+}\,b &\text{ if and only if } ma\,\mathscr{R}^+\,nb,\\ a\,\mathscr{J}^{*^+}\,b &\text{ if and only if } ma\,\mathscr{J}^+\,nb,\\ \mathscr{H}^{*^+}=\mathscr{L}^{*^+}\cap\mathscr{R}^{*^+} &\text{ and } \mathscr{D}^{*^+}=\mathscr{L}^{*^+}o\,\mathscr{R}^{*^+}, \end{aligned}$$

where m and n are the smallest positive integers such that ma and nb respectively are additively regular.

Definition 2.5. (cf. [4]) A quasi completely regular semiring $(S,+,\cdot)$ is called *completely Archimedean* if $\mathscr{J}^{*^+}=S\times S$.

Definition 2.6. (cf. [4]) A semiring S is said to be a *quasi skew-ring* if S has a subskew-ring R such that for every $a\in S$, there exists a positive integer n with $na\in R$.

Definition 2.7. A congruence ρ on a semiring S is called a *b-lattice congruence* (*idempotent semiring congruence*) if S/ρ is a b-lattice (respectively, an idempotent semiring). A semiring S is called a *b-lattice* (*idempotent semiring*) Y of semirings $S_\alpha(\alpha\in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ρ on S such that $Y=S/\rho$ and each S_α is a ρ-class.

Theorem 2.8. (cf. [4]) *The following conditions on a semiring $(S,+,\cdot)$ are equivalent.*

1. *S is a quasi completely regular semiring.*
2. *Every $\mathscr{H}^{*^+}$-class is a quasi skew-ring.*
3. *S is (disjoint) union of quasi skew-rings.*
4. *S is a b-lattice of completely Archimedean semirings.*

5. *S is an idempotent semiring of quasi skew-rings.*

Definition 2.9. (cf. [5]) Let $(S,+,\cdot)$ be a semiring. A nonempty subset I of S is said to be a *bi-ideal* of S if $a \in I$ and $x \in S$ imply that $a+x,\ x+a,\ ax,\ xa \in I$.

Definition 2.10. (cf. [5]) Let I be a bi-ideal of a semiring S. We define a relation ρ_I on S by $a\,\rho_I\,b$ if and only if either $a,b \in I$ or $a=b$ where $a,b \in S$. It is easy to verify that ρ_I is a congruence on S. This congruence is said to be the *Rees congruence* on S and the quotient semiring S/ρ_I contains a strongly infinite element, namely I. This quotient semiring S/ρ_I is said to be the *Rees quotient semiring* and is denoted by S/I. In this case the semiring S is said to be an *ideal extension* or simply an *extension* of I by the semiring S/I. An ideal extension S of a semiring I is said to be a *nil-extension* of I if for any $a \in S$ there exists a positive integer n such that $na \in I$.

Theorem 2.11. (cf. [5]) *A semiring S is a quasi skew-ring if and only if S is a nil-extension of a skew-ring.*

Theorem 2.12. (cf. [5]) *The following conditions on a semiring are equivalent:*

1. *S is a completely Archimedean semiring.*
2. *S is a nil-extension of a completely simple semiring.*
3. *S is Archimedean and quasi completely regular.*

3. Main results

In this section we characterize b-lattice of nil-extensions of rectangular skew-rings.

Definition 3.1. An idempotent semiring $(S,+,\cdot)$ is said to be a *rectangular band semiring* if $(S,+)$ is a rectangular band.

Definition 3.2. A semiring $(S,+,\cdot)$ is said to be a *rectangular skew-ring* if it is a completely simple semiring and $(S,+)$ is an orthodox semigroup, i.e., $E^+(S)$ is a subsemigroup of the semigroup $(S,+)$.

Theorem 3.3. *A semiring is a rectangular skew-ring if and only if it is isomorphic to direct product of a rectangular band semiring and a skew-ring.*

Proof. Let $(S,+,\cdot)$ be a rectangular skew-ring. Then by definition $(S,+,\cdot)$ is a completely simple semiring and $E^+(S)$ is a subsemigroup of the semigroup $(S,+)$. Let $S = \mathscr{M}(I,R,\Lambda;P)$. As $E^+(S)$ is a subsemigroup of $(S,+)$ then we have $(i,-p_{\lambda,i},\lambda)+(j,-p_{\mu,j},\mu) = (i,-p_{\mu,i},\mu)$. This implies $-p_{\lambda,i}+p_{\lambda,j}-p_{\mu,j} = -p_{\mu,i}$, i.e., $p_{\lambda,i}-p_{\mu,i}+p_{\mu,j} = p_{\lambda,j}$. Putting $j=o$, we get $p_{\lambda,i} = p_{\mu,i}$. Similarly, putting $i=o$ we get $p_{\lambda,j} = p_{\mu,j}$. Obviously, $E^+(S)$ is a rectangular band semiring.

Now we define a mapping $\varphi : S \to E^+(S) \times R$ by

$$\varphi(i,a,\lambda) = \Big((i,-p_{\lambda,i},\lambda), p_{\lambda,i}+a\Big) \text{ for all } (i,a,\lambda)\in S.$$

Now,

$$\begin{aligned}\varphi(i,a,\lambda)+\varphi(j,b,\mu) &= \Big((i,-p_{\lambda,i},\lambda),p_{\lambda,i}+a\Big)+\Big((j,-p_{\mu,j},\mu),p_{\mu,j}+b\Big)\\ &= \Big((i,-p_{\lambda,i},\lambda)+(j,-p_{\mu,j},\mu),p_{\lambda,i}+a+p_{\mu,j}+b\Big)\\ &= \Big((i,-p_{\mu,i},\mu),p_{\mu,i}+a+p_{\lambda,j}+b\Big)\\ &= \varphi(i,a+p_{\lambda,j}+b,\mu)\\ &= \varphi\Big((i,a,\lambda)+(j,b,\mu)\Big).\end{aligned}$$

Again,

$$\begin{aligned}\varphi(i,a,\lambda)\cdot\varphi(j,b,\mu) &= \Big((i,-p_{\lambda,i},\lambda),p_{\lambda,i}+a\Big)\cdot\Big((j,-p_{\mu,j},\mu),p_{\mu,j}+b\Big)\\ &= \Big((i,-p_{\lambda,i},\lambda)(j,-p_{\mu,j},\mu),(p_{\lambda,i}+a)(p_{\mu,j}+b)\Big)\\ &= \Big((ij,-p_{\lambda\mu,ij},\lambda\mu),ab\Big).\end{aligned}$$

Also,

$$\begin{aligned}\varphi\Big((i,a,\lambda)\cdot(j,b,\mu)\Big) &= \varphi(ij,-p_{\lambda\mu,ij}+ab,\lambda\mu)\\ &= \Big((ij,-p_{\lambda\mu,ij},\lambda\mu),p_{\lambda\mu,ij}-p_{\lambda\mu,ij}+ab\Big)\\ &= \Big((ij,-p_{\lambda\mu,ij},\lambda\mu),ab\Big).\end{aligned}$$

So $\varphi(i,a,\lambda)\cdot\varphi(j,b,\mu) = \varphi\Big((i,a,\lambda)\cdot(j,b,\mu)\Big)$.

Hence φ is a homomorphism. It is easy to show that φ is one-one and onto. Consequently, φ is an isomorphism from a rectangular skew-ring S onto a direct product of a rectangular band semiring $E^+(S)$ and a skew-ring R.

Converse part is obvious. □

Theorem 3.4. *The following conditions on a semiring S are equivalent:*

- (i) *S is a b-lattice of nil-extensions of rectangular skew-rings.*
- (ii) *S is a quasi completely regular semiring and for every $e,f\in E^+(S)$, there exists $n\in\mathbb{N}$ such that $n(e+f)=(n+1)(e+f)$.*
- (iii) *S is additively quasi regular, $b^2\,\mathscr{H}^{*^+}\,b$ for all $b\in S$ and $a=a+x+a$ implies $a=a+2x+2a$.*

Proof. $(i)\Rightarrow(ii)$: Let S be a b-lattice of nil-extensions of rectangular skew-rings. Then S is a b-lattice of nil-extensions of orthodox completely simple semirings. So by Theorem 2.12, it follows that S is a b-lattice of completely Archimedean semirings and hence by Theorem 2.8, we have S is a quasi completely regular semiring. Again, $(S,+)$ is a semilattice of nil-extensions of orthodox completely simple semigroups, i.e., $(S,+)$ is a semilattice of nil-extensions of rectangular groups. Then by Theorem X.2.1 [1], for every $e,f\in E^+(S)$, there exists $n\in Z^+$ such that $n(e+f)=(n+1)(e+f)$.

$(ii) \Rightarrow (iii)$: Let $(S, +, \cdot)$ be a quasi completely regular semiring and for every $e, f \in E^+(S)$, there exists $n \in Z^+$ such that $n(e+f) = (n+1)(e+f)$. Hence S is an additively quasi regular semiring and $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$. Let $a = a+x+a$. Then a is additively regular and hence by Theorem X.2.1 [1], it follows that $a = a + 2x + 2a$.

$(iii) \Rightarrow (i)$: Let S be additively quasi regular, $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$ and $a = a + x + a$ implies $a = a + 2x + 2a$. Then by Theorem X.2.1 [1], $(S, +)$ is a semilattice of nil-extensions of rectangular groups. This implies $(S, +)$ is a GV-semigroup. To complete the proof, it suffices to show that every additively regular element of S is completely regular. For this let $a = a+x+a$. Then a is regular and hence a is completely regular in the semigroup $(S, +)$. So there exists an element $y \in S$ such that $a = a+y+a$ and $a+y = y+a$. Now since $\mathscr{H}^{*^+}$ is a congruence on $(S, \cdot)$ and $(a+y) \mathscr{H}^{*^+} a$, it follows that $a(a+y) \mathscr{H}^{*^+} a^2 \mathscr{H}^{*^+} a \mathscr{H}^{*^+} (a+y)$. Since each $\mathscr{H}^{*^+}$-class contains a unique additive idempotent and $a(a+y) \mathscr{H}^{*^+} (a+y)$, it follows that $a(a+y) = a+y$. Hence S is a quasi completely regular semiring. Consequently, S is a b-lattice of nil-extensions of rectangular skew-rings. □

Theorem 3.5. *A semiring S is a nil-extension of a rectangular skew-ring if and only if S is a completely Archimedean semiring and $E^+(S)$ is a subsemigroup of $(S, +)$.*

Proof. Let $(S, +, \cdot)$ be a nil-extension of a rectangular skew-ring $(K, +, \cdot)$. Then $(K, +, \cdot)$ is a completely simple semiring and $E^+(K)$ is a subsemigroup of $(K, +)$. Hence S is a completely Archimedean semiring. Clearly, $E^+(S) = E^+(K)$ and thus $E^+(S)$ is a subsemigroup of $(S, +)$.

Conversely, let $(S, +, \cdot)$ be a completely Archimedean semiring and $E^+(S)$ be a subsemigroup of $(S, +)$. Then $(S, +, \cdot)$ is a nil-extension of a completely simple semiring $(K, +, \cdot)$. Also, $E^+(S) = E^+(K)$ implies $(K, +, \cdot)$ is a completely simple semiring such that $(K, +)$ is orthodox. Thus K is a rectangular skew-ring and hence S is nil-extension of a rectangular skew-ring K. □

Theorem 3.6. *The following conditions are equivalent on a semiring $(S, +, \cdot)$:*

(i) *S is a quasi completely regular semiring and $E^+(S)$ is a subsemigroup of $(S, +)$.*

(ii) *S is additively quasi regular, $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$, $a = a + x + a$ implies $a = a + 2x + 2a$ and Reg^+S is a subsemigroup of $(S, +)$.*

(iii) *S is a b-lattice of nil-extensions of rectangular skew-rings and $E^+(S)$ is a subsemigroup of $(S, +)$.*

Proof. $(i) \Rightarrow (ii)$: Let S be a quasi completely regular semiring and $E^+(S)$ be a subsemigroup of $(S, +)$. Then clearly S is a b-lattice of nil-extensions of rectangular skew-rings. Hence by Theorem 3.4, it follows that S is additively quasi regular,

$b^2 \mathscr{H}^{*^+} b$ for all $b \in S$ and $a = a + x + a$ implies $a = a + 2x + 2a$. Also $E^+(S)$ is a subsemigroup of $(S, +)$. Then by Proposition X.2.1 [1], for any $a, b, x, y \in S$, $a = a+x+a$ and $b = b+y+b$ imply $a+b = a+b+y+x+a+b$, i.e., $a, b \in Reg^+S$ implies $a + b \in Reg^+S$. Hence Reg^+S is a subsemigroup of $(S, +)$.

$(ii) \Rightarrow (iii)$: Let S be additively quasi regular, $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$, $a = a + x + a$ implies $a = a + 2x + 2a$ and Reg^+S is a subsemigroup of $(S, +)$. Then by Theorem 3.4, S is a b-lattice of nil-extensions of rectangular skew-rings. Let $e, f \in E^+(S) \subset Reg^+(S)$. Then $e + f \in Reg^+(S)$ and let $x \in S$ be an additive inverse of $e + f$. Then $f + x + e = f + (x + e + f + x) + e = 2(f + x + e)$. Now $(e+f)+(f+x+e)+(e+f) = e+f+x+e+f = e+f$ implies $e+f = (e+f)+2(f+x+e)+2(e+f) = (e+f)+(f+x+e)+(e+f)+(e+f) = (e+f)+(e+f) = 2(e+f)$. Hence we have $E^+(S)$ is a subsemigroup of $(S, +)$.

$(iii) \Rightarrow (i)$: It is obvious by Theorem 3.4. □

Theorem 3.7. *The following conditions are equivalent on a semiring* $(S, +, \cdot)$:

- (i) *S is a quasi completely regular semiring and for every $e, f \in E^+(S)$, $e + f = f + e$.*
- (ii) *S is a b-lattice of quasi skew-rings and for every $e, f \in E^+(S)$, $e + f = f + e$.*
- (iii) *S is additively quasi regular and Reg^+S is a subsemigroup of $(S, +)$ which is a b-lattice of skew-rings.*

Proof. $(i) \Rightarrow (ii)$: Let S be a quasi completely regular semiring and for every $e, f \in E^+(S)$, $e + f = f + e$. Then $\mathscr{H}^{*+}$ is a b-lattice congruence on S. Hence S is a b-lattice of quasi skew-rings and for every $e, f \in E^+(S)$, $e + f = f + e$.

$(ii) \Rightarrow (i)$: This part is obvious.

$(i) \Rightarrow (iii)$: Let S be a quasi completely regular semiring such that for every $e, f \in E^+(S)$, $e+f = f+e$. Then obviously S is additively quasi regular. Now let $a, b \in Reg^+S$. Then there exist $x, y \in S$ such that $a = a+x+a$ and $b = b+y+b$. So $a+x, x+a, b+y, y+b \in E^+(S)$. Now, $a+b = a+x+a+b+y+b = a+b+y+x+a+b$ implies $a + b \in Reg^+S$. Hence Reg^+S is a subsemigroup of $(S, +)$. Since in a quasi completely regular semiring, every additively regular element is completely regular, it follows that Reg^+S is a completely regular semiring. Thus by Theorem 3.6 [9], Reg^+S can be regarded as a b-lattice Y of completely simple semirings R_α $(\alpha \in Y)$, where $Y = S/\mathscr{J}^+$ and each R_α is a $\mathscr{J}^+$-class in S. Let e and f be two additive idempotents in R_α. Then $e + f = f + e$ and $e \mathscr{J}^+ f$. Since R_α is a completely simple semiring then $(R_\alpha, +)$ is a completely simple semigroup and so $e \mathscr{D}^+ f$ and thus we have $e = f$. This shows that each R_α contains a single additive idempotent, so that $(R_\alpha, +)$ is a group and hence $(R_\alpha, +, \cdot)$ is a skew-ring. In other words, we have shown that Reg^+S is a b-lattice Y of skew-rings R_α.

$(iii) \Rightarrow (i)$: Let S be an additively quasi regular semiring and Reg^+S be a subsemigroup of $(S, +)$ which is a b-lattice of skew-rings. Then clearly S is a quasi completely regular semiring. Moreover, $(Reg^+(S), +)$ is a Clifford semigroup.

Let $e, f \in E^+(S)$. Then e and f are two idempotents in the Clifford semigroup $(Reg^+(S), +)$. Hence $e + f = f + e$. □

Definition 3.8. An idempotent semiring $(S, +, \cdot)$ is said to be a *left zero semiring* if $(S, +)$ is a left zero band.

Definition 3.9. *A semiring S is said to be a* left skew-ring *if it is isomorphic to a direct product of a left zero semiring and a skew-ring.*

We recall that a left group is isomorphic to a direct product of a left zero semigroup and a group. Thus, if a semiring $(S, +, \cdot)$ is a left skew-ring then the semiring $(S, +, \cdot)$ is a rectangular skew-ring and the semigroup $(S, +)$ is a left group and conversely.

Theorem 3.10. *The following conditions are equivalent on a semiring $(S, +, \cdot)$:*

(i) *S is a b-lattice of nil-extensions of left skew-rings.*

(ii) *S is a quasi completely regular semiring and for every $e, f \in E^+(S)$, there exists $n \in \mathbb{N}$ such that $n(e + f) = n(e + f + e)$.*

(iii) *S is additively quasi regular, $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$ and $a = a + x + a$ implies $a + x = a + 2x + a$.*

Proof. (i) ⇒ (ii): Let S be a b-lattice Y of nil-extensions of left skew-rings S_α ($\alpha \in Y$). Then S is a b-lattice of nil-extensions of rectangular skew-rings. Hence by Theorem 3.4, S is a quasi completely regular semiring. Again $(S, +)$ is a semilattice of nil-extensions of left groups $(S_\alpha, +)$ ($\alpha \in Y$). Hence by Theorem X.2.2 [1], it follows that for every $e, f \in E^+(S)$, there exists a positive integer n such that $n(e + f) = n(e + f + e)$.

(ii) ⇒ (iii): Let S be a quasi completely regular semiring and for every $e, f \in E^+(S)$, there exists $n \in \mathbb{N}$ such that $n(e + f) = n(e + f + e)$. Then clearly S is additively quasi regular and $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$. Again $(S, +)$ is a GV-semigroup such that for every $e, f \in E^+(S)$, there exists $n \in \mathbb{N}$ such that $n(e + f) = n(e + f + e)$. Hence by Theorem X.2.2 [1], it follows that $a = a + x + a$ implies $a + x = a + 2x + a$.

(iii) ⇒ (i): Let S be an additively quasi regular semiring such that $b^2 \mathscr{H}^{*^+} b$ for all $b \in S$ and $a = a + x + a$ implies $a + x = a + 2x + a$. Then $a = a + x + a = (a + x) + a = (a + 2x + a) + a = a + 2x + 2a$. So by Theorem 3.4, it follows that S is a b-lattice Y of nil-extensions of rectangular skew-rings S_α ($\alpha \in Y$) and, by Theorem X.2.2 [1], we have $(S_\alpha, +)$ is a left group. Hence S_α is a left skew-ring. Consequently, S is a b-lattice of nil-extensions of left skew-rings. □

Theorem 3.11. *The following conditions are equivalent on a semiring $(S, +, \cdot)$:*

(i) *S is a b-lattice of nil-extensions of left skew-rings and $E^+(S)$ is a subsemigroup of $(S, +)$.*

(*ii*) *S is a quasi completely regular semiring and for every* $e, f \in E^+(S)$, $e+f = e+f+e$.

Proof. $(i) \Rightarrow (ii)$: Let S be a b-lattice of nil-extensions of left skew-rings and $E^+(S)$ is a subsemigroup of $(S,+)$. Then by Theorem 3.10, it follows that S is a quasi completely regular semiring and for every $e, f \in E^+(S)$, there exists a positive integer n such that $n(e+f) = n(e+f+e)$. Now since $E^+(S)$ is a subsemigroup of $(S,+)$, we have $e+f, e+f+e \in E^+(S)$. Hence $e+f = n(e+f) = n(e+f+e) = e+f+e$.

$(ii) \Rightarrow (i)$: Let S be a quasi completely regular semiring and for every $e, f \in E^+(S)$, $e+f = e+f+e$. Then by Theorem 3.10, we have S is a b-lattice of nil-extensions of left skew-rings. Also for every $e, f \in E^+(S)$, $2(e+f) = e+f+e+f = e+f+f = e+f$ which implies that $e+f \in E^+(S)$, i.e., $E^+(S)$ is a subsemigroup of $(S,+)$. □

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