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Cayley graphs of gyrogroups

Limpapat Bussaban, Attapol Kaewkhao and Suthep Suantai

Abstract. Gyrogroup is a generalization of group. It is well-known that any group can be viewed as a gyrogroup with trivial gyroautomorphism. In this article, the Cayley graphs of gyrogroups are discussed and some well-known properties in Cayley graphs of groups will be proved for Cayley graphs of gyrogroups.

1. Introduction

Cayley graph or *Cayley colour graph*, named for the famous mathematician Arthur Cayley, of a group G relative to a generating set $S \subseteq G$, denoted by $Cay(G, S)$, is a digraph with vertex set G and edge set $E(G)$ defined by $(x, y) \in E(G)$ if $y = sx$ for some $s \in S$, i.e., the edge from x to y is labeled by the colour s. Cayley graphs of groups have been extensively studied and many interesting results have been obtained, see [2, 4, 6], for examples. Recall the well-known properties of the Cayley graphs of a group as follows: the Cayley graph $Cay(G, S)$ is undirected if and only if the generating S is symmetric, i.e., $S = S^{-1}$, $S^{-1} = \{s^{-1} | s \in S\}$; the Cayley graph $Cay(G, S)$ is connected if and only if the group G can be generated by S, i.e., $G = \langle S\rangle$ and every Cayley graph $Cay(G, S)$ is vertex-transitive.

Gyrogroup, a group-like structure, first arose by A. A. Ungar [10] in the study of Einstein's velocity addition in the special theory of relativity. Gyrogroups play an important role in studying non-associative algebraic structure and hyperbolic geometry, just as groups play an important role in studying associative algebraic structure and Euclidean geometry. It motivated from the c-ball of relativistically admissible velocities, $\mathbb{R}^3_c = \{v \in \mathbb{R} : \|v\| < c\}$ such that c is a positive constant representing the speed of light in vacuum and Einstein velocity addition $\oplus_E$ in $\mathbb{R}^3_c$ is given by

$$u \oplus_E v = \frac{1}{1 + \frac{\langle u,v\rangle}{c^2}} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} \langle u, v\rangle\, u \right\}$$

where γ_u is the Lorentz factor given by $\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}$.

In [9], Ungar showed that the system $(\mathbb{R}^3_c, \oplus_E)$, called Einstein gyrogroup, does not form a group since $\oplus_E$ is neither associative nor commutative. The breakdown of associativity in $(\mathbb{R}^3_c, \oplus_E)$ is remedied by the space rotations $gyr[u, v]$, called

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gyroautomorphism, i.e.,

$$u \oplus_E (v \oplus_E w) = (u \oplus_E v) \oplus_E gyr[u, v]w$$

$$(u \oplus_E v) \oplus_E w = u \oplus_E (v \oplus_E gyr[v, u]w).$$

The resulting system $(\mathbb{R}^3_c, \oplus_E)$ forms a gyrocommutative gyrogroup.

Gyrogroup, generalized algebraic structure of group, was intensively studied in many papers [1, 3, 5, 8], any group can be observed as a gyrogroup with trivial gyroautomorphism. However, gyrogroups share remarkable analogies with groups. The algebraic properties of gyrogroups were studied by Suksumran [7], including Cayley's theorem, Lagrange's theorem, and isomorphism theorem for gyrogroups.

In this article, the concept of Cayley graphs of gyrogroups will be discussed, and we continue to prove some well-known properties of Cayley graphs of groups for finite gyrogroups, including the direction and the connectivity. Moreover, we show that there exists a Cayley graph of some gyrogroup which is not vertex-transitive.

2. Preliminaries

For the basic theory of gyrogroups and its algebraic properties, the reader is refered to [7, 10] and the basic terminologies of algebraic graph theory, the reader is refered to [2]. Let G be a nonempty set and $\oplus$ be a binary operation in G. The pair $(G, \oplus)$ is called *groupoid* if its binary operation is closed. A groupoid $(G, \oplus)$ is called *loop* if it contains an identity element 0.

Definition 2.1. A groupoid $(G, \oplus)$ is called a *gyrogroup* if its binary operation satisfies the following axioms:

(G1) there is $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$;

(G2) for any $a \in G$, there is $b \in G$ such that $b \oplus a = 0$;

(G3) for any $a, b \in G$, there is an automorphism $gyr[a, b] : G \to G$ such that for any $c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c;$$

(G4) for any $a, b \in G$, $gyr[a, b] = gyr[a \oplus b, b]$.

Throughout this paper, 0 in (G1) is called an *identity* of G and the element b in (G2) is called an *inverse* of a, the notation of inverse of a is denoted by $\ominus a$.

Definition 2.2. Let $(G, \oplus)$ be a gyrogroup with gyrogroup operation (or, addition) $\oplus$. The *gyrogroup cooperation* (or, *coaddition*) $\boxplus$ is a second binary operation in G given by the equation

$$a \boxplus b = a \oplus gyr[a, \ominus b]b$$

for all $a, b \in G$. Note that $a \boxminus b = a \boxplus \ominus b$.

Proposition 2.3. (cf. [10]) *Let* $(G, \oplus)$ *be a gyrogroup and let* $a, b, c \in G$. *The following indentities are satisfied:*

1. $a \oplus (\ominus a \oplus b) = b$ [left cancellation]
2. $(b \ominus a) \boxplus a = b$ [right cancellation]
3. $(b \boxminus a) \oplus a = b$ [right cancellation]
4. $(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$ [right gyroassociative law]
5. $\ominus(a \boxplus b) = (\ominus b) \boxplus (\ominus a)$ [cogyroautomorphic inverse]
6. $gyr[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$
7. $gyr[\ominus a, \ominus b] = gyr[a, b]$ [even symmetry]
8. $gyr^{-1}[a, b] = gyr[b, a]$ [inverse symmetry]
9. $gyr[a, b \oplus a] = gyr[a, b]$ [right loop property]

A nonempty subset H of a gyrogroup G is called a *subgyrogroup* of G if it forms gyrogroup under the binary operation of G restricted to H.

Proposition 2.4. ([7], Proposition 26) *Let* A *be a nonempty subset of a gyrogroup* $(G, \oplus)$. *There exists a unique smallest subgyrogroup generated by* A *of* G, *denoted by* $\langle A \rangle$. *In case of* A *singleton, i.e.,* $A = \{a\}$, *the smallest subgyrogroup* $\langle a \rangle$, *instead of* $\langle \{a\} \rangle$, *forms a group under operation* $\oplus$.

Definition 2.5. Let $(G, \oplus)$ be a gyrogroup. For each $a \in G$, a *left gyrotranslation by* a is a self-map L_a of G given by $L_a(x) = a \oplus x$, for all $x \in G$.

Theorem 2.6. (cf. [7]) *A loop* $(G, \oplus)$ *is a gyrogroup if and only if the following conditions hold:*

1. *for any* $a, b \in G$, *there exists a bijection* $gyr[a, b] : G \to G$ *such that*

$$gyr[a, b] \circ L_x = L_{gyr[a,b]x} \circ gyr[a, b]$$

   for all $x \in G$,

2. *for any* $a, b \in G$, *there exists* $c \in G$ *such that* $L_a \circ L_b = L_c \circ gyr[a, b]$,
3. *for any* $a, b \in G$, *there exists* $c \in G$ *such that* $L_{\ominus c \oplus a} = L_{\ominus(c \oplus b) \oplus b}$.

Theorem 2.7. (cf. [10]) *A groupoid* $(G, \oplus)$ *forms a gyrogroup if and only if it satisfies the following properties:*

(g1) *There is* $0 \in G$ *such that* $a \oplus 0 = a$ *and* $0 \oplus a = a$ *for all* $a \in G$; [two-sided identity]

$(g2)$ *For each $a \in G$, there is $b \in G$ such that $a \oplus b = 0, b \oplus a = 0$.* [two-sided inverse]

For $a, b, c \in G$, define [gyrator identity]

$$gyr[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)),$$

then

$(g3)$ $gyr[a, b] \in Aut(G, \oplus)$; [gyroautomorphism]

$(g3a)$ $a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$; [left gyroassociative law]

$(g3b)$ $(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$; [right gyroassociative law]

$(g4a)$ $gyr[a, b] = gyr[a \oplus b, b]$; [left loop property]

$(g4b)$ $gyr[a, b] = gyr[a, b \oplus a]$. [right loop property]

Let Γ be a graph. The set of vertices of a graph Γ is denoted by $V(\Gamma)$ and the set of edges of a graph Γ is denoted by $E(\Gamma)$. A graph Γ is called *undirected* if every pair of adjecent vertices has a bidirectional edge. A (directed) graph Γ is called *connected* if there exist a directed path from u to v and a directed path from v to u for any pair of vertices (u, v). A mapping $f : V(\Gamma) \to V(\Gamma)$ is called *endomorphism* if $(f(x), f(y)) \in E(\Gamma)$ for any $(x, y) \in E(\Gamma)$. An endomorphism map f is called *automorphism* if f is bijective. The set of all automorphisms of a graph Γ is denoted by $Aut(\Gamma)$. A graph Γ is called a *vertex-transitive graph* if for any $x, y \in V(\Gamma)$, there exists $f \in Aut(\Gamma)$ such that $f(x) = y$.

Definition 2.8. Let $(G, \oplus)$ be a gyrogroup and $\emptyset \neq S \subseteq G \setminus \{0\}$. The *Cayley digraph*, or simply *Cayley graph*, $Cay(G, S)$ is the simple directed graph whose vertex set and edge set are

$$V(Cay(G, S)) = G; E(Cay(G, S)) = \{(u, v) \in G \times G : v = s \oplus u \text{ for some } s \in S\}.$$

Remark 2.9. $(u, v) \in E(Cay(G, S))$ is denoted by $u \to v$, if $v \boxminus u \in S$.

Remark 2.10. If $S = \{s_1, s_2, \dots s_n\}$, then the Cayley graph $Cay(G, S)$ is the union of Cayley graphs $Cay(G, \{s_i\}), i = 1, 2, \dots, n$, i.e.,

$$V(Cay(G, S)) = \bigcup_{s_i \in S} V(Cay(G, \{s_i\})) \text{ and } E(Cay(G, S)) = \bigcup_{s_i \in S} E(Cay(G, \{s_i\})).$$

Indeed, $(u, v) \in E(Cay(G, S))$ is equivalent to $(u, v) \in E(Cay(G, \{s\})$ for some $s \in S$, i.e., to $(u, v) \in \bigcup_{s_i \in S} E(Cay(G, \{s_i\}))$

Remark 2.11. The Cayley graph of a gyrogroup is regular since the outdegree and the indegree of every vertex of $Cay(G, S)$ equal to $|S|$.

Lemma 2.12. *Let $(G, \oplus)$ be a gyrogroup and $S \subseteq G$. Then $(0, s) \in Cay(G, S)$ for all $s \in S$.*

Proof. Let $s \in S$. It is obtained by the right identity property that $s = s \oplus 0$. Thus, $(0, s) \in E(Cay(G, S)$, that is $0 \to s$ for all $s \in S$. ☐

3. Main Results

Recall that $S^{-1} = \{\ominus s : s \in S\}$ and a set S is called symmetric if $S = S^{-1}$.

Theorem 3.1. *Let $(G, \oplus)$ be a gyrogroup and let S be a nonempty subset of G. Then, $Cay(G, S)$ is undirected if and only if S is symmetric.*

Proof. Let $x, y \in G$ such that $x \to y$. Then $y \boxminus x \in S$. Since $x \boxminus y = \ominus(y \boxminus x) \in S^{-1} = S$, $Cay(G, S)$ is undirected.

Conversely, for any $s \in S$, we have $0 \to s$. By assumption, there is $t \in S$ such that $0 = t \oplus s$. Hence $\ominus s = t \in S$. That is $S = S^{-1}$. □

It is well-known that the Cayley graph of a group is connected if and only if the generating set spans a group. However, this fact need not be satisfied for the Cayley graph of a gyrogroup. The following example show that the spaning condition does not guarantee connectedness of Cayley graphs of gyrogroups.

Example 3.2. Let $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with addition $\oplus$ and gyration table which are defined as follows:

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	5	6	0	7	1	2	4
4	4	2	1	7	0	6	5	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	A	A	A	A	I	I
2	I	A	I	A	A	I	A	I
3	I	A	A	I	I	A	A	I
4	I	A	A	I	I	A	A	I
5	I	A	I	A	A	I	A	I
6	I	I	A	A	A	A	I	I
7	I	I	I	I	I	I	I	I

where a mapping $A : G \to G$ is given by

$$\begin{array}{ll} 0 \mapsto 0 & 4 \mapsto 4 \\ 1 \mapsto 6 & 5 \mapsto 2 \\ 2 \mapsto 5 & 6 \mapsto 1 \\ 3 \mapsto 3 & 7 \mapsto 7 \end{array}$$

Since $A \circ L_a = L_{A(a)}$ for all $a \in G$, the mapping A is an automorphism. By using Theorem 2.7, we obtain that $(G, \oplus)$ is a gyrogroup. Let $S = \{1, 2\}$. Then $\langle S \rangle = G$ since $3 = 1 \oplus 2, 4 = 1 \oplus ((1 \oplus 2) \oplus 1), 5 = (1 \oplus 2) \oplus 1, 6 = (1 \oplus 2) \oplus 2$ and $7 = 2 \oplus ((1 \oplus 2) \oplus 1)$. By Definition 2.8, the graph in Figure 1 is not connected.

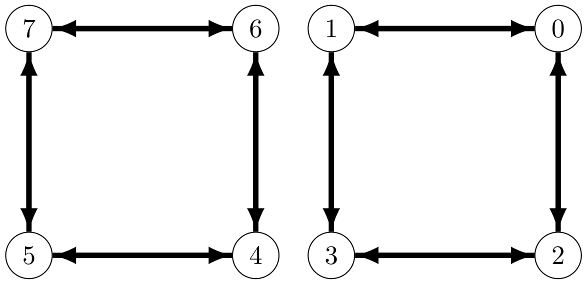

Figure 1: $Cay(G, \{1, 2\})$

Let S be a nonempty subset of a gyrogroup G. The left-generating subset by S of G, denoted by $(S\rangle$, is defined by

$$(S\rangle = \{s_n \oplus (\ldots \oplus (s_3 \oplus (s_2 \oplus s_1))\ldots) : n \in \mathbb{N}, s_1, s_2, s_3, \ldots, s_n \in S\}.$$

Note that $(S\rangle$ is a subset of a subgyrogroup $\langle S\rangle$ of a gyrogroup G but, in general, the subset $(S\rangle$ need not form a subgyrogroup of G, such as $(\{1, 2\}\rangle = \{0, 1, 2, 3\}$ does not form a subgyrogroup of G in Example 3.2 since $1, 3 \in (\{1, 2\}\rangle$ while $3 \oplus 1 \notin (\{1, 2\}\rangle$. The connectedness of Cayley graphs of gyrogroups is assured by the following theorem.

Theorem 3.3. *Let G be a gyrogroup and S be a nonempty subset of G such that S is symmetric. Then $Cay(G, S)$ is connected if and only if $(S\rangle = G$.*

Proof. Assume that $Cay(G, S)$ is connected. Then $(S\rangle \subseteq \langle S\rangle = G$ by Theorem 3.5. For each $x \in G$. By connectedness of $Cay(G, S)$, there are $s_1, s_2, \ldots, s_n \in S$ such that $y_1 = s_1 \oplus 0, y_2 = s_2 \oplus y_1, \ldots, y_n = s_n \oplus y_{n-1}$. Hence,

$$0 \to y_1 \to y_2 \to \cdots \to y_n \to x$$

. That is $x = s \oplus y_n = s \oplus (s_n \oplus \ldots \oplus (s_3 \oplus (s_2 \oplus s_1)))$ for some $s \in S$. Thus, $x \in (S\rangle$, that is $(S\rangle = G$.

Conversely, we assume that $(S\rangle = G$. It is sufficiently to see that $0 \to x$ and $x \to 0$ for any $x \in G$. Let $x \in G$. By assumption, there exist $s_1, s_2, s_3, \ldots, s_n \in S$ such that $x = s_n \oplus \ldots \oplus (s_3 \oplus (s_2 \oplus s_1))$. By using Lemma 2.12, we can see that $0 \to s_1 \to s_2 \oplus s_1 \to \cdots \to x$. Since S is symmetric, by Theorem 3.1, we also obtain that $x \to 0$. Thus, $Cay(G, S)$ is connected. □

Example 3.4. The Cayley graph of gyrogroup $(G, \oplus)$ defined in Example 3.2 with the generating set $S = \{1, 3\}$ is connected since $(S\rangle = G$.

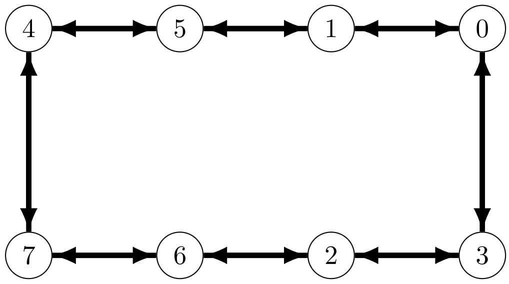

Figure 2: $Cay(G, \{1, 3\})$

Note that if G forms a group, then $(S) = \langle S \rangle$ and the connectness of the Cayley graphs of gyrogroups and groups are homologous by above theorem. However, in the case of gyrogroup G, the following corollaries result from Theorem 3.3.

Corollary 3.5. *Let* $(G, \oplus)$ *be a gyrogroup and* $S \subseteq G$*. If* $Cay(G, S)$ *is connected, then* $\langle S \rangle = G$*.*

Corollary 3.6. *Let* G *be a gyrogroup. If* $Cay(G, \{a\})$ *is connected for some* $a \in G$*, then* G *is group.*

Recall that a graph Γ is vertex-transitive if for all $x, y \in V(\Gamma)$, there exists $f \in Aut(\Gamma)$ such that $f(x) = y$.

Example 3.7. The Cayley graph of gyrogroup $(G, \oplus)$ defined in Example 3.2 with the generating set $S = \{1, 2, 3\}$ is not vertex-transitive.

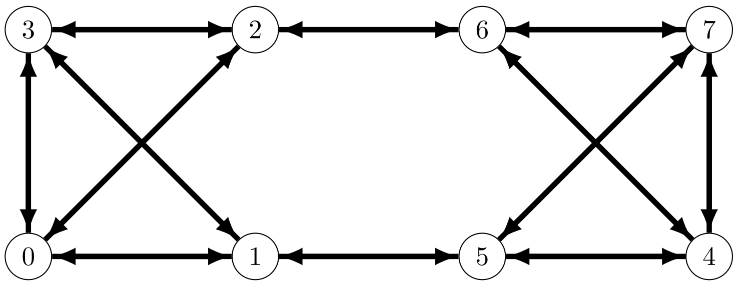

Figure 3: $Cay(G, \{1, 2, 3\})$

Proof. Suppose that the graph in Figure 3, denoted by Γ, is vertex-transitive. There exists an automorphism $f : G \to G$ such that $f(2) = 0$. Then $f(0), f(3), f(6)$ possibly belong to $\{1, 2, 3\}$ since $(2, 0), (2, 3), (2, 6) \in E(\Gamma)$. If $f(0) = 2$, then $f(1)$ must be 6 since f bijective which is a contradiction to $(1, 3) \in E(\Gamma)$ but $(f(1), f(3)) \notin E(\Gamma)$. In another cases can be proved analogous. Thus, $\Gamma = Cay(G, \{1, 2, 3\})$ is not vertex-transitive. □

4. Conclusion

In this paper, we studied Cayley graphs of gyrogroups and its well-known properties, including the direction and the connectivity. Moreover, we conclude that Cayley graph of a gyrogroup need not be a vertex-transitive graph.

Problem: *When a Cayley graph of a gyrogroup is vertex-transitive?*

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L. Bussaban and S. Suantai
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand
E-mail: lim.bussaban@gmail.com, suthep.s@cmu.ac.th

A. Kaewkhao
Center of Excellence in Mathematics and Applies Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand
E-mail: akaewkhao@yahoo.com